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MATHEMATICAL MODELLING FOR RENAL FLUID FLOWS

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Abstract:

This paper presents mathematical model for fluid flow which takes place in renal tubules in kidneys. The characteristic feature of this fluid flow is that, in addition to axial velocity, there is a radial velocity because of reabsorption of water and other low molecular weight substances along the walls of the renal tubule. Here we study two special cases in which the radial velocity varies either linearly or exponentially along the wall of the renal tubule. We have also discussed maximum and minimum values for radial and axial velocities along with limitations of pressure drop over the tube length L.

Keywords: Kidney, renal tubule, radial velocity, axial velocity, pressure drop.

Introduction

In recent years various researchers have shown their keen interest in mathematical modeling for flows in biofluid dyamics [1-5]. During last decades an extensive research work has been done on the fluid dynamics of biological fluids in presence and absence of magnetic fluid due to bio engineering and medical application [6-8]. Mathematical model of the renal tubule are classified as epithelial, tubular or multi tubular. The tubular models allow prediction of the effect of epithelial transport to modify the luminal solution and conversely the effect of altered luminal composition on the trans epithelial fluxes. Here the conservation equation constitute asset of ODE with initial data that must be integrated along tubule length. The multitubular models are mathematically the most complex and have character of BVP, where transport along an entire tubule contributes to the interstitial composition. Frank and Stephenson [9] have analysed the model in which a force of unspecified origin drives the fluid from descending thin limb (DTL) to interstitial vascular space, thus concentrating the solution in DTL. Layton et al. [10] have solved hyperbolic PDE of renal model explicitly for both dynamic and steady state. Pitman [11] have studied mass conservation in a dynamic numerical method for model of the urine concentrating mechanism. Anita et al. [12] have studied the dynamics of coupled nephrons.

Infact the functional unit of the kidney is called the nephron or renal tubule, and each kidney has about 1 million of these tubules. One major part of a nephron is the glomerular tuft through which blood coming from the renal artery and afferent arterioles is filtered. The glomerular filtrate is essentially identical to plasma, and no chemical separation occurs up to this point. If the kidneys gives this filtrate for excretion, the body loses many important materials, like water, at a faster rate than the one at which they can be provided by synthesis of feeding. The rest of the nephron therefore recovers these useful materials and give them back to the blood. Thus nearly 80 percent of the filtrate is reasbsorbed and of the remaining, about 95 percent is again reabsorbed by the end of the remaining, about 95 percent is again reabsorbed by the end of the collecting ducts.

This reabsorption or seepage create a redial component of the velocity in the cylindrical tubule, which must be considered along with the axial component of the velocity (Seen Fig. 1)

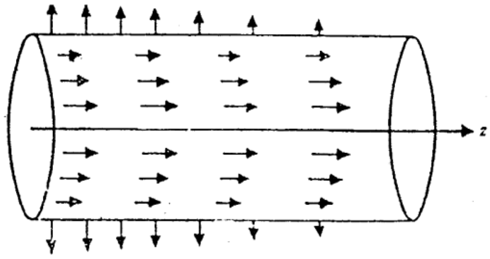

Fig. 1 Two-dimensional flow in renal tubule

Due to loss of fluid from the walls, both the radial and axial velocities decrease with z. Mathematically, we have to find solution for flow of fluid with viscosity in a circular cylinder when there are axial and radial components of velocity and the radial velocity at all points on the cylindrical surface is prescribed and is a decreasing function □(z) of z.

2. Basic Equations and Boundary Conditions

At the outset, we may not that the equation of motion can be simplified since the inertial term in relation to the viscous terms can be neglected. The average tubular radius is about 10^{-3} cm, the average velocity is about 10^{-1} cm/sec, and the fluid viscosity is about 7×10^{-3} dyness sec/cm^2, this gives a Reynolds number of about 10^{-3} and, since this is very much less than one, we neglect the inertial terms to get the following equations of continuity and motion

(2.1) $$\frac{I}{r}\frac{\partial}{\partial r}(rv_r)\frac{\partial v_z}{\partial z}=0$$

(2.2) $$\frac{1}{\mu}\frac{\partial p}{\partial r}=\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)+\frac{\partial^2 v_r}{\partial z^2}$$

(2.3) $$\frac{1}{\mu}\frac{\partial p}{\partial z}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{\partial^2 v_z}{\partial z^2}$$

The boundary conditions are

(2.4) $$\frac{\partial v_z}{\partial r}=0, v_r=0, v_z=\text{finite at } r=0$$

(2.5) $v_z = 0$, $v_r =$ □(z) at r = R

(2.6) $p = p_0$ at z = 0

$P = p_L$ at $z = L$

Eliminating p between (2.2) and (2.3), we get

(2.7) $$\frac{\partial^2}{\partial r \partial z}\left[\frac{1}{r}\frac{\partial}{\partial z}(rv_r)\right]+\frac{\partial^3 v_r}{\partial z^3}=\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right]+\frac{\partial^3 v_z}{\partial z^2 \partial r}$$

Taking the partial derivative of this equation with respect to z and substituting from (2.1), we get

(2.8) $$\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)\right)\right]+2\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial z^2}\right)\right]+\frac{1}{r}\frac{\partial^4}{\partial z^4}\right\}(rv_r)=0$$

Alternatively, we can satisfy (2.1) by taking

(2.9) $$v_r=\frac{1}{r}\frac{\partial \psi}{\partial z},\; v_z=-\frac{1}{r}\frac{\partial \psi}{\partial r}$$

Substituting (2.9) in (2.7), we get

(2.10) $$D^2(D^2\psi)=0$$

where the operation D^2 is defined by

(2.11) $$D^2=\left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)$$

if

(2.12) $v_r = f(r)\, g(z)$,

then the form of (2.8) suggests that an analytical solution may be possible if

(2.13) $g(z) = A_0 + A_1 z$ or $g(z) = A_2 e^{-y2}$,

From (2.5) since v_r = □(z) when r = R, we find

(2.14) f(R) g(z) = □(z).

This suggests that we may get an analytical solution when the radial part of velocity on the surface of the cylinder is written as

(2.15) □(z) = $a_0 + a_1 z$ or □(z) = ce^{-yz}

3. Solutions : Below we consider these two cases in sections as case I and case II

Case I : When Radial Velocity at Wall varies (decreases) as a linear function of z.

Fro (2.10), we assume the solution as

(3.1 $$\psi(r,z)=f(r)a_0 z+\frac{1}{2}a_1 z^2)+G(r)$$

so that using (2.9), we find

(3.2) $$v_r=\frac{1}{r}F(r)(a_0+a_1 z),$$

$$v_Z = -\frac{1}{r}F^*(r)\left(a_0 z + \frac{1}{2}a_1 z^2\right) - \frac{1}{r}G^*(r),$$

(3.3) $$D^2\psi = \left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{r}\right)F(r)\left(a_0 z + \frac{1}{2}a_1 z^2\right) + \left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right)$$
$$G(r) + a_1 F(r)$$

(3.4) $$D^2(D^2\psi) = \left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right)^2 F(r)\left(a_0 z + \frac{1}{2}a_1 z^2\right) + \left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right)^2$$
$$G(r) + 2a_1\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right)F(r) = 0$$

Using equations (2.10) and (3.4), we obtain

(3.5) $$\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right)^2 F(r) = 0$$

(3.6) $$\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right)^2 G(r) + 2a_1\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right)F(r) = 0$$

Equation (3.5) gives

(3.7) $$\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right)H(r) = 0, \quad \left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right)F(r) = H(r)$$

Solving (3.7), we get

(3.8) $H(r) = Ar^2 + B$

(3.9) $$r^2\frac{d^2F}{dr^2} - r\frac{dF}{dr} = Ar^4 + Br^2$$

Integrating (3.9), we obtain

(3.10) $$F(r) = C + Dr^2 + \frac{Ar^4}{8} + \frac{Br^2}{2}\ln r$$

From (3.6) and (3.10)

(3.11) $$\left(\frac{d^2}{dr^2} - \frac{1}{2}\frac{d}{dr}\right)\left[\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right)G(r) + 2a_1 F(r)\right] = 0$$

Using (3.7) and (3.8), we get

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(3.12) $\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)G(r)+2a_1F(r)=Mr^2+N$

Now from (3.4), (2.5) &(3.2)

(3.13) $\frac{d}{dr}\left[F^*(r)\right]=0 \text{ at } r=0,$

$\frac{d}{dr}\left[\frac{1}{r}G^*(r)\right]=0 \text{ at } r=0,$

(3.14) $\frac{1}{r}F(r)=0 \text{ at } r=0,$

(3.15) $\frac{1}{r}F^*(r) \text{ and } \frac{1}{r}G^*(r)$ are finite at r = 0,

(3.16) $F^*(r)=0,\ \ G^*(R)=0,\ \ F(R)=R$

From (3.10), (3.14) and (3.15), we find

(3.17) C = 0, B = 0

From (3.10), (3.16) and (3.17), we get

(3.18) $2DR+\frac{1}{2}AR^3=0,\ \ Dr^2+\frac{AR^4}{8}=R$

so that

(3.19) $F(r)=\frac{2r^2}{R}-\frac{r^4}{R^3}=R\left[2\left(\frac{r}{R}\right)^2-\left(\frac{r}{R}\right)^4\right]$

Substituting (3.19) in (3.12), we get

(3.20) $\frac{d^2G}{dr^2}-\frac{1}{r}\frac{dG}{dr}=Mr^2+N-4a_1\frac{r^2}{R}+2a_1\frac{r^4}{R^3}$

Integrating (3.20), we obtain

(3.21) $G(r)=M_1r^2+N_1+\frac{Mr^4}{8}+\frac{Nr^2\ln r}{2}-\frac{a_1}{2}\frac{r^4}{R}+\frac{a_1}{12}\frac{r^6}{R^3}$

From (3.15) and (3.21)

(3.22) N = 0

From equations (3.16) and (3.21), we find

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(3.23) $2M_1R+\frac{1}{2}MR^3-\frac{3a_1}{2}R^2=0$

Equation (3.23) can determine only one of the two unknown constants M and M_1. To determine both these, we need one more relation. Using (3.2), we get

(3.24) $Q(z)=\int_0^R 2\sigma r v_z(r,z)dr$

$$=2\int_0^R\left[\left(\frac{4r^3}{R^3}-\frac{4r}{R}\right)\left(a_0z+\frac{1}{2}a_1z^2\right)-2M_1r-\frac{Mr^3}{2}-\frac{2a_1}{R}r^3+\frac{a_1r^5}{2R^3}\right]dr$$

(3.25) $\frac{Q_0}{2\pi R^2}=\frac{MR^2}{8}-\frac{a_1}{3}R,$

(3.26) $M=\frac{8}{R^2}\left(\frac{Q_0}{2\pi R^2}+\frac{a_1R}{3}\right)$

(3.27) $M_1=\frac{Q_0}{\pi R^2}+\frac{a_1R}{12}$

From (3.21), (3.22), (3.26) and (3.27), we get

(3.28) $G(r)=\left(\frac{a_1R}{12}-\frac{Q_0}{\pi R^2}\right)r^2+N_1+\frac{1}{R^2}\left(\frac{Q_0}{2\pi R^2}+\frac{a_1R}{3}\right)r^4$

$$-\frac{a_1}{2}\frac{r^4}{R}+\frac{a_1}{12}\frac{r^6}{R^3}$$

The constant N_1 need not be determined since $\psi(r,z)$ can always contain anarbitrary constant without affecting the velocity components.

(3.29) $v_r(r,z)=\left[2\frac{r}{R}-\left(\frac{r}{R}\right)^3\right](a_0+a_1z),$

(3.30) $v_Z(r,z)=-4\left(\frac{r}{R}-\frac{r^3}{R^3}\right)\left(a_0z+\frac{1}{2}a_1z^2\right)-$

$$2\left(\frac{a_1R}{12}-\frac{Q_0}{\pi R^2}-\frac{4}{R^2}\frac{Q_0}{2\pi R^2}+\frac{a_1R}{3}\right)r^2+2a_1\frac{r^2}{R}-\frac{a_1}{2}\frac{r^4}{R^3}$$

$$=\left(1-\frac{r^2}{R^2}\right)\left[\frac{2Q_0}{\pi R^2}-\frac{2}{R}(2a_0z+a_1z^2-\frac{a_1R}{2}\left(\frac{1}{3}-\frac{r^2}{R^2}\right)\right]$$

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Differentiating (3.24), we obtain

(3.31) $$\frac{dQ}{dz} = 8\pi(a_0 + a_1 z)\int_0^R \left(\frac{r^3}{R^3} - \frac{r}{R}\right) dr = -2\pi R(a_0 + a_1 z)$$

so that the decrease of flux is equal to the amount of the fluid coming out of the cylinder per unit length per unit time. Integrating (3.31), we get

(3.32) $$Q(z) = Q_0 - \pi R(2a_0 z + a_1 z^2),$$

From (3.30) and (3.32), we find

(3.33) $$v_Z = \left(1 - \frac{r^2}{R^2}\right)\left[\frac{2Q(z)}{\pi R^2} - \frac{a_1 R}{2}\left(\frac{1}{8} - \frac{r^2}{R^2}\right)\right]$$

For Hagen-Poiseuille flow in a circular tube, we have

(3.34) $$v_z = \left(1 - \frac{R^2}{R^2}\right)\frac{2Q}{\pi R^2}$$

Comparing (3.33) and (3.34), we have two changes (i) Q is replaced by the variable Q(z), and (ii) there is further distortion due to the varying nature of the radial flow.

Using (2.2), (2.3), (3.29) & (3.34) & (3.32), we find

(3.35) $$\frac{\partial p}{\partial r} = \frac{8\mu r}{R^3}(a_0 + a_1 z),$$

(3.36) $$\frac{\partial p}{\partial z} = -\frac{4a_1\mu}{R}\left[\frac{r^2}{R^2} + \frac{2Q(z)}{a_1\pi R^3} + \frac{1}{3}\right]$$

Integrating (3.35), we obtain

(3.37) $$p(r,z) = -\frac{4\mu r^2}{R^3}(a_0 + a_1 z) + K(z)$$

Differentiating (3.37) partially with respect to z and then substituting $\partial p/\partial z$ (3.36), we get

(3.38) $$K^*(z) = -\frac{4a_1\mu}{R}\left(\frac{1}{3} + \frac{2Q(z)}{a_1\pi R^3}\right)$$

so that

(3.39) $$K(z) = -\frac{4a_1\mu}{R}\left(\frac{1}{3}z + \frac{2\bar{Q}(z)}{a_1\pi R^3}\right) + K_0$$

where

(3.40) $Q(z) = \int_0^2 Q(z)dz$

Substituting from (3.39) in (3.38), we get

(3.41) $p(r,z) - p(0,0) = -\frac{4\mu}{R}(a_0 + a_1 z)\frac{r^2}{R^2} - \mu\left(\frac{4a_1}{3R} + \frac{8Q}{\pi R^4}\right)z$

The average pressure $\bar{p}(z)$ at any section is given by

(3.42) $\bar{p}(z) = \frac{\int_0^R p(r,z)2\pi r\, dr}{\int_0^R 2\pi r\, dr} = -\mu\left[\frac{2a_0}{R} + \left(\frac{8Q(z)}{\pi R^4} + \frac{10a_1}{3R}\right)z\right]$

Thus the pressure drop over the tube length L is

(3.43) $\Delta\bar{p} = \bar{p}(0) - \bar{p}(L) = \mu\left(\frac{8\bar{Q}(L)}{\pi R^4} + \frac{10a_1}{3R}\right)L$

Case II : When Radial Velocity at Wall varies (decreases) Exponentially with z.
We assume a solution of the form

(3.44) $\psi(r,z) = f(r)e^{-\alpha z} + g(r)$

so that the equation

(3.45) $D^2(D^2\psi) = \left[\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right)^2 + 2\alpha^2\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right) + \alpha^4\right]$

$$f(r)e^{-\alpha z} + \left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right)^2 g(r) = 0$$

gives

(3.46) $\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \alpha^2\right)f(r) = 0$

(3.47) $\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)g(r) = 0$

Those solutions of (3.46) and (3.47) which are finite at r = 0 are

(3.48) $f(r) = A_1 rJ_1(\alpha r) + A_2 r^2 J_2(\alpha r)$

(3.49) $g(r) = A_3 r^4 + A_4 r^2 + A_5$

Using (2.9) in these equations, we get

(3.50) $v_r(r,z) = -\frac{f(r)}{r}\alpha e^{-\alpha z}$

$$v_2(r,z) = \frac{f^*(r)}{r}e^{-\alpha z} - \frac{g^*(r)}{r}$$

The boundary conditions (2.4) and (2.5) with $\phi(z) = v_0 e^{-\alpha z}$ give

(3.51) $\frac{d}{dr}\left[\frac{f^*(r)}{r}\right] = 0, \frac{d}{dr}\left[\frac{g^*(r)}{r}\right] = 0, \frac{f(r)}{r} = 0$

(3.52) $\frac{f^*(R)}{R} = 0, \frac{g^*(R)}{R} = 0, -\frac{f(R)}{R}\alpha = v_0$

(3.53) $A_1 J_0(\alpha R) + A_2 R J_1(\alpha R) = 0$

(3.54) $A_1 J(\alpha R) + A_2 R J_2(\alpha R) + v_0/\alpha = 0$

(3.55) $4A_3R^3 + 2A_4R = 0$

(3.56) $v_r(r,z) = \frac{J_1(\alpha R)J_1(\alpha r) - (r/r)J_0(\alpha R)J_2(\alpha r)}{J_1^2(\alpha R) - J_2(\alpha r)J_0(\alpha r)} v_{0e^{-\alpha z}},$

(3.57) $v_z(r,z) = \frac{J_1(\alpha R)J_0(\alpha r) - (r/R)J_0(\alpha R)J_1(\alpha r)}{J_1^2(\alpha R) - J_2(\alpha R)J_0(\alpha R)}$

$$v_0 e^{-\alpha z} + 4A_2R^2\left(1 - \frac{r^2}{R^2}\right)$$

where the constant A_3 has to be determined in terms of Q_0 which is the initial flux at z = 0, Hence, using (2.3.65), (2.3.68) and (3.2.70) we get

(3.58) $Q_0 = \int_0^R 2\pi r v_z(r,0)dr = 2\pi\int_0^R [-f^*(r) - g^* r)]$

$$r = 2\pi\left[f(0) + g(0) - f(R) - g(R)\right]$$

$$= 2\pi\left[-A_1RJ_1(\alpha R) - A_2R^2J_2(\alpha R) - A_3R^4 - A_4R^2\right]$$

$$= 2\pi\left(\frac{v_0R}{\alpha} + A_3R^4\right)$$

so that

(3.59) $A_3R^2 = \frac{Q_0}{2\pi R^2} - \frac{v_0}{\alpha R}$

From (2.2), (3.16) and (3.50), we have

(3.60) $$\frac{1}{\mu}\frac{\partial}{\partial r} = \frac{\partial}{\partial r}\left\{\frac{\partial}{\partial r}\frac{\partial}{\partial r}\left[-\alpha f(r)e^{-\alpha z}\right]\right\} - \frac{a^3 f(r)}{r}e^{-\alpha z} = -e^{-\alpha z}\left[\alpha\frac{\partial}{\partial r}\left(\frac{f^*(r)}{r}\right) + \alpha^3\frac{r(r)}{r}\right],$$

(3.61) $$\frac{1}{\mu}\frac{\partial}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\left[\frac{f^*(r)}{r}e^{-\alpha z} + \frac{g^*(r)}{r}\right]\right\} - \alpha^2\frac{f^*(r)}{r}e^{-\alpha z}$$
$$= -e^{-\alpha z}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\frac{\left(f^*(r)\right)}{r}\right] + \frac{x^2 f^*(r)}{r}\right\} - 16A,$$

Integrating (3.61), we obtain

(3.62) $$p(r,z) - p(0,0) = (1 - e^{-\alpha z})\left[\alpha\frac{f^*(r)}{r} + \alpha^3\int_0^r \frac{f(r)}{r}dr\right] - 16A_3 z$$

4. Extremum values (For Radial and Axial Velocities)

Case I : (a) Radial velocity : We know the condition that the maximum and minimum values of an expression is given by

(4.1) If $f'(c) = 0$ and $f''(c) < 0$

Then f(x) has a maximum at x = c

If $f'(c) = 0$ and $f''(c) > 0$

Then f(x) has a minimum at x = c

Differentiating equation (3.29) w.r.t. r, we find

(4.2) $$\frac{d\upsilon_r}{dr} = \left[\frac{2}{R} - \frac{3r^2}{R^3}\right](a_0 + a_1 z)$$

From the condition (4.1) equation (4.2) becomes

(4.3) $$\frac{2}{R} - \frac{3r^2}{R^3} = 0$$

From (4.3), we get

(4.4) $$\frac{r}{R} = \pm\sqrt{\frac{2}{3}}$$

Differentiating again (4.2), we get

(4.5) $$\frac{d^2\upsilon_r}{dr^2} = -\frac{6r}{R^3}(a_0 + a_1 z)$$

Clearly $\frac{d^2\upsilon_r}{dr^2} < 0$ when $\frac{r}{R} = \sqrt{\frac{2}{3}}$

The maximum radial velocity exists at $\frac{r}{R} = (2/3)^{1/2}$

Then the maximum radial velocity is given by

$$(\upsilon_r)_{max} = \left[2\left(\frac{2}{3}\right)^{1/2} - \left(\frac{2}{3}\right)^{3/2}\right](a_0 + a_1 z)$$

(4.6) $$\Rightarrow (\upsilon_r)_{max} = \frac{4}{3}\sqrt{\frac{2}{3}}(a_0) + (a_1 z)$$

Also from equation (4.5), we see

$$\frac{d^2\upsilon_r}{dr^2} > 0 \text{ when } \frac{r}{R} = -\left(\frac{2}{3}\right)^{1/2}$$

Then the minimum radial velocity exists at $\frac{r}{R} = -\left(\frac{2}{3}\right)^{1/2}$

$$(\upsilon_r)_{min} = \left[2\left(-\sqrt{\frac{2}{3}}\right) - \left(-\sqrt{\frac{2}{3}}\right)^3\right](a_0 + a_1 z)$$

which implies

(4.7) $$(\upsilon_r)_{min} = -\frac{4}{3}\sqrt{\frac{2}{3}}(a_0 + a_1 z)$$

Case I (b) : Axial velocity :

Differentiating equation (3.30), we find

(4.8) $$\frac{d\upsilon_z}{dz} = -\frac{2}{R}\left(1 - \frac{r^2}{R^2}\right)(2a_0 + 2a_1 z)$$

From condition (4.1), equation (4.8) provides

(4.9) $$2a_0 + 2a_1 z = 0$$

which gives

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(4.10) $z = -\dfrac{a_0}{a_1}$

Also differentiating equation (4.8) w.r.t. z, we find

(4.11) $\dfrac{d^2\upsilon_Z}{dz^2} = \dfrac{4a_1}{R}\left(1 - \dfrac{r}{R^2}\right)$

We get the four conditions for the extremum values

(i) If $a_1 0$ and $\left(\dfrac{r}{R}\right)^2 < 1$

Then $\dfrac{d^2\upsilon_Z}{dz^2} < 0$, so v_Z has max at $z = -\dfrac{a_0}{a_1}$

(ii) If $a_1 > 0$ and $\left(\dfrac{r}{R}\right)^2 > 1$

Then $\dfrac{d^2\upsilon_Z}{dz^2} > 0$ and therefore υ_Z has minimum at $z = -\dfrac{a_0}{a_1}$

(iii) If $a_1 < 0$ and $\left(\dfrac{r}{R}\right)^2 < 1$

Then $\dfrac{d^2\upsilon_Z}{dz^2} > 0$, so υ_Z has minimum value at $z = -\dfrac{a_0}{a_1}$

(iv) If $a_1 > 0$ and $\left(\dfrac{r}{R}\right)^2 > 1$

Then $\dfrac{d^2\upsilon_Z}{dZ^2} > 0$, there υ_Z has maximum at $Z = -\dfrac{a_0}{a_1}$

The pressure drop for uniform reabsorption rate :

The pressure drop over the tube length L is written as

(4.12) $\Delta\bar{p} = \mu\left(\dfrac{8\bar{Q}(L)}{\pi R^4} + \dfrac{10a_1}{3R}\right)L$

For the uniform reabsorption rate, the pressure drop over the tube length L is given by

(4.13) $\Delta\bar{p} = \dfrac{\mu 8\bar{Q}(L)}{\pi R^4}$

where

$$\bar{Q}(z) = \int_0^Z Q(Z)dz$$

For the uniform reabsorption rate Q(Z) is given by

(4.14) $Q(Z) = Q_0 - 2a_0\pi RZ$

Then $\bar{Q}(Z)$ is given by

(4.15) $\bar{Q}(Z) = Q_0 Z - a_0\pi RZ^2$

From equation (4.13) we find

$$\bar{Q}(Z) = Q_0 Z - \frac{[Q_0 - Q(Z)]}{2} Z$$

which gives

(4.16) $\bar{Q}(Z) = \frac{[Q_0 + Q(Z)]}{2} Z$

and

$Q(0) = Q_0$ at $Z = 0$

From equation (4.13), we get

(4.17) $\Delta\bar{p} = \frac{8\mu}{\pi R^4} \frac{[Q(0) + q(L)]}{2} L$

From equation (4.17) it is clear that

(4.18) $\Delta\bar{p} \geq \frac{4\mu Q(0)}{\pi R^4} L$

From equation (4.14) at Z = L

(4.19) $Q(Z) = Q_0 - 2a_0\pi RL$

From equation (4.17) and (4.19) we find

(4.20) $\Delta\bar{p} = \frac{8\mu L}{\pi R^4} Q(0) - 2\pi Ra_0 Z$

Obviously from (4.20), we have

(4.21) $\Delta\bar{p} \leq \frac{8\mu L}{\pi R^4} Q(0)$

Therefore from (4.18) and (4.21) we conclude

(4.22) $\frac{4\mu Q(0)}{\pi R^4} L \leq \Delta\bar{p} \leq \frac{8\mu L}{\pi R^4} Q(0)$

In case II :

We can calculate as in case I for extremum values (maximum or minimum) when radial velocity at wall varies exponenliatly with z along with pressure drop over the tube Length L.

5. Conclusion and Application

Here we have discussed mathematical model for two dimensional fluid flows in renal tubules (nephron) and have found solution in two cases.

(i) When radial velocity at wall decreases linearly with z
(ii) When radial velocity at wall decreases exponentially with z.

Along with radial velocity, we have also found axial velocity. Here we have also discussed extremum (maximum and minimum) values of these velocities. It is found that radial velocity υ_Z is max, when $\frac{r}{R} = \sqrt{\frac{2}{3}}$ and min when $\frac{r}{R} = -\sqrt{\frac{2}{3}}$.

Also axial velocity υ_Z depends on the constants a_0 and a_1 and also on $\left(\frac{r}{R}\right)^2$ (when it is less than or greater than one). By using equation (4.22) we can find limitation on pressure drop for tube length L.

Our study in mathematical modeling for two dimensional flow in renal tubule (nephron) is very useful and relevant for investigations in biomathematics and specially to under stand kidney functioning.

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