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PYTHAGOREAN PICTURE FUZZY SET: ITS INTUITIVE MEANING AND DERIVED DISTANCE

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ABSTRACT. In this paper, we set voting as an example and give an intuitive explanation for the Pythagorean picture fuzzy Set (PPFS).We also construct a natural distance measure from PPFS's intuitive meaning.By measuring the distance from any Pythagorean picture fuzzy numbers (PPFNs) to the ideal PPFNs,we can rank PPFNs in descending order.Then, we develop a new multiple attribute decision-making method under Pythagorean picture environment.Finally, an illustrative example is given to demonstrate the practicality and effectiveness of our new method.

KEYWORDS: Pythagorean picture fuzzy set, Pythagorean picture fuzzy number, distance measure, multiple attribute decision-making.

1.INTRODUCTION

In order to collect various types of data, different data structures are introduced.As we all know, classic Cantor set can well describe distinct information.Due to the complexity and fuzziness of the real society, fuzzy set[23] and its various expansions are proposed.For example, Atanassov proposed intuitionistic fuzzy set(IFS) [1], which allowed each element x equipped with both membership degree $\mu(x)$ and no-membership degree $\nu(x)$.In order to capture more information, Cuong[3,4] described each element with four kinds of degree: positive, neutral, negative, abstain, which generalized IFS to PFS.

Ever since its appearance, picture fuzzy theory which contains picture fuzzy operators and picture fuzzy relations as well as picture fuzzy soft sets, has received much attentions. In [14], Singh introduced the weighted correlation coefficient on PFS aiming at clustering information.Wei[18]developed some picture fuzzy aggregation operators based on arithmetic and geometric operations.Basing on arithmetic and geometric operations, Garg[9]also proposed different kinds of picture fuzzy aggregation operators.From the angle of probability, wang[20] constructed another kind of picture fuzzy aggregation operators.Le[13] propose a generalized picture distance measure and constructed a new hierarchical picture fuzzy clustering method.

Picture fuzzy relation was also defined by Hong[12].In [8,19], some similarity measures have been introduced for picture fuzzy sets.Some main fuzzy logic operators such as negations, conjunctions, disjunctions and implications have been investigated on picture fuzzy sets[6]. PFS was also applied to the health care support system[17] and weather nowcasting[10,15].

Yager[21,22] relaxed the constraint $\mu(x)+\nu(x)\leq 1$ into $\mu^2(x)+\nu^2(x)\leq 1$ and then he modified IFS to Pythagorean fuzzy set[21],which expanded the scope of its application and brought many interesting changes.Inspired by Yager's idea, researches also have relaxed the constraint $\mu(x)+\eta(x)+\nu(x)\leq 1$ into $\mu^2(x)+\eta^2(x)+\nu^2(x)\leq 1$ and generalize PFS to Pythagorean picture fuzzy set(PPFS) and discussed its applications[7,11,16].Nevertheless, what is the essential difference between PPFS (or PFS) and IFS?Is the difference just limited to constraints?What is the intuitive meaning of PPFS (or PFS)?Few attentions have been paid to these problems.To highlight the features of PPFS, we set Pythagoras's voting as an real example to explain its intuitive meaning vividly.As an interesting result, we conclude a natural simple distance measure on PPFS.

To facilitate our discussion, the remainder of this paper is organized as follows.In the next section, we review some basic concepts related to PFS.Section 3 gives an intuitive explanation of PPFS.In Section 4, a new distance measure on PPFS is introduced and its basic properties are discussed.In Section 5, based on our distance, we develop a new method for multiple attribute decision-making with PPFS.An illustrative example is also given to show the effectiveness of the new approach in Section 6.In Section 7, we give the conclusion and some remarks.

2 Preliminaries

Let us first define some basic concepts related to PFS.

FS was first proposed by Zadeh[23]in 1965 as the following.

Definition 2.1[23]Let X be an universe of discourse, then a fuzzy set is defined as: $A=\{\langle x,\mu_A(x)\rangle | x\in X\}$ which is characterized by a membership function $\mu_A : X\to[0,1]$, where μ_A denotes the degree of membership ofthe element x to the set A.

Atanassov[1] generalized FS to IFS as below.

Definition2.2[1]An IFS in X is given by

$A=\{\langle x,\mu_A(x),\nu_A(x)\rangle | x\in X\}$

which is characterized by a membership function $\mu_A : X\rightarrow[0,1]$ and anon-membership function $\nu_A : X\rightarrow[0,1]$, with the condition

$0\le\mu_A(x)+\nu_A(x)\le 1,\forall x\in X$

where the numbers $\mu_A(x)$ and $\nu_A(x)$ represent the degree of membership and the degree of non-membership of the element x to the set A, respectively.We can denote $(\mu_A(x),\nu_A(x))$ as an intuitionistic fuzzy number(IFN).If $\pi_A(x)=1-\mu_A(x)-\nu_A(x)=0,\forall x\in X$, then the IFS is reduced to a common fuzzy set.

Yager[21,22] changed the linear form of logical negation $Neg(a)=1-a$ in IFS to Pythagorean negation $Neg(a)=\sqrt{1-a^2}$,and provided a related class of non-standard fuzzy sets called Pythagorean fuzzy set.

Definition2.3 [24] Let X be a universe of discourse. The Pythagorean fuzzy set P can be represented as

$P=\{\langle x,\mu_P(x),\nu_P(x)\rangle | x\in X, 0\le\mu_P^2(x)+\nu_P^2(x)\le 1\}$,

where, the numbers $\mu_P(x)$ and $\nu_P(x)$ represent the degree of membership and the degree of non-membership of the element x to the set P, respectively.

Voting was given as a good example in [5].Human voters can be divided into four groups: vote for, neutral, vote against, refusal of the voting.In order to describe this situation, Cuonget. al. [3-5] generalized FS and IFS to PFS as the new concept for computaional intelligence problems.

Definition 2.4 [5] A picture fuzzy set A on a universe X is an object of the form

$A=\{\langle x,\mu_A(x),\eta_A(x),\nu_A(x)\rangle | x\in X\}$

where $\mu_A(x)(\in[0,1])$ is called the "degree of positive membership of x in A", $\eta_A(x)(\in[0,1])$ is called the "degree of neutral membership of x in A" and $\nu_A(x)(\in[0,1])$ is called the "degree of negative membership of x in A".Besides, $\mu_A(x),\eta_A(x),\nu_A(x)$ satisfy the following condition:

$\forall x\in X, \mu_A(x)+\eta_A(x)+\nu_A(x)\le 1.$

Then for $\forall x \in X$, $\rho_A(x) = 1 - \mu_A(x) - \eta_A(x) - \nu_A(x)$ could be called the "degree of refusal membership of x in A". For convenience, we can call $(\mu_A(x), \eta_A(x), \nu_A(x))$ a picture fuzzy number.Obviously, when $\eta_A(x) = 0, \forall x \in X$, the PFS is reduced to IFS.

In order to rank picture fuzzy numbers, score function was proposed as below[9].

Definition2.5 [9]For A picture fuzzy number $\alpha = (\mu, \eta, \nu)$, its score function is defined as

$S(\alpha) = \mu - \eta - \nu.$

Here, $S(\alpha)$ can be interpreted as net winning points.

Inspired by Yager[21,22],PFS can be naturally extend to Pythagorean picture fuzzy set (PPFS) as the following.

Definition 2.6A Pythagorean picture fuzzy set A on a universe X is an object of the form

$A = \{\langle x, \mu_A(x), \eta_A(x), \nu_A(x) \rangle | x \in X\}$

where $\mu_A(x)(\in[0,1])$ is called the "degree of positive membership of x in A", $\eta_A(x)(\in[0,1])$ is called the "degree of neutral membership of x in A" and $\nu_A(x)(\in[0,1])$ is called the "degree of negative membership of x in A".Besides, $\mu_A(x), \eta_A(x), \nu_A(x)$ satisfy the following condition:

$\forall x \in X, \mu_A^2(x) + \eta_A^2(x) + \nu_A^2(x) \le 1.$

Then for $\forall x \in X$, $\rho_A(x) = 1 - \mu_A(x) - \eta_A(x) - \nu_A(x)$ could be called the "degree of refusal membership of x in A". $P(\mu_A(x), \eta_A(x), \nu_A(x))$ can be called as Pythagorean picture fuzzy number (PPFN) which denotes a voting result.

3 An intuitive explanation for PPFS

What is the intuitive meaning of PPFS?what is the essential difference between PPFS and IFS?In this section, we try to answer these questions.

In order to simulate the decision-making procedure of the decision maker, we set Pythagoras as a simple example.As everyone knows, Pythagoras is a great thinker, philosopher, mathematician, scientist.Naturally, we can simulated Pythagoras's thinking process by a group of experts hiding in Pythagoras's brainto vote for meetings.So, we choose voting as a realistic example to interpret PPFS and PPFN.When Pythagoras received a decision task, for example, "whether Midas is a competent king?"He can carry out his work following several steps:

Step 0: preparatory work.Pythagoras first assesses the risk of decision. For example, once he makes a false assessment, the Greek Empire will decline. So, Pythagoras has to determines a proper validityor called accuracy $r=\sqrt{\mu^2+\eta^2+\nu^2}$ of thevoting. Or, in an equal way, the hesitant degree of decision-making $\pi=\sqrt{1-\mu^2-\eta^2-\nu^2}$ .Here, r is the degree of confidence in the decision, that is, the degree of precision. π is the maximum allowable absence rate.When π increase, more experts become absent from the meeting, then decision's precision r decrease.

The precision r, incidentally, can also be defined by $\mu+\eta+\nu$ which has the same monotonicity as $\sqrt{\mu^2+\eta^2+\nu^2}$ .In order not to interfere with the experts' decision, Pythagoras concealed the parameter r .

Step 1: the first round of voting.

Then, Pythagoras gathers all the n experts and holds a meeting for the first round of voting.

To make a joke, these experts have been living in Pythagoras's brain.Faced with limited data,

these experts hiding in Pythagoras's brain begin to use their professional knowledge to assess Midas,and raise their hands to vote giving the support or opposition.The first round of voting's result comes out, and Pythagoras counts the results of the vote.There are x_1 experts vote for the king Midas to be competent, y_1 experts nertral, z_1 experts vote against, $n-x_1-y_1-z_1$ experts refuse to vote.Pythagoras calculates the vote's positive rate, neutral rate, and negative rate as $\mu_1=\frac{x_1}{n},\eta_1=\frac{y_1}{n},\nu_1=\frac{z_1}{n}$, then he records the result of the meeting by a PPFN $P_1=(\mu_1,\eta_1,\nu_1)$.

Step 2: the second round of voting.Pythagoras begins to analyze the results of the vote.

After calculating and comparing, he finds the trouble comes: $\sqrt{\mu_1^2+\eta_1^2+\nu_1^2}<r$ .The reasons maybe: some experts are absent or some experts refuse to vote due to timid and overcautious.

Then Pythagoras encourages experts to actively express their opinion and begins another meeting.

Similar to the first round of voting,there are $x_2 \geq x_1$ experts vote for the king Midas to be competent, $y_2 \geq y_1$ experts netrual, $z_2 \geq z_1$ experts vote against, $n-x_2-y_2-z_2 \leq n-x_1-y_1-z_1$

experts refuse to vote.Pythagoras calculates the vote's positive rate, neutral rate, and negative rate as $\mu_2=\frac{x_2}{n},\eta_2=\frac{y_2}{n},\nu_2=\frac{z_2}{n}$,then he records the result of the meeting by another PPFN $P_2=(\mu_2,\eta_2,\nu_2)$.If the vote does not meet the accuracy requirements, that is $\sqrt{\mu_2^2+\eta_2^2+\nu_2^2}<r$,Pythagoras has to hold the next round of voting.

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Step n: the n-th round of voting. If the (n-1)-th round of voting does not satisfy Pythagoras, that is to say $\sqrt{\mu_{n-1}^2+\eta_{n-1}^2+\nu_{n-1}^2}<r$,Pythagoras will start the n-th round of voting.We omit the voting process and only denote the voting result by a PPFN $P_n=(\mu_n,\eta_n,\nu_n)$.

Step $(n+1)$: Achieve the precision requirement and end the voting.After several rounds of voting, Pythagoras finally reached a satisfactory result $P=(\mu,\eta,\nu)$ with $\sqrt{\mu^2+\eta^2+\nu^2}=r$.

It is worth noting that, every $P_i(i=1,2,\cdots)$ is meaningful and interesting, because it reflectsspecific assessment results under certain accuracyconstraint $r_i=\sqrt{\mu_i^2+\eta_i^2+\nu_i^2}$.All these voting results $P_i(i=1,2,\cdots)$ forms a PPFS $\{P_i=(\mu_i,\eta_i,\nu_i)|r_1,r_2,\cdots,r_i,\cdots,\to r\}$.By statistical theory, when the sample is large enough, ratios of components in the sample are almost equal to the total's.The estimated values of the samples fluctuate up and down around the parameters of the population, showing a normal distribution.As a result, a more daring and rough assumption is that $\mu_1:\eta_1:\nu_1=\mu_2:\eta_2:\nu_2=\cdots=\mu_n:\eta_n:\nu_n=\cdots$.In other words, it is reasonableto forecast that $P_1=(\mu_1,\eta_1,\nu_1),P_2=(\mu_2,\eta_2,\nu_2),\cdots,P_n=(\mu_n,\eta_n,\nu_n)$ growth synchronously.So, under the precision constraint r, Pythagoras's decision process can be modified by a sequence of points $P_1,P_2,\cdots,P_n\to P$,which is shown in Figure 1. For the sake of simplicity, we only draw two-dimensional images instead of three-dimensional image.

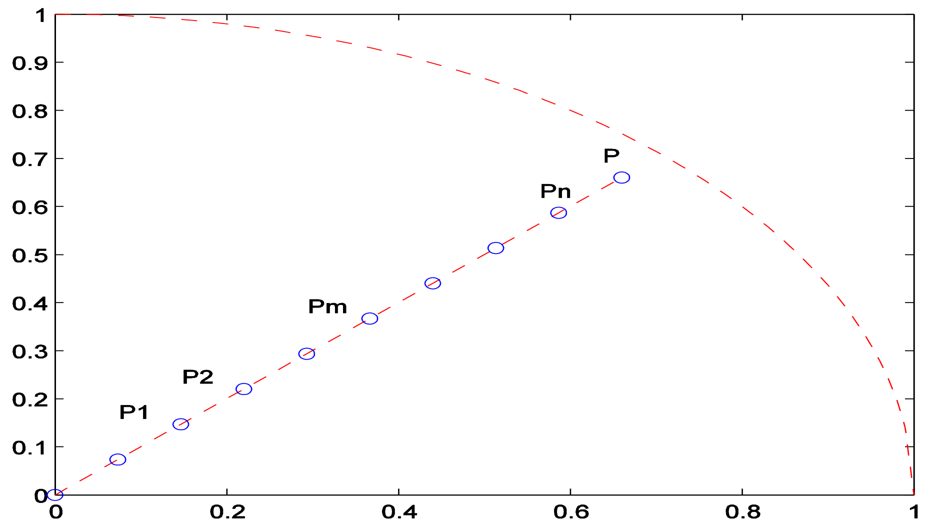

Figure 1: Under Pythagoras's external encourage, the experts keep voting along straight line OP.

Here, our core idea is to consider Pythagoras's voting as a progressive process. Voting results gradually change from P_1 to $P_2, \cdots, P_n, \cdots$.The change is motivated by Pythagoras's constant external pressure which forces the experts to express their opinions and gradually refrain from refusing vote.

4 A new distance measure on PPFS

In this section, we will construct a new distance measure on PPFS.As shown in Figure 2, there are two PPFNs P and Q.Naturally, there comes a question: how to measure the distance between P and Q?

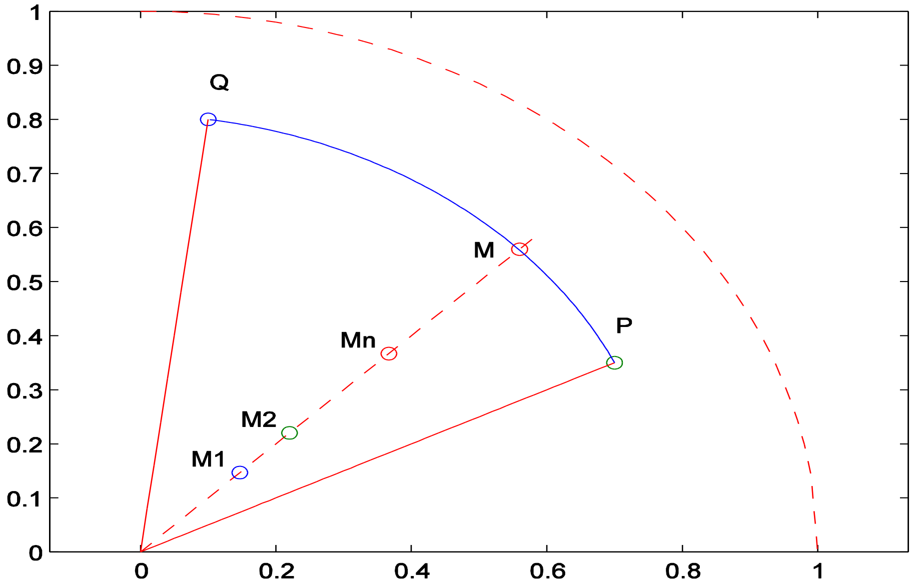

Figure 2: The changing information provides internal impetus which urges experts to adjust their voting, so that the voting results gradually change from point Pto pointQ.

To facilitate our explanation, we also set Pythagoras's voting for Midas as an example.In Figure 2, $P=(\mu_P,\eta_P,\nu_P)$ and $Q=(\mu_Q,\eta_Q,\nu_Q)$ represents different voting result, respectively.For convenience, we convert the Cartesian coordinates above to polar coordinates.According to the definition of trigonometric function

$$r=\sqrt{x^2+y^2+z^2},0\le r\le 1$$

$$\theta=\arcsin\frac{z}{r},0\le\theta\le\frac{\pi}{2}$$

$$\phi=\arcsin\frac{y}{x},0\le\phi\le\frac{\pi}{2}$$

Then, we smoothly get $P=(r_P,\theta_P,\phi_P),Q=(r_Q,\theta_Q,\phi_Q)$.We will soon see the benefits of polar coordinates.Suppose, Pythagoras just reach the voting result P now.As we have seen in Section 3, this team can only run along straight line OP under Pythagoras's precision pressure.Nevertheless, three points O P Q are not collinear, and Pythagoras's experts team can not go straight from point P to point Q automatically.How can Pythagoras lead his experts team to hold new meetings and arrive the new result Q?The answer is that: as time goes on, the information gathered by team is gradually updated and changed.This is the internal driving force which makes the team's attitude gradually change.Relative to Pythagoras's external precision pressure, the changing information is the internal factor that urges the expert's attitude to change consciously.Now, we gradually realize that

Pythagoras's voting is progressive process which change with precision r and time t.Let's track the gradual transition from P to Q as following.

Step 0: start stare P.Based on current information, Pythagoras leads his team to vote many rounds, and reach the current situation P.

Step 1: the first adjustment.With time slips away, nothing remains the same.When tomorrow's sun rises, we will gather more detailed information about Midas.Then Pythagoras will hold meetings to evaluate Midas again.The latest news about Midas arrives continuously.The international and domestic situation is also constantly changing.These factors make Pythagoras's team change their ideas gradually.Pythagoras gives adjusted precision r after he evaluates the current situation, and gathers the experts in his team together.That is to say, Pythagoras changes the validity r for next voting.Based on these latest information, Pythagoras begin a new round of voting.The process of the meeting can be denoted by PPFNs $M_1, M_2, \cdots, M_n, \cdots, M$,which can be seen in Figure 2.Each $M_i, i = 1, 2, \cdots$ records the outcome of certain conference vote under the

accuracy control condition $r_{M_i} \le r$.When Pythagoras reach M with $r_M = r$, he ends the first round of voting.This is the end of the first adjustment.

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Step n: then-th adjustment.With time flying, details about Midas continuously become rich.

Pythagoras has organized many rounds of meetings.The n-th adjustment is similarly to the first judgement,so we omit its procedure.

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Step (n+1): reach the end state and pause.After several rounds of adjustment, Pythagoras lead his team to Q.That is to say, the latest voting result is $Q = \left(r_Q, \theta_Q, \phi_Q\right)$.

Now, we will ask Pythagoras a question: what is the distance between P and Q ?Perhaps, I guess, Pythagoras will answer with a smile: you can count how many meetings we have held from P to Q.Obviously, Pythagoras has held infinite meetings from P to Q, each point marked by M_i in Figure 2corresponds to a meeting.Naturally, we can turn to compute the volume of the region which is composed by these points.

Here comes another question: how to determine the volume of region $\overline{OPQ}$?A simple hypothesis is : r_M are linear functions with respect to θ_M, ϕ_M.In other words, the surface $\overline{PQ}$ can be described by parameter equation $r = r\left(\theta, \phi\right) = a\theta + b\phi + c$.Certainly, we have already known the starting point $P=(r_P,\theta_P,\phi_P)$ and end point $Q=(r_Q,\theta_Q,\phi_Q)$ of the surface. So there are two constraint conditions:

$$r(\theta_P,\phi_P)=a\theta_P+b\phi_P+c=r_P,$$

$$r(\theta_Q,\phi_Q)=a\theta_Q+b\phi_Q+c=r_Q.$$

In order to determine parameters a,b,c, we need another constraint.For the sake of symmetry, let $r(\theta_P,\phi_P)=r(\theta_Q,\phi_Q)$:

$$a\theta_P+b\phi_P+c=a\theta_Q+b\phi_Q+c.$$

Solve three equations above, we get

$$a=\frac{r_P-r_Q}{2(\theta_P-\theta_Q)},$$

$$b=\frac{r_P-r_Q}{2(\phi_P-\phi_Q)},$$

$$c=r_P-a\theta_P-b\phi_P.$$

Therefore, we can easily calculate the volume of region $\overline{OPQ}$ by integral:

$$V_{\overline{OPQ}}=\int_{\overline{OPQ}}dV=\left|\int_{\Omega=\{(r,\theta,\phi)|0\le r\le a\theta+b\phi+c,\theta_P\le\theta\le\theta_Q,\phi_P\le\phi\le\phi_Q\}}r^2\sin\phi\,dr\,d\phi\,d\theta\right|=\left|\int_{\theta_P}^{\theta_Q}d\theta\int_{\phi_P}^{\phi_Q}\sin\phi\,d\phi\int_0^{a\theta+b\phi+c}r^2dr\right|$$

In order to transform the above distance into $[0,1]$, we divide $V_{\overline{OPQ}}$ by its maximum $\frac{\pi}{6}$,then we get the following definition.

Definition 4.1 Let $P=(r_P,\theta_P,\phi_P)$ and $Q=(r_Q,\theta_Q,\phi_Q)$ be PPFNs, which are expressed by polar coordinates.The distance between P and Q can be defined by

$$d(P,Q)=\frac{6}{\pi}\left|\int_{\theta_P}^{\theta_Q}d\theta\int_{\phi_P}^{\phi_Q}\sin\phi\,d\phi\int_0^{a\theta+b\phi+c}r^2dr\right|,$$

where $a=\frac{r_P-r_Q}{2(\theta_P-\theta_Q)}$, $b=\frac{r_P-r_Q}{2(\phi_P-\phi_Q)}$, $c=r_P-a\theta_P-b\phi_P$.

Next, we will discuss some basic properties of our newly defined distance measure d on PPFS.

Apparently, $0\le d(P,Q)\le 1$, so the following proposition holds.

Theorem 4.2(Boundness) Let $P=(r_P,\theta_P,\phi_P)$ and $Q=(r_Q,\theta_Q,\phi_Q)$ be PPFNs expressed by polar coordinates, then the distance between P and Q ranges from 0 to 1: $0\le d(P,Q)\le 1$.

Proof. By definition 4.1, combine $0\le r\le 1$, $|\theta_P-\theta_Q|\le\frac{\pi}{2}$ and $|\cos\phi_P-\cos\phi_Q|\le 1$, we have

$$d(P,Q)=\frac{6}{\pi}\left|\int_{\theta_P}^{\theta_Q}d\theta\int_{\phi_P}^{\phi_Q}\sin\phi d\phi\int_{0}^{a\theta+b\phi+c}r^2dr\right|\le\frac{6}{\pi}\left|\int_{\theta_P}^{\theta_Q}d\theta\int_{\phi_P}^{\phi_Q}\sin\phi\left[\frac{r^3}{3}\right]_{r=0}^{r=1}d\phi\right|$$

$$\le\frac{6}{\pi}\left|\int_{\theta_P}^{\theta_Q}\frac{1}{3}\left|\cos\phi_P-\cos\phi_Q\right|d\theta\right|\le\frac{6}{\pi}\left|\theta_P-\theta_Q\right|\times\frac{1}{3}\left|\cos\phi_P-\cos\phi_Q\right|\le\frac{6}{\pi}\times\frac{\pi}{2}\times\frac{1}{3}=1,$$

which completes the proof of Theorem 4.2.

Ttheorem 4.3 (Symmetry) Let $P=(r_P,\theta_P,\phi_P)$ and $Q=(r_Q,\theta_Q,\phi_Q)$ be PPFNs,then we have $d(P,Q)=d(Q,P)$.

Proof. As we have an absolute value operation in our new distance, the proof is straightforward.

Theorem 4.4 (Triangle inequality)Suppose $P=(r_P,\theta_P,\phi_P)$, $R=(r_R,\theta_R,\phi_R)$ and $Q=(r_Q,\theta_Q,\phi_Q)$ be three PPFNs expressed by polar coordinates.Let $a=\frac{r_P-r_Q}{2(\theta_P-\theta_Q)}$, $b=\frac{r_P-r_Q}{2(\phi_P-\phi_Q)}$, $c=r_P-a\theta_P-b\phi_P$.If $\theta_P\le\theta_Q\le\theta_Q$, $\phi_P\le\phi_Q\le\phi_Q$, and $r_P\ge a\theta_R+b\phi_R+c$,then we have $d(P,Q)=d(P,R)+d(R,Q)$.

Proof. Denote the integral domain in $d(P,Q),d(P,R),d(R,Q)$ as $\Omega_{OPQ},\Omega_{OPR},\Omega_{ORQ}$,respectively.Then, combine the linear properties of $r(\theta,\phi)=a\theta+b\phi+c$ with $r_R\ge a\theta_R+b\phi_R+c$, we have

$$\begin{aligned}\Omega_{OPQ}&=\left\{(r,\theta,\phi)\middle|0\le r\le a\theta+b\phi+c,\theta_P\le\theta\le\theta_Q,\phi_P\le\phi\le\phi_Q\right\}\\&=\left\{(r,\theta,\phi)\middle|0\le r\le a\theta+b\phi+c,\theta_P\le\theta\le\theta_R,\phi_P\le\phi\le\phi_R\right\}\\&\quad\cup\left\{(r,\theta,\phi)\middle|0\le r\le a\theta+b\phi+c,\theta_R\le\theta\le\theta_Q,\phi_R\le\phi\le\phi_Q\right\}\\&\subseteq\Omega_{OPR}\cup\Omega_{ORQ}\end{aligned}$$

Therefore

$$d(P,Q)=\frac{6}{\pi}\int_{\Omega_{OPQ}}dV\le\frac{6}{\pi}\int_{\Omega_{OPR}}dV+\frac{6}{\pi}\int_{\Omega_{ORQ}}dV=d(P,R)+d(R,Q),$$

which completes the proof of Theorem 4.4.

5 A modified TOPSIS method for multiple attribute decision-making under Pythagorean picture fuzzy environment

Basing on our new distance measure on PPFSs, we will propose a modifiedTOPSIS method for multiple attribute decision-making (MADM) problems under Pythagorean picture fuzzy environment.As is known to us, TOPSIS method is based on the concise principle that the optimal alternative should have the shortestdistance from the positive ideal solution marked by A^+ and the farthest distance from thenegative ideal solution denoted by A^-.

A multiple attribute decision-making problem under Pythagorean picture fuzzy environment can be expressed as a decision matrix whose elements represents the evaluation values of all alternatives with respect to each attribute.Let $A=\{A_1,A_2,\cdots,A_m\}$ be a discrete set of m feasible alternatives, $G=\{G_1,G_2,\cdots,G_n\}$ be a finite set of n attributes,and $w=\{w_1,w_2,\cdots,w_n\}$ be the weight vector of all attributes,satisfying $0\le w_i\le 1$ and $\sum_{i=1}^{n} w_i=1$ .We suppose the decision makers provide evaluation value for the alternative A_i under the attribute G_j with anonymity,the value can be considered as a Pythagorean picture fuzzy number $G_j(A_j)=P(r_{ij},\theta_{ij},\phi_{ij})$.Then the decision matrix $R=(G_j(A_j))_{m\times n}$ is a Pythagorean picture fuzzy decision matrix.Therefore, the MADM problem with PPFSs can be represented as the following matrix form:

$$R=(G_j(A_j))_{m\times n}=\begin{bmatrix} P(r_{11},\theta_{11},\phi_{11}) & P(r_{12},\theta_{12},\phi_{12}) & \cdots & P(r_{1n},\theta_{1n},\phi_{1n}) \\ P(r_{21},\theta_{21},\phi_{21}) & P(r_{22},\theta_{22},\phi_{22}) & \cdots & P(r_{2n},\theta_{2n},\phi_{2n}) \\ \vdots & \vdots & \vdots & \vdots \\ P(r_{m1},\theta_{m1},\phi_{m1}) & P(r_{m2},\theta_{m2},\phi_{m2}) & \cdots & P(r_{mn},\theta_{mn},\phi_{mn}) \end{bmatrix}$$

In what follows, we propose a modified TOPSIS method to MADM with PPFSs.

Step 1. We construct the decision matrix $R=(G_j(A_j))_{m\times n}$, where each element $G_j(A_j)=P(r_{ij},\theta_{ij},\phi_{ij})$ represents information expressed by PPFNs.

Step 2. Search positive ideal solution A^+ and negative ideal solution A^- [24] as:

$$A^+=\left\{\left\langle G_j,\max_i\left\langle S(G_j(A_i))\right\rangle\right\rangle|j=1,2,\cdots,n\right\},\ A^-=\left\{\left\langle G_j,\min_i\left\langle S(G_j(A_i))\right\rangle\right\rangle|j=1,2,\cdots,n\right\}.$$

Step 3. According to Definition 4.1, we can calculate the distance between A_i and A^+ denoted as $d(A_i,A^+)=\sum_{j=1}^{n} w_j d(G_j(A_i),G_j(A^+)),i=1,2,\cdots,m$. Likewise, we can compute the distance between A_i and A^- denoted as $d(A_i,A^-)=\sum_{j=1}^{n} w_j d(G_j(A_i),G_j(A^-)),i=1,2,\cdots,m$.

Step 4. Calculate the revised closeness[2] $\varsigma(A)$ for each alternative A_i as:

$$\varsigma(A_i)=\frac{d(A_i,A^-)}{\max_{1\le i\le m} d(A_i,A^-)}-\frac{d(A_i,A^+)}{\max_{1\le i\le m} d(A_i,A^+)},i=1,2,\cdots,m.$$

Step 5. Rank all the alternatives $A_i(i=1,2,\cdots,m)$ in accordance with the revised closeness $\varsigma(A_i)$ and select the best one(s) A^* such that $\varsigma(A^*)=\max_{1\le i\le m}\varsigma(A_i)$.

Step 6. End.

6 Illustrative example

In this section, our new method will be illustrated with a numerical example[9]. Suppose a multinational company is planning itsfinancial strategy for the next year. Four alternatives are obtained as below: A_1: Southern Asian markets; A_2: Eastern Asian markets; A_3: Northern Asian markets; A_4: Local markets.In order to evaluate these alternatives, four attributes are chosen: G_1 represents thegrowth analysis; G_2 is the risk analysis; G_3 corresponds the social politicalimpact analysis and G_4 is the environmental impactanalysis, whose weight vector is $w=(0.2,0.3,0.1,0.4)^T$ .The decision matrix $R=\left(G_j\left(A_j\right)\right)_{4\times4}$ is given by experts as following table 1.

Table 1:Pythagorean Picture Fuzzy Decision Matrix

	G_1	G_2	G_3	G_4
A_1	$P(0.2,0.1,0.6)$	$P(0.5,0.3,0.1)$	$P(0.5,0.1,0.3)$	$P(0.4,0.3,0.2)$
A_2	$P(0.1,0.4,0.4)$	$P(0.6,0.3,0.1)$	$P(0.5,0.2,0.2)$	$P(0.2,0.1,0.7)$
A_3	$P(0.3,0.2,0.2)$	$P(0.6,0.2,0.1)$	$P(0.4,0.1,0.3)$	$P(0.3,0.3,0.4)$
A_4	$P(0.3,0.1,0.6)$	$P(0.1,0.2,0.6)$	$P(0.1,0.3,0.5)$	$P(0.2,0.3,0.2)$

Take the element $G_1(A_1)=P(0.2,0.1,0.6)$ for example. When the committee evaluate A_1 with respect to G_1,there are 20 percent experts vote for, 10 percent experts vote neutral, 60 percent experts vote against and 10 percent experts are absent or refuse to vote. Other PPFNs in table 1 have the similar meanings.

In the following, we will utilize our modified TOPSIS method to solve this decision-making problem.Since the decision matrix $R=\left(G_j\left(A_j\right)\right)_{4\times4}$ has been constructed already,

We begin from step 2 to seek positive ideal solution A^+ and negative ideal solution A^- as:

$$A^+=\left\{\left\langle G_j,\max_i\left\langle S\left(G_j\left(A_i\right)\right)\right\rangle\right\rangle | j=1,2,3,4\right\}$$
$$=\left\{P(0.3,0.2,0.2),P(0.6,0.2,0.1),P(0.5,0.2,0.2),P(0.4,0.3,0.2)\right\},$$

$$A^-=\left\{\left\langle G_j,\min_i\left\langle S\left(G_j\left(A_i\right)\right)\right\rangle\right\rangle | j=1,2,3,4\right\}$$
$$=\left\{P(0.2,0.1,0.6),P(0.1,0.2,0.6),P(0.1,0.3,0.5),P(0.2,0.1,0.7)\right\}.$$

Step 3. Calculate the distance between $A_i\left(i=1,2,3,4\right)$ and A^+ denoted by $d\left(A_i,A^+\right)$ according to Definition 4.1. Similarly, we can acquire the distance between $A_i\left(i=1,2,3,4\right)$ and A^- denoted by $d\left(A_i,A^-\right)\left(i=1,2,3,4\right)$. Here, we only take $d\left(A_1,A^+\right)$ for example. We should convert the Cartesian coordinates to polar coordinates during our computation.

$$d\left(A_1,A^+\right)=\sum_{j=1}^4 w_j d\left(G_j\left(A_1\right),G_j\left(A^+\right)\right)$$

$$=w_1 d\left(\left(0.640312,1.21406,0.463648\right),\left(0.412311,0.506445,0.588003\right)\right)$$
$$+w_2 d\left(\left(0.591608,0.169846,0.54042\right),\left(0.640312,0.156816,0.321751\right)\right)$$
$$+w_3 d\left(\left(0.591608,0.531808,0.197396\right),\left(0.574456,0.355603,0.380506\right)\right)$$
$$+w_4 d\left(\left(0.538516,0.380506,0.643501\right),\left(0.538516,0.380506,0.643501\right)\right)$$

$$=0.2\times\frac{6}{\pi}\left|\int_{1.21406}^{0.506445}d\theta\int_{0.463648}^{0.588003}\sin\phi d\phi\int_0^{0.161105\theta-0.916738\phi+0.869764}r^2dr\right|$$
$$+0.3\times\frac{6}{\pi}\left|\int_{0.169846}^{0.156816}d\theta\int_{0.54042}^{0.321751}\sin\phi d\phi\int_0^{-1.86885\theta-0.111366\phi+0.969209}r^2dr\right|$$
$$+0.1\times\frac{6}{\pi}\left|\int_{0.531808}^{0.355603}d\theta\int_{0.197396}^{0.380506}\sin\phi d\phi\int_0^{0.0486696\theta-0.0468342\phi+0.57497}r^2dr\right|$$
$$+0.4\times 0$$
$$=0.2\times 0.00414491+0.3\times 0.000176094+0.1\times 0.00115587+0.4\times 0=0.000997396$$

Homoplastically, we can obtain the results which are listed in table 2.

Table 2:The distance between A_i and ideal solutions

	$d\left(A_i,A^+\right)$	$d\left(A_i,A^-\right)$	$\varsigma\left(A_i\right)$	Ranking
A_1	0.000997396	0.028868	0.706686	3
A_2	0.00795906	0.0392154	0.764959	2
A_3	0.00168679	0.0369587	0.892641	1
A_4	0.0338624	0.0125491	-0.679995	4

Step 4. Reckon the revised closeness $\varsigma\left(A_i\right)$ for the alternative $A_i, i=1,2,3,4$ .Here, we take $\varsigma\left(A_1\right)$ for example

$$\varsigma\left(A_1\right)=\frac{d\left(A_1,A^-\right)}{\max_{1\le i\le m} d\left(A_i,A^-\right)}-\frac{d\left(A_1,A^+\right)}{\max_{1\le i\le m} d\left(A_i,A^+\right)}=\frac{0.028868}{0.0392154}-\frac{0.000997396}{0.0338624}=0.706686.$$

Similarly, we can acquire other results which are also listed in table 2.

Step 5. Rank all the alternatives $A_i(i=1,2,3,4)$ in accordance with the revised closeness $\varsigma(A_i)$,we acquire

$A_3 \succ A_2 \succ A_1 \succ A_4$.

Thus, the best alternative is A_3.

Step 6. End.

Harish Garg[9] proposed some series of the aggregation operators for the picture fuzzysets. Basing on those operators, Harish Garg[9] also presented a new decision-making approach and solved the numerical example above as: $A_2 \succ A_1 \succ A_3 \succ A_4$ .We have the same rankings with $A_2 \succ A_1 \succ A_4$,while our difference is the comparison between A_2 and A_3 .Which alternative is better? A_2 or A_3?Let us analyze this evaluation carefully from four attributes.

(1) With respect to attribute G_1, the performance of A_2 is $G_1(A_2)=R(2,1)=(0.1,0.4,0.4)$ .By Definition 2.5, the score is $S(G_1(A_2))=0.1-0.4-0.4=-0.7$. While, the performance of A_3 under attribute G_1 can be described as $S(G_1(A_3))=0.3-0.2-0.2=-0.1$ .As $S(G_1(A_3))>S(G_1(A_2))$, we can infer that A_3 performs better than A_2 so far.Weight $w_1=0.2$ indicates that the importance for this comparison is 20 percent.Besides, $S(G_1(A_3))-S(G_1(A_2))=0.6$ also tells us, A_3 wins much in this round of competition.

(2) With respect to attribute G_2, the comparison between A_2 and A_3 is similar. We briefly compute here. $S(G_2(A_3))=0.6-0.2-0.1=0.3>S(G_2(A_2))=0.6-0.3-0.1=0.2$ .So we know that A_3 wins in this round of competition.The importance this round of competition is $w_2=0.3$.Besides, $S(G_2(A_3))-S(G_2(A_2))=0.1$ also tells us, A_3 wins a little in this round of competition.

(3) With respect to attribute G_3, $S(G_3(A_3))=0.4-0.1-0.3=0<S(G_3(A_2))=0.5-0.2-0.2=0.1$.So we know that A_3 lost in this round of competition.The importance this round of competition is the least weight $w_3=0.1$,which indicates that this competition is not important.Besides, $S(G_3(A_3))-S(G_3(A_2))=-0.1$ also tells us, A_3 does not lost much in this round of competition.

(4) With respect to attribute G_4, $S\left(G_4\left(A_3\right)\right)=0.3-0.3-0.4=-0.4>S\left(G_4\left(A_2\right)\right)=0.2-0.1-0.7=-0.6$. So we can see that A_3 win in this round of competition.The importance this round of competition is the maximal weight $w_4=0.4$, which indicates that this competition is the most important.Besides, $S\left(G_4\left(A_3\right)\right)-S\left(G_4\left(A_2\right)\right)=0.2$ also tells us, A_3 wins much in this round of competition.

As clearly seen above, A_3 performs better than A_2.Therefore, our method is more reasonable here.As a result, our newly defined distance is effective and meaningful.

7 CONCLUSION

In this paper, we set voting as a plain and intuitive example to explain the Pythagorean picture fuzzy set,which makes the PPFS easy to understand.Follow this intuitive explanation, we deeply analyze the process and internal mechanism of decision-making.Then, a new distance measure is naturally derived.Based on this intuitive and interesting distance, a modified TOPSIS method is developed for multiple attribute decision-makingunder Pythagorean picture fuzzy environment.Finally, an illustrative example for financial strategy is given to demonstrate its effectiveness and practicality.In the future, we will try to develop new methods for multiple attribute decision-making under Pythagorean picture fuzzy environment.

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