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Common Fixed Point Theorem in Menger Space through Weak Compatibility

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Abstract: In the present paper, a common fixed point theorem for five self-mapping has been proved under more general t-norm (H-type norm) in Menger space through weak compatibility. A corollary is also derived from the obtained result. The theorem is supported by providing a suitable example.

Keywords. Menger space, fixed point theorem, distribution function, weakly compatible, converge.

Introduction

Fixed point theory in Menger space can be considered as a part of Probabilistic Analysis, which is very dynamical area of mathematical research.Menger[1] introduce the notion of Menger spaces as a generalization of core notion of metric spaces. It is observed by many authors that contraction condition in metric space may be exactly translated into PM- Space endowed with min norm. Shegal and Bharucha-Reid [2] obtained a generalization of Banach Contraction principle on a complete Menger space which is a milestone in developing fixed point theorems in Menger Space and initiated the study of fixed points in PM- spaces. Further, Schweizer-Sklar[3] expanded the study of these spaces.Mishra [4] introduce the notion of compatible mapping in PM-space. Jungck [5] enlarged the concept of compatible maps. Sessa [6] initiated the tradition of improving commutativity in fixed point theorems by introducing the notion of weakly compatible commuting maps in Metric spaces.

In the present paper, a common fixed point theorem for five self-mapping has been proved under more general t-norm (H-type norm) in Menger space through weak compatibility. A corollary is also derived from the obtained result. The theorem is supported by providing a suitable example.

Distribution Function

A distribution function is a function $F : R \to [0,1]$ which is left continuous on R, non-decreasing and

$F(-\infty) = 0\,,\ F(\infty) = 1$

Let $\Delta = \{F : F\ is\ distribution\ function\}$

And $H \in \Delta(also\ known\ as\ Heaviside\ Function)$ defined by

$$H(y) = \begin{cases} 0 & if\ y \le 0 \\ 1 & if\ y > 0 \end{cases}$$

Triangular Norm

A triangular norm (t-norm) is a binary operation on the interval [0,1] ($t: [0,1] \times [0,1] \rightarrow [0,1]$) such that $\forall p, q, r, s \in [0,1]$ the following condition are satisfied ;

(i) $t(p, 1) = p$
(ii) $t(p, q) = t(q, p)$
(iii) $t(r, s) \leq t(p, q)$ for $r \leq p$ *and* $s \leq q$
(iv) $t(p, t(q, r)) = t(t(p, q), r)$

Example: - The following are the four basic t-norms

(i) The minimum t-norms : $t_m(a, b) = \min\{a, b\}$
(ii) The product t-norms : $t_p(a, b) = ab$
(iii) The Lukasiewiez t-norms : $t_L(a, b) = \max\{0 , a + b - 1.0\}$
(iv) The weakest t-norm, the Drastic t-norm

$$t_D(a, b) = \begin{cases} a \text{ if } b = 1 \\ b \text{ if } a = 1 \\ 0 \text{ otherwise} \end{cases}$$

Probabilistic Metric Space (PM-Space)

The ordered pair (Y,G) is called a Probabilistic Metric Space (PM-Space) if Y is a non-empty set G is a probabilistic distance function satisfy the following condition : $\forall a, b, c \in Y$ *and* $r, s > 0$

(i) $G_{a,b}(r) = 1 \Leftrightarrow a = b$
(ii) $G_{a,b}(0) = 0$
(iii) $G_{a,b}(r) = G_{b,a}(r)$
(iv) $G_{a,c}(r) = 1,\ G_{c,b}(s) = 1 \Rightarrow G_{a,b}(r + s) = 1$

Menger Space

The order triplet (Y, G, t) is called a Menger space if (Y,G) is a PM-space, t is a t-norm and $\forall a, b, c \in Y$ *and* $r, s > 0$

$G_{a,b}(r + s) \geq t(G_{a,c}(r),\ G_{c,b}(s))$

This inequality is known as Menger's inequality (Schweizer and Sklar [3])

Preposition 1

Let (Y, d) be a metric space. Then the metric d induces a distribution function G defined by

$G_{u,v}(t) = H(t - d(u, v))$

$\forall u, v \in Y$ *and* $t > 0$(Sehgal and Bharucha-Reid [2])

If t-norm is given by, t(a, b) = min {a, b} ∀ a, b ∈ [0,1] then(Y, G, t) is a Menger space. Further (Y, G, t) is complete Menger space if (Y, d) is complete.

Definitions :-

Let(Y, G, t) be a Menger space and t be a continuous t-norm.

(i) A sequence $\{y_n\}$ in (Y, G, t) is said to be convergent to a point y $\in Y$ if for every $\varepsilon > 0$ *and* $\lambda > 0$, there exists an integer $\mathrm{N} = \mathrm{N}(\varepsilon, \lambda)$ such that $y_n \in U_n(\varepsilon, \lambda)\ \forall\, n \geq N$ or equivalently $G_{y_n, y}(\varepsilon) > 1 - \lambda \forall\, n \geq N$

where $U_n(\varepsilon, \lambda) = \{v \in \mathrm{Y} : G_{u,v}(\varepsilon) > 1 - \lambda\}$ is (ε, λ) neighborhood of $u \in \mathrm{Y}$ and $\lambda \in (0,1)$

(ii) A sequence $\{y_n\}$ in (Y, G, t) is said to be Cauchy sequence if for every $\varepsilon > 0$ *and* $\lambda > 0$, there exists an integer $\mathrm{N} = \mathrm{N}(\varepsilon, \lambda)$ such that

$$G_{y_n, y_m}(\varepsilon) > 1 - \lambda \forall\, n, m \geq N$$

A Menger Space (Y, G, t) with continuous t-norm is said to be complete if every Cauchy sequence in Y converges to a point in Y. (Singh et al [7])

Weakly Compatible

In a Menger Space (Y, G, t) two self mappingF_1 and F_2 aresaid to be weakly compatible or coincidently commuting if they commute at their coincidence points. i.e. if $F_1(\mathrm{y}) = F_2(y)$ for some y $\in Y$ then $F_2F_1(\mathrm{y}) = F_1F_2(y)$(Singh and Jain [8])

Compatible

In a Menger Space (Y, G, t) two self mappingF_1 and F_2 aresaid to be compatible If$G_{F_1F_2(y_n), F_2F_2(y_n)}(t) \to 1\ \forall\ t > 0$, whenever $\{y_n\}$ is a sequence in Y: $F_1(y_n)$, $F_2(y_n) \to y$, for y $\in$ Y, as n $\to \infty$(Mishra [4])

Preposition 2

In a Menger Space (Y, G, t) two self mappingF_1 and F_2 arecompatible then they are weakly compatible. But converse of the result is not true. (Singh and Jain [8])

Example:-Let (Y, d) be a metric space $\mathrm{Y} = [0,3]$and $(\mathrm{Y, G, t})$be a induced Menger space with$G_{u,v}(t) = H(t - d(u, v))\forall \mathrm{u, v} \in Y$ *and* $t > 0$.

Define self mappingsF_1 and F_2 as follows :

$$F_1(y) = \begin{cases} 3 - y & if\ \ 0 \leq y < 2 \\ 3 & if\ \ 2 \leq y \leq 3 \end{cases}$$

$$F_2(y) = \begin{cases} y - 1 & if\ \ 0 \leq y < 2 \\ 3 & if\ \ 2 \leq y \leq 3 \end{cases}$$

Take $y_m = 2 - \frac{1}{m}$. *Now,*

$G_{F_1(y_m),1}(t) = H\left(t - \frac{1}{m}\right)$; therefore $\lim_{m\to\infty} G_{F_1(y_m),1}(t) = H(t) = 1$

Hence $F_1(y_m) \to 1$as $m \to \infty$. Similarly Hence $F_2(y_m) \to 1$ as $m \to \infty$

$G_{F_1F_2(y_m), F_2F_1(y_m)}(t) = H(t - 2)$;

$\lim_{m\to\infty} G_{F_1F_2(y_m), F_2F_1(y_m)}(t) = H(t - 2) \neq 1\ \forall t > 0$.

Hence the pair (F_1 , F_2) is not compatible. Also set of coincidence points of F_1 and F_2 is [2, 3].
Now for any $y \in [2,3], F_1(y) = F_2(y) = 3$
$and\ F_1F_2(y) = F_1(3) = 3 = F_2(3) = F_2F_1(y)$.
Thus F_1 and F_2 are weakly compatible but not compatible.

Preposition 3

In a Menger Space (Y, G, t) , if $t(a,a) \geq a \forall a \in [0,1]$
Then $t(a,b) = \min\{a,b\} \forall a,b \in [0,1]$(Singh and Jain [8])

Lemma 1. Let $\{y_n\}$ be a sequence in a Menger (Y, G, t) with continuous t-norm $t(a,a) \geq a \forall a \in [0,1]$, $\exists\ c \in (0,1)\ \forall t > 0\ and\ n \in N$,
$G_{y_n,y_{n+1}}(ct) \geq G_{y_{n-1},y_n}(t)$
Then $\{y_n\}$ is a Cauchy sequence in Y.(Singh and Pant [9])

Lemma 2. let (y, G, t) be a Menger space. If $c \in (0,1)$ such that for $a, b \in Y$.
$G_{a,b}(ct) \geq G_{b,a}(t)$
Then $a = b$. (Singh and Jain [8]).

Theorem :- Let $F_1, F_2, F_3, F_4\ and\ F_5$ are self mapping on a complete Menger space (Y, G, t) with $t(a,a) \geq a \forall a \in [0,1]$, satisfying;

(i) $F_4 \subseteq F_1F_2\ and\ F_5 \subseteq F_3$;
(ii) $F_1F_2 = F_2F_1,\ F_4F_2 = F_2F_4,\ F_3F_2 = F_2F_3\ and\ F_1F_5 = F_5F_1$
(iii) $Either\ F_3\ or\ F_4$ is continuous.
(iv) (F_4 , F_3) is compatible and (F_5 , F_1F_2) is weakly compatible.
(v) $\exists\ c \in (0,1)$
$G_{F_4(a),F_5(b)}(cy) \geq \min\{G_{F_3(a),F_4(a)}(y);\ G_{F_1F_2(b),F_5(b)}(y); G_{F_1F_2(b),F_4(a)}(\alpha y);$
$G_{F_3(a),F_5(b)}((2-\alpha)y);\ G_{F_3(a),F_1F_2(b)}(y)\}; \forall a, b \in Y, \alpha \in (0,2)$ and y>0.

Proof :- Let $y_0 \in Y$. From condition (i) $\exists y_1, y_2 \in Y$:

$$F_4(y_0) = F_1F_2(y_1) = z_0$$
$$F_5(y_1) = F_3(y_2) = z_1$$

Inductively we can construct sequences $\{y_n\}\ and\ \{z_n\}$ in Y such that

$$F_4(y_{2n}) = F_1F_2(y_{2n+1}) = z_{2n}$$
$$F_5(y_{2n+1}) = F_3(y_{2n+2}) = z_{2n+1} \text{ for } n = 0,1,2,3 \ldots$$

Step 1.Putting $a = y_{2n}$, $b = y_{2n+1}\ and\ \alpha = 1 - q\ with\ q \in (0,1)\ in\ condition$ (v), we get :

$$G_{F_4(y_{2n}),F_5(y_{2n+1})}(cy) \geq \min\{G_{F_3(y_{2n}),F_4(y_{2n})}(y);\ G_{F_1F_2(y_{2n+1}),F_5(y_{2n+1})}(y);$$
$$G_{F_1F_2(y_{2n+1}),F_4(y_{2n})}((1-q)y); G_{F_3(y_{2n}),F_5(y_{2n+1})}((1+q)y);\ G_{F_3(y_{2n}),F_1F_2(y_{2n+1})}(y)\}$$

…….(1)

$$\Rightarrow G_{z_{2n},\ z_{2n+1}}(cy) \geq \min\{G_{z_{2n-1},\ z_{2n}}(y);\ G_{z_{2n},\ z_{2n+1}}(y); G_{z_{2n},z_{2n}}((1-q)y);$$
$$G_{z_{2n-1},\ z_{2n+1}}((1+q)y);\ G_{z_{2n-1},\ z_{2n}}(y)\}$$
$$\geq \min\{G_{z_{2n-1},\ z_{2n}}(y);\ G_{z_{2n},\ z_{2n+1}}(y); 1;\ G_{z_{2n-1},\ z_{2n+1}}((1+q)y);\ G_{z_{2n-1},\ z_{2n}}(y)\}$$
$$\geq \min\{G_{z_{2n-1},\ z_{2n}}(y);\ G_{z_{2n},\ z_{2n+1}}(y); G_{z_{2n},\ z_{2n+1}}(qy);\ G_{z_{2n-1},\ z_{2n}}(y)\}$$
$$\geq \min\{G_{z_{2n-1},\ z_{2n}}(y);\ G_{z_{2n},\ z_{2n+1}}(y); G_{z_{2n},\ z_{2n+1}}(qy)\}$$
$$As\ t-norm\ is\ continous,\ letting\ q \rightarrow 1, \text{ we get};$$
$$G_{z_{2n},\ z_{2n+1}}(cy) \geq \min\{G_{z_{2n-1},\ z_{2n}}(y);\ G_{z_{2n},\ z_{2n+1}}(y); G_{z_{2n},\ z_{2n+1}}(y)\}$$
$$= \min\{G_{z_{2n-1},\ z_{2n}}(y);\ G_{z_{2n},\ z_{2n+1}}(y)\}$$

Hence, $G_{z_{2n},\ z_{2n+1}}(cy) \geq \min\{G_{z_{2n-1},\ z_{2n}}(y);\ G_{z_{2n},\ z_{2n+1}}(y)\}$

Similarly $G_{z_{2n+1},\ z_{2n+2}}(cy) \geq \min\left\{G_{z_{2n},\ z_{2n+1}}(y);\ G_{z_{2n+1},\ z_{2n+2}}(y)\right\}$.

Therefore, for all even or odd we have;

$$G_{z_n,\ z_{n+1}}(cy) \geq \min\{G_{z_{n-1},\ z_n}(y);\ G_{z_n,\ z_{n+1}}(y)\}$$

Consequently: $G_{z_n,\ z_{n+1}}(y) \geq \min\left\{G_{z_{n-1},\ z_n}(c^{-1}y);\ G_{z_n,\ z_{n+1}}(c^{-1}y)\right\}$

By repeated application of above inequality, we get :

$$G_{z_n,\ z_{n+1}}(y) \geq \min\{G_{z_{n-1},\ z_n}(c^{-1}y);\ G_{z_n,\ z_{n+1}}(c^{-m}y)\}$$

Since $G_{z_n,\ z_{n+1}}(c^{-m}y) \rightarrow 1$ as $m \rightarrow \infty$, it follows that

$G_{z_n,\ z_{n+1}}(cy) \geq G_{z_{n-1},\ z_n}(y) \forall y > 0\ and\ n \in N$

Therefore by Lemma 1.$\{z_n\}$ is a Cauchy sequence in Y, which is complete. Hence $\{z_n\} \rightarrow z \in Y$. Also its subsequence converges as follows :

$$\{F_5(y_{2n+1})\} \rightarrow z \quad and \quad F_1F_2(y_{2n+1}) \rightarrow z$$
$$\{F_4(y_{2n})\} \rightarrow z \quad and \quad F_3(y_{2n}) \rightarrow z$$

Case 1 :- F_3 *is continuous*

As F_3 *is continuous*, $(F_3)^2(y_{2n}) \rightarrow F_3(z)$ *and* $(F_3)F_4(y_{2n}) \rightarrow F_3(z)$. As (F_4, F_3) is compatible, we have $(F_4)F_3(y_{2n}) \rightarrow F_3(z)$

Step 2. Putting a $= F_3(y_{2n})$, b $= y_{2n+1}$ $and\ \alpha = 1\ in\ condition$ (v), we get :

$$G_{F_4(F_3(y_{2n})),F_5(y_{2n+1})}(cy) \geq \min\{G_{F_3(F_3(y_{2n})),F_4(F_3(y_{2n}))}(y);\ G_{F_1F_2(y_{2n+1}),F_5(y_{2n+1})}(y);$$
$$G_{F_1F_2(y_{2n+1}),F_4(F_3(y_{2n}))}(y); G_{F_3(F_3(y_{2n})),F_5(y_{2n+1})}(y);\ G_{F_3(F_3(y_{2n})),F_1F_2(y_{2n+1})}(y)\} \quad \ldots\ldots(2)$$

Letting n $\rightarrow \infty$, we get

$$G_{F_3(z),z}(cy) \geq \min\{G_{F_3(z),F_3(z)}(y);\ G_{z,z}(y); G_{z,F_3(z)}(y); G_{F_3(z),z}(y); G_{F_3(z),z}(y)\}$$

i.e. $G_{F_3(z),z}(cy) \geq G_{z,F_3(z)}(y)$.

$$Therefore,\ by\ lemma\ 2,\ we\ get;\quad F_3(z) = z$$

Step 3. Putting a $= z,$ b $= y_{2n+1}$ $and\ \alpha = 1\ in\ \ condition$ (v), we get :

$$G_{F_4(z),F_5(y_{2n+1})}\,(cy) \geq \min\{G_{F_3(z),F_4(F_3(z))}(y);\ G_{\,F_1F_2(y_{2n+1}),F_5(y_{2n+1})}(y);$$

$$G_{\,F_1F_2(y_{2n+1}),F_4(z)}(y); G_{F_3(z),F_5(y_{2n+1})}(y);\ G_{F_3(z),F_1F_2(y_{2n+1})}\,(y)\} \qquad \ldots\ldots(3)$$

Letting n → ∞ , we get

$$G_{F_4(z),z}(cy) \geq \min\{G_{z,\ \ F_4(z)}(y);\ G_{z,z}(y); G_{z,F_4(z)}(y); G_{z,z}(y); G_{z,z}(y)\}$$

$$\Rightarrow \qquad G_{F_4(z),z}(cy) \geq G_{F_4(z),\ z}(y)$$

Which gives $F_4(z) = z$. Therefore $F_3(z) = F_4(z) = z$

Step 4. Putting a $= F_2(z),$ b $= y_{2n+1}$ $and\ \alpha = 1\ in\ \ condition$ (v), we get :

$$G_{F_4(F_2(z)),F_5(y_{2n+1})}\,(cy) \geq \min\{G_{F_3(F_2(z)),F_4(F_2(z))}(y);\ G_{\,F_1F_2(y_{2n+1}),F_5(y_{2n+1})}(y);$$

$$G_{\,F_1F_2(y_{2n+1}),F_4(F_2(z))}(y); G_{F_3(F_2(z)),F_5(y_{2n+1})}(y);\ G_{F_3(F_2(z)),F_1F_2(y_{2n+1})}\,(y)\} \quad \ldots\ldots(4)$$

As $F_4F_2 = F_2F_4,\ \ F_3F_2 = F_2F_3$ so

$$F_4(F_2(z)) = F_2(F_4(z)) = F_2(z),\ \ F_3(F_2(z)) = F_2(F_3(z)) = F_2(z)$$

Letting n → ∞ , we get

$$G_{F_2(z),\ \ z}(cy) \geq \min\{G_{F_2(z),F_2(z)}(y);\ G_{z,z}(y); G_{z,F_2(z)}(y); G_{F_2(z),\,z}(y);\ G_{F_2(z),\,z}(y)\}$$

$$\Rightarrow \qquad G_{F_2(z),\ \ z}(cy) \geq G_{F_2(z),\,z}(y)$$

Which gives $F_2(z) = z$. Therefore

$$F_2(z) = \ F_3(z) = F_4(z) = z$$

Step 5. $F_4(Y)\ \subseteq\ F_1F_2(Y)$; $\exists\, u \in Y; z = F_4(z) =\ F_1F_2(u)$

Putting a $= y_{2n}$, b $= u\ \ and\ \alpha = 1\ in\ \ condition$ (v), we get :

$$G_{F_4(y_{2n}),F_5(u)}\,(cy) \geq \min\{G_{F_3(y_{2n}),F_4(y_{2n})}(y);\ G_{\,F_1F_2(u),F_5(u)}(y);$$

$$G_{F_1F_2(u),F_4(y_{2n})}(y); G_{F_3(y_{2n}),F_5(u)}(y);\ G_{F_3(y_{2n}),F_1F_2(u)}\,(y)\}\ldots\ldots(5)$$

Letting n → ∞ , we get

$$G_{z\,,\ \ F_5(u)}(cy) \geq \min\{G_{z,\ \ z}(y); G_{F_5(u),\,F_5(u)}(y); G_{F_5(u),\ z}(y); G_{z,\ \ F_5(u)}(y);\ G_{z,\,F_5(u)}(y)\}$$

$$\Rightarrow \qquad G_{z\,,\ \ F_5(u)}(cy) \geq G_{z\,,\ \ F_5(u)}(y)$$

$$Therefore,\ by\ lemma\ 2\ ,\ we\ get\ ; \quad F_5(u) = z$$

Hence $F_5(u) =\ F_1F_2(u) = z.\ \ As\ (F_5,\ F_1F_2)\ is\ weakly\ compatible,\ we\ have$

$$F_1F_2F_5(u) =\ F_5F_1F_2(u).\ \ Thus\ \ F_5(u) =\ F_1F_2(z)$$

Step 6. Putting a $= y_{2n}$, b $= z\ \ and\ \alpha = 1\ in\ \ condition$ (v), we get :

$$G_{F_4(y_{2n}),F_5(z)}\,(cy) \geq \min\{G_{F_3(y_{2n}),F_4(y_{2n})}(y);\ G_{\,F_1F_2(z),F_5(z)}(y);$$

$$G_{F_1F_2(z),F_4(y_{2n})}(y); G_{F_3(y_{2n}),F_5(z)}(y);\ G_{F_3(y_{2n}),F_1F_2(z)}\,(y)\}\ldots\ldots(6)$$

Letting n → ∞ , we get

$$G_{z\,,\ \ F_5(z)}(cy) \geq \min\{G_{z,\ \ z}(y); G_{F_5(z),\,F_5(z)}(y); G_{F_5(z),\ z}(y); G_{z,\ \ F_5(z)}(y);\ G_{z,\,F_5(z)}(y)\}$$

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$$\Rightarrow \quad G_{z,\ F_5(z)}(cy) \geq G_{z,\ F_5(z)}(y)$$

$$Therefore,\ by\ lemma\ 2,\ we\ get;\quad F_5(z) = z$$

Therefore

$$F_2(z) = F_3(z) = F_4(z) = F_5(z) = z$$

Step 7. Putting a $= y_{2n}$, b $= F_1(z) and\ \alpha = 1\ in\ condition$ (v), we get :

$$G_{F_4(y_{2n}),F_5(F_1(z))}(cy) \geq \min\{G_{F_3(y_{2n}),F_4(y_{2n})}(y);\ G_{F_1F_2(F_1(z)),F_5(F_1(z))}(y);$$

$$G_{F_1F_2(F_1(z)),F_4(y_{2n})}(y); G_{F_3(y_{2n}),F_5(F_1(z))}(y);\ G_{F_3(y_{2n}),F_1F_2(F_1(z))}(y)\}\ldots\ldots(7)$$

As $F_1F_2 = F_2F_1\ and\ F_5F_1 = F_1F_5$ so

$$F_2(F_1(z)) = F_1(F_2(z)) = F_1(z),\ F_5(F_1(z)) = F_1(F_5(z)) = F_1(z)$$

Letting n $\rightarrow \infty$, we get

$$G_{z,\ F_1(z)}(cy) \geq \min\{G_{z,\ z}(y); G_{F_1(z),\,F_1(z)}(y); G_{F_1(z),\ z}(y); G_{z,\ F_1(z)}(y);\ G_{z,F_1(z)}(y)\}$$

$$\Rightarrow \quad G_{z,\ F_1(z)}(cy) \geq G_{z,\ F_1(z)}(y)$$

Which gives $F_1(z) = z$. Therefore

$$F_1(z) = F_2(z) = F_3(z) = F_4(z) = F_5(z) = z$$

Thus we obtain that z the common fixed point of the five maps in this case.

Case 2 :- $F_4\ is\ continuous$

As $F_4\ is\ continuous$, $(F_4)^2(y_{2n}) \rightarrow F_4(z)\ and\ (F_4)F_3(y_{2n}) \rightarrow F_4(z)$. As (F_4, F_3) is compatible, we have $(F_3)F_4(y_{2n}) \rightarrow F_4(z)$

Step 8. Putting a $= F_4(y_{2n})$, b $= y_{2n+1}\ and\ \alpha = 1\ in\ condition$ (v), we get :

$$G_{F_4(F_4(y_{2n})),F_5(y_{2n+1})}(cy) \geq \min\{G_{F_3(F_4(y_{2n})),F_4(F_4(y_{2n}))}(y);\ G_{F_1F_2(y_{2n+1}),F_5(y_{2n+1})}(y);$$

$$G_{F_1F_2(y_{2n+1}),F_4(F_4(y_{2n}))}(y); G_{F_3(F_4(y_{2n})),F_5(y_{2n+1})}(y);\ G_{F_3(F_4(y_{2n})),F_1F_2(y_{2n+1})}(y)\}$$

$$\ldots\ldots(8)$$

Letting n $\rightarrow \infty$, we get

$$G_{F_4(z),z}(cy) \geq \min\{G_{F_4(z),F_4(z)}(y);\ G_{z,z}(y); G_{z,F_4(z)}(y); G_{F_4(z),z}(y); G_{F_4(z),z}(y)\}$$

i.e. $G_{F_4(z),z}(cy) \geq G_{z,F_4(z)}(y)$.

$$Therefore,\ by\ lemma\ 2,\ we\ get;\quad F_4(z) = z$$

Now steps $5 - 7$ gives us $F_1(z) = F_4(z) = F_5(z) = F_1F_2(z) = z$

Step 9. $F_5(Y) \subseteq F_3(Y)$; $\exists\ w \in Y$; $z = F_5(z) = F_3(w)$

Putting a $=$ w, b $= y_{2n}\ and\ \alpha = 1\ in\ condition$ (v), we get :

$$G_{F_4(w),F_5(y_{2n})}(cy) \geq \min\{G_{F_3(w),\ F_4(w)}(y);\ G_{F_1F_2(y_{2n}),\ F_5(y_{2n})}(y);$$

$$G_{F_1F_2(y_{2n}),F_4(w)}(y); G_{F_3(w),F_5(y_{2n})}(y);\ G_{F_3(w),F_1F_2(y_{2n})}(y)\}\ldots\ldots(9)$$

Letting n $\rightarrow \infty$, we get

$$G_{F_4(w),\ z}(cy) \geq \min\{G_{z,\ F_4(w)}(y);\ G_{z,\ z}(y); G_{z,\ F_4(w)}(y); G_{z,\ z}(y);\ G_{z,\ z}(y)\}$$

$$\Rightarrow \quad G_{F_4(w),\ z}(cy) \geq G_{F_4(w),\ z}(y)$$

Which gives $F_4(w) = F_3(w) = z$

As (F_4, F_3) *is weakly compatible, we have* $F_4(z) = F_3(z)$

Also, $F_2F_1 = F_1F_2$. *Thus* $F_1(z) = z$

So, $F_2F_1(z) = F_1F_2(z) = F_2(z) = z$

Hence $F_1(z) = F_2(z) = F_3(z) = F_4(z) = F_5(z) = z$

Thus we obtain that z the common fixed point of the five maps in this case.

Uniqueness :-

Let v be another common fixed point of F_1, F_2, F_3, F_4 *and* F_5 ; then

$F_1(v) = F_2(v) = F_3(v) = F_4(v) = F_5(v) = v.$

Putting $a = z,\ b = v$ *and* $\alpha = 1$ *in condition* (v), we get :

$$G_{F_4(z),F_5(v)}(cy) \geq \min\{G_{F_3(z),F_4(z)}(y);\ G_{F_1F_2(v),F_5(v)}(y); G_{F_1F_2(v),F_4(z)}(y);$$
$$G_{F_3(z),F_5(v)}(y);\ G_{F_3(z),F_1F_2(v)}(y)\};$$
$$\Rightarrow \quad G_{z,\ v}(cy) \geq \min\{G_{z,\ z}(y);\ G_{v,\ v}(y); G_{v,\ z}(y);\ G_{z,\ v}(y);\ G_{z,\ v}(y)\};$$

$$\Rightarrow \quad G_{z,\ v}(cy) \geq G_{z,\ v}(y);$$

Which gives $z = v$. Therefore zis unique common fixed point of $F_1, F_2,$ F_3, F_4 *and* F_5.

Corollary :- If we take $F_2 = I$, the identity map on Y in theorem 3.1, then we get ;Let F_1, F_3, F_4 *and* $F_5$are self mapping on a complete Menger space (Y, G, t) with $t(a, a) \geq a \forall a \in [0,1]$, satisfying;

(i) $F_4(Y) \subseteq F_1(Y)$ *and* $F_5(Y) \subseteq F_3(Y)$;

(ii) $F_1F_5 = F_5F_1$

(iii) *Either* F_3 *or* F_4 is continuous.

(iv) (F_4, F_3) is compatible and (F_5, F_1) is weakly compatible.

(v) $\exists$ c $\in$ (0,1)

$$G_{F_4(a),F_5(b)}(cy) \geq \min\{G_{F_3(a),F_4(a)}(y);\ G_{F_1(b),F_5(b)}(y); G_{F_1(b),F_4(a)}(\alpha y);$$
$G_{F_3(a),F_5(b)}((2-\alpha)y);\ G_{F_1(a)F_3(b)}(y)\}; \forall$ a, b $\in Y$, $\alpha \in (0,2)$ and y>0.

Then F_1, F_3, F_4 *and* F_5 have a unique common fixed point in Y.

Example:- Let $Y = [0,1]$ with the metric d defined by $d(a, b) = |a - b|$ and defined $G_{a,\ b}(t) = H(t - d(a,\ b)) \forall a, b \in Y$ *and* $t > 0$.Clearly (Y, G, t) is a complete Menger Space where t-norm is defined by $t(a, b) = \min\{a, b\} \forall a, b \in [0,1]$.

Let F_1, F_2, F_3, F_4 *and* F_5be maps from Y into itself defined as

$$F_1(y) = y,\ F_2(y) = \frac{y}{3},\ F_3(y) = \frac{y}{7},\ F_4(y) = 0\ and\ F_5(y) = \frac{y}{9}\ \forall\ y \in Y$$

Then $F_4(Y) = \{0\} \subseteq [0,\ \frac{1}{3}] = F_1F_2(Y)$ and $F_5(Y) = [0,\ \frac{1}{9}] \subseteq [0,\ \frac{1}{7}] = F_3(Y)$

Clearly $F_1F_2 = F_2F_1,\ F_4F_2 = F_2F_4,\ F_3F_2 = F_2F_3,\ F_1F_5 = F_5F_1$ and $F_3,\ F_4$ are continuous. If we take $c = \frac{1}{3}\ and\ y = 1$, we see that the condition (v) of the main theorem is satisfied. Moreover the maps $F_4\ and\ F_3$ is compatible.

If $\lim_{n\to\infty} y_n = 0$, where $\{y_n\}$ is a sequence in Y such that

$$\lim_{n\to\infty} F_4(y_n) = 0 = \lim_{n\to\infty} F_3(y_n)$$

$F_5\ and\ F_1F_2$ is weakly compatible at 0. thus all condition of the main theorem are Satisfied and 0 is unique common fixed point of $F_1,\ F_2,\ F_3,\ F_4\ and\ F_5$.

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