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DISTRIBUTED ORDER EQUATIONS AS BOUNDARY VALUE PROBLEM

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ABSTRACT

A Boundary value problem is an arrangement of conventional differential equations with arrangement and subsidiary qualities indicated at more than one point. Most normally, the arrangement and subsidiaries are indicated at only two focuses (the boundarys) characterizing a two-point boundary value problem. A boundary condition is a remedy a few mixes of estimations of the obscure arrangement and its subordinates at more than one point. In this paper the author has talked about boundary value problems in detail including various problems to make the concept more clear. There has been a new expansion in interest in the utilization of conveyed request differential conditions (especially for the situation where the subordinates are given in the Caputo sense) to demonstrate different wonders. The recent papers have given insight into the mathematical estimation of the arrangement, and a few outcomes on presence and uniqueness have been demonstrated. For each situation, the portrayal of the arrangement depends, among different boundaries, on Caputo-type beginning conditions. In this paper, we examine the presence and uniqueness of arrangements and we propose a mathematical technique for their estimate for the situation where the underlying conditions are not known and, all things being equal, some Caputo-type conditions are parted with from the beginning

INTRODUCTION

A two-point boundary value problem (BVP) of all out request n on a boundary interim [a,b] might be composed as an unequivocal first request arrangement of ordinary differential equations (ODEs) with boundaryvalues assessed at two focuses as

$y'(x)=f(x,y(x)),x\in(a,b),g(y(a),y(b))=0$(1)

Here, $y,f,g\in Rn$ and the framework is called express on the grounds that the subordinate y' shows up unequivocally. The n boundary conditions characterized by g must be autonomous; that is, they can't be communicated as far as one another (if g is straight the boundary conditions must be directly free).

By and by, most BVPs don't emerge legitimately in the structure (1) however rather as a blend of conditions characterizing different requests of subordinates of the factors which aggregate to n . In an unequivocal BVPramework, the boundary conditions and the correct hand sides of the ordinary differential equations (ODEs) can include the subsidiaries of every arrangement variable up to a request one not exactly the most elevated subsidiary of that variable showing up on the left hand side of the ODE characterizing the variable..

The words two-point allude to the way that the boundary condition work g is assessed at the arrangement at the two interim endpoints an and b not at all like for beginning worth problems (IVPs) where the n starting conditions are altogether assessed at a solitary point. Sporadically, problems emerge where the capacity g is additionally assessed at the arrangement at different focuses in (a,b) . In these cases, we have a multipoint BVP. As appeared in Ascher et al. (1995), a multipoint problem might be changed over to a two-point problem by characterizing separate arrangements of factors for each subinterval between the focuses and including boundary conditions which guarantee congruity of the factors over the entire interim. Like revising the first BVP in the minimized structure (1), revamping a multipoint problem as a two-point problem may not prompt anproblem with the most effective computational arrangement.

Most for all intents and purposes emerging two-point BVPs have isolated boundary conditions where the capacity g might be part into two sections (one for every endpoint):

$ga(y(a))=0, gb(y(b))=0.$

Here, $ga\in Rs$ and $gb\in Rn-s$ for some value s with $1<s<n$ and where each of the vector functions ga and gb are independent. However, there are well-known, commonly arising, boundary conditions which are not separated; for example, consider periodic boundary conditions which, for a problem written in the form of equation (1), are

$y(a)-y(b)=0.$

With initial value problems we had a differential equation and value of the solution are specified and an appropriate number of derivatives at the same point (collectively called initial conditions). For instance, for a second order differential equation the initial conditions are,

$$y(t0)=y0y'(t0)=y'0$$

With boundary value problems there is a differential equation and the function are specified and/or derivatives at *different* points, which are called boundary values. For second order differential equations any of the following can be used for boundary conditions.

$y(x0)=y0y(x1)=y1$ (1)

$y'(x0)=y0y'(x1)=y1$ (2)

$y'(x0)=y0y(x1)=y1$ (3)

$y(x0)=y0y'(x1)=y1$ (4)

We'll be looking pretty much exclusively at second order differential equations. We will also be restricting ourselves down to linear differential equations. So, for the purposes of our discussion here we'll be looking almost exclusively at differential equations in the form,

$y''+p(x)y'+q(x)y=g(x)$ (5)

along with one of the sets of boundary conditions given in (1) –, (4). We will, look at some different boundary conditions but the differential equation will always be on that can be written in this form.

We'll see initial value problems will not hold here. We can,solve (5) provided the coefficients are constant and for a few cases in which they aren't. None of that will change. The changes (and perhaps the problems) arise when we move from initial conditions to boundary conditions.

One of the first changes is definition, a differential equation was homogeneous if $g(x)=0g(x)=0$ for all x. Here we will say that a boundary value problem is homogeneous if in addition to $g(x)=0g(x)=0$ we also have $y0=0y0=0$ and $y1=0y1=0$(regardless of the boundary conditions we use). If any of these are not zero we will call the BVP nonhomogeneous.

The biggest change that we see here comes when we solve the boundary value problem. When solving linear initial value problems a unique solution will be guaranteed under very mild conditions. We only looked at this idea for first order IVP's but the idea does extend to higher order IVP's. Here we saw that all we needed to guarantee a unique solution was some basic continuity conditions. With boundary value problems we will often have no solution or infinitely many solutions even for very nice differential equations that would yield a unique solution if we had initial conditions instead of boundary conditions.

Before we get into solving some of these let's next address the question of why we're even talking about these in the first place. As we'll see in the next chapter in the process of solving some partial differential equations we will run into boundary value problems that will need to be solved as well. In fact, a large part of the solution process there will be in dealing with the solution to the BVP. In these cases, the boundary conditions will represent things like the temperature at either end of a bar, or the heat flow into/out of either end of a bar. Or maybe they will represent the location of ends of a vibrating string. So, the boundary conditions there will really be conditions on the boundary of some process.

So, with some of basic stuff out of the way let's find some solutions to a few boundary value problems. Note as well that there really isn't anything new here yet. We know how to solve the differential equation and we know how to find the constants by applying the conditions. The only difference is that here we'll be applying boundary conditions instead of initial conditions.

DISTRIBUTED ORDER EQUATIONS

In this paper, we shall consider distributed order linear equations of the form

m

$$\beta(r) D^r y(t) dr = f(t), \quad 0 \leq t \leq T \quad (1)$$

where $D^r y(t)$ is the derivative of $y(t)$ in the Caputo sense, that is

$$D^r y(t) = {}^{RL}D^r (y - T[y])(t),$$

$T[y]$ is the Taylor polynomial of degree $\lfloor r \rfloor$ for y centred at 0, and ${}^{RL}D^r$ is the Riemann–Liouville derivative of order r, given

$$RLDr := D\lceil r\rceil J\lceil r\rceil - r$$

with $J^{\gamma} y$ being the Riemann–Liouville integral operator defined by:

$$J^{\gamma} y\, t \;=\frac{1}{\quad}\; {}^{t}\, t-s^{\,\gamma-1} y\, s\, ds$$

We shall also assume, without loss of generality, that $m > 0$ is an integer; if m is not an integer, we can always define $\beta(r)$ 0 on the interval $(m, m\ ]$ and use m instead of m in (1). Where appropriate, we extend the functions f and β to functions defined on R^{+} by setting their values to zero outside the intervals 0, T and 0, m , respectively.

We assume further the following:

(H1) β is an absolutely integrable function on the interval [0, m] and satisfies

$$\overset{m}{\quad} \beta(r)s^{r}\, dr /= 0, \quad \text{for } Re(s) > 0,$$

(H2) $f \in L^{1}[0, \infty)$,

(H3) y is such that $D^{r} y < M$ for $t \in [0, \infty)$ and for every $r \in [0, m]$.

In, Caputo provided the basic analysis for simple distributed order equations of the form

$$\int_{\alpha}^{\gamma} \beta(r) D^{m+r} y(t) dr = f(t),$$

where $0 < \alpha < \gamma < 1$ and m N. This distributed order equation is, in fact, a particular case of Eq. (1), where there are no integer order derivatives, since in this case all the orders of the derivatives belong to the interval $(m, m\ 1)$. In [10] Diethelm and Ford, proceeding similarly to Caputo in [9] established the existence and uniqueness of the solution of Eq. (1) and proved the following theorem:

Theorem 1. *If conditions (H1)–(H3) hold, then Eq. (1) has a unique solution given by:*

$$y(t) = y(0) + \left(f * \mathcal{L}^{-1}\left[\frac{1}{\int_0^m \beta(r)(\cdot)^r dr}\right]\right)(t) + \sum_{k=1}^{m-1} y^k(0)\mathcal{L}^{-1}\left[\frac{\int_k^m \beta(r)(\cdot)^{r-k-1} dr}{\int_0^m \beta(r)(\cdot)^r dr}\right](t), \quad (2$$

where $*$ denotes the standard convolution operation

$$(g * h)(t) = \int_0^t g(t-\tau) h(\tau) d\tau,$$

for suitable functions g and h, and L−1 is the inverse Laplace transform.

Expression (2) gives an indication of the ways in which the solution depends on the right-hand side function f , on the Caputo-type initial conditions yk(0), k 0, . . . , m 1 and on β, the distribution of the derivatives.

In [11,10] the authors also proposed a numerical method to solve the distributed order differential equation (1), given the initial conditions yk(0), k 0, . . . , m 1.

Here we will follow a similar approach but we assume that the Caputo-type initial conditions are not known and instead, the following boundary conditions are given

$y(k)(a) = yk$, $k = 0, . . . , m - 1$, for a certain a, $0 < a \leq T$. (3)

If a T then (3) defines a boundary condition in the usual sense. If a $< T$ then one can regard (3) as a generalised boundary condition or interior condition.

The paper is organised in the following way: because distributed order differential equations may be considered as generalisations of single or multi-term fractional differential equations, in Section 2 we briefly review the recently obtained results for single and multi-term fractional boundary value problems. In Section 3 we discuss the existence and uniqueness of the solution of problem (1) and (3). In Section 4 we propose a numerical method to approximate the solution of the distributed order equation. We finish by presenting some numerical results that illustrate the good performance of our proposed method and we end with some conclusions.

2. Single and multi-term fractional differential equations

As pointed out in [11,10], distributed order differential equations may be regarded as a generalisation of single-term fractional differential equations

$D^r y(t) = f (t, y(t))$

or multi-term fractional differential equations

Therefore, in this section, we will briefly recall the main results obtained for single and multi-term fractional differential equations as boundary value problems.

Concerning single-term fractional boundary value problems (FBVPs) of the form

$D^{\alpha}{}_{*}y(t) = f(t, y(t)), \quad y(a) = y_a, a > 0,$

we have proved recently [12] that if

(i) $0 < \alpha < 1$,

(ii) the function f is continuous and satisfies a Lipschitz condition with Lipschitz constant $L > 0$ with respect to its second argument,

(iii) $\frac{2La^{\alpha}}{\Gamma(\alpha+1)} < 1$,

then the FBVP is equivalent to the following integral equation

$$y(t) = y(0) + \left(f * \mathcal{L}^{-1}\left[\frac{1}{\int_0^m \beta(r)(\cdot)^r dr}\right]\right)(t) + \sum_{k=1}^{m-1} y^k(0)\mathcal{L}^{-1}\left[\frac{\int_k^m \beta(r)(\cdot)^{r-k-1}dr}{\int_0^m \beta(r)(\cdot)^r dr}\right](t), \quad (2)$$

We have also proved that if conditions (i)–(iii) are satisfied, then the solution of the FBVP exists and is unique on a certain interval $0, T, T\ a$.

In that paper we also used a shooting algorithm as in Ref. [13] to find the value of $y(0)$ for which the solution of the initial value problem

$$D^{a}y(t) = f(t, y(t)), \quad y(0) = y_0,$$

satisfies the boundary condition $y(a) y_a$.

Very recently, in [14], we extended these results to multi-term FBVPs of the form

$$D^{a}y(t) = f(t, y(t), D^{\beta_1}y(t), D^{\beta_2}y(t), \ldots, D^{\beta_n}y(t)), \quad (4)$$

$$y^{(k)}(a) = y^{(k)}, \ k = 0, 1, \ldots, \lceil a \rceil, \ a > 0, \quad (5)$$

where $a > \beta_1 > \beta_2 > \quad > \beta_n \quad Q$.

Defining M as the least common multiple of the denominators of $a, \beta_1, \ldots, \beta_n, \gamma \quad 1/M$ and $N \quad aM$ we proved that the multi-term FBVP (4)–(5) is equivalent to the following system of N equations

$$D^{\gamma} y_1(t) = y_2(t)$$

$$D^{\gamma} y_2(t) = y_3(t)$$

.

$$D^{\gamma} y_{N-1}(t) = y_N(t)$$
$$D^{\gamma} y_N(t) = f\left(t, y_{\frac{\beta_1}{\gamma}+1}(t), \ldots, y_{\frac{\beta_n}{\gamma}+1}(t)\right),$$

together with conditions

$$y_j(a) = \begin{cases} y_a^{(k)}, & \text{if } j = kM+1 \text{ for some } k \in \mathbb{N} \\ y_a^{(j)}, & \text{else,} \end{cases} \quad (7)$$

in the following sense: whenever Y = (y1, . . . , yN)T with y ∈ C ⌈ α⌉ [0, T], for some T ≥ a, is the solution of the system of Eqs. (6), then the function y = y1 solves the multi-term Eq. (4) and satisfies the boundary conditions (5); on the other hand, whenever y ∈ C ⌈ α⌉ [0, T] is a solution of the multi-term Eq. (4) satisfying the boundary conditions (5), then

Y = (y1, . . . , yN) = y, D y, D y, . . . , D − y, satisfies the system (6) and the conditions (7), for suitable constants

y(j). It should be noted that the values of y(j) are known only in the cases where j = kM + 1 for some k ∈ N, and that the remaining ones are not known.

Taking this into account, and because each of the equations in this system is a single-term fractional differential equation with order 0 < γ < 1, we proved easily that if

(iv) α, β1, . . . , βn Q,

(v) the function f satisfies a uniform Lipschitz condition, with Lipschitz constant L, in all its arguments except for the first

on a suitable domain D,

(vi) 2Laγ < 1,

Γ (γ +1)

then, the multi-term FBVP (4)–(5) has a unique continuous solution on an interval 0, T of the real line, T a.

With respect to the numerical approximation of the solution of (4)–(5), we also proposed a shooting algorithm based on the equivalence between the FBVP (4)–(5) and the corresponding system of equations. As explained fully in that paper, we considered the initial value problem

$D^{\gamma} y_1(t) = y_2(t)$

$D^{\gamma} y_2(t) = y_3(t)$

$$\vdots$$
$$D^{\gamma} y_{N-1}(t) = y_N(t)$$
$$D^{\gamma} y_N(t) = f\left(t, y_{\frac{\beta_1}{\gamma}+1}(t), \ldots, y_{\frac{\beta_n}{\gamma}+1}(t)\right),$$

together with conditions

$$y_j(0) = \begin{cases} y_0^{(k)}, & \text{if } j = kM + 1 \text{ for some } k \in \mathbb{N} \\ 0, & \text{else,} \end{cases}$$

where the values of $y_j(0)$ are not known, whenever $j = kM + 1$, for some $k \in$ N. For the remaining cases we will have
$y_j(0) = 0$ due to the following lemma proved in:

Lemma 2. *Let* $y \in C^k[0, T]$ *for some* $T > 0$ *and some* $k \in$ N, *and let* $0 < q < k$, $q \notin$ N. *Then* $D^q y(0) = 0$.

We have then fixed the unknowns $y_j(0)$ and $y_j(a)$ in order to ensure that the solution of the initial value problem satisfies the boundary conditions imposed at $t = a$.

1. **Numerical method and results**

In this section we present a numerical method to approximate the solution of problems of the form (1) and (3). The idea is to follow the approach used in the previous papers [11,10]. First, we discretise the integral term in the distributed order equation using a quadrature formula. As we will see, this will result on a multi-term fractional differential equation satisfying the boundary conditions imposed at t a, (3). Then, we apply a numerical method to solve the resulting multi-term FBVP as explained in Section 2.

As was pointed out when we have integer orders of the derivatives we will need to take into account the jumps of $D^r y(t)$, when r is an integer, if we intend to use, for example, the trapezium rule to approximate the integral term in the distributed order equation. In that paper [10] the authors proved the following result

Lemma 3. *Let* $y \in C^p[0, T]$ *with some* $p \in$ N *and* $T > 0$. *For every fixed* $t \in (0, T]$, *consider* $D^r y(t) = z(r)$ *as a function of* r.

- *z is a C^{∞} function on $\cup_{j=1}^{p}(j, j+1]$;*
- *At the integer argument $j = 1, 2, \ldots, p-1$ the function z has a jump discontinuity, since*

$$\lim_{r \to j^+} z(r) - \lim_{r \to j^-} z(r) = -y^{(j)}(0).$$

Note that there is no jump discontinuity if and only if $y^{(j)}(0) = 0$.

whose analytical solution is $y(t)$ t^3 $2t$ 4.

Note that in this example, we will have integer order derivatives in the problem since the derivatives lie in the interval 0, 2 . According to Lemma 3 and once the exact solution is known, we will have jump discontinuities as described in that lemma since we do not have homogeneous initial conditions. Therefore we cannot expect the method used in the previous example to perform in the same way with this problem. The phenomenon may be observed in Table 1, where we have used the trapezium rule and the backward difference method to solve the multi-term fractional differential equation. But if we insist, as was done in, that every integer value on the integral interval (in this case 0, 2) is a grid point of the quadrature formula, we can take into account the jumps described in Lemma 3 by using the so-called modified trapezium rule explained in. Some numerical results for this example, using this quadrature method, combined once again with the backward difference method to solve the obtained multi-term differential equation are reported in Table 2.

Table 1
Absolute errors in $y(0.5)$ for Example 2 using the standard trapezium rule with n subintervals and the backward difference method with stepsize h.

h	n		
	2	4	8
0.1	1.26385	0.535302	0.253611
0.05	1.28935	0.536039	0.245587
0.025	1.30012	0.532421	0.240642
0.0125	1.30487	0.529266	0.238199
0.00625	1.30706	0.527318	0.236982
0.003125	1.30810	0.526228	0.236357

Table 2

Absolute errors at $y(0.5)$ for Example 2 using the modified trapezium rule with n subintervals and the backward difference method with stepsize h.

h	n		
	2	4	8
0.1	0.0512000	0.00542023	0.00015709
0.05	0.0286896	0.00603624	0.00418375
0.025	0.0174860	0.00535740	0.00329024
0.0125	0.0119594	0.00451476	0.00215495
0.00625	0.0092399	0.00397546	0.00152362
0.003125	0.0079010	0.00368779	0.00121633

CONCLUSION

So, there are probably several natural questions that can arise at this point. Do all BVP's involve this differential equation and if not why did we spend so much time solving this one to the exclusion of all the other possible differential equations?

The answers to these questions are fairly simple. First, this differential equation is most definitely not the only one used in boundary value problems. It does however exhibit all of the behavior that we wanted to talk about here and has the added bonus of being very easy to solve. So, by using this differential equation almost exclusively we can see and discuss the important behavior that we need to discuss and frees us up from lots of potentially messy solution details and or messy solutions. We will, on occasion, look at other differential equations in the rest of this chapter, but we will still be working almost exclusively with this one.

There is another important reason for looking at this differential equation. When we get to the next chapter and take a brief look at solving partial differential equations we will see that almost every one of the examples that we'll work there come down to exactly this differential equation. Also, in those problems we will be working some "real" problems that are actually solved in places and so are not just "made up" problems for the purposes of examples. Admittedly they will have some simplifications in them, but they do come close to realistic problem in some cases.

In this paper we have investigated the existence, uniqueness and the approximation of the solution of distributed order equations in the case where the initial values for the solution are not known. Results on existence and uniqueness had been established in the previous works of Caputo and the papers of Diethelm and Ford, all of them providing a representation of the solution in terms of the initial values. Based on some previous recent papers where we have studied boundary value problems for fractional differential equations, we have established the existence and uniqueness results by showing that a similar representation of the solution can be obtained in the case where the initial values are not given. Based on quadrature we have also proposed a numerical scheme in order to approximate the solution of the considered problems.

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