INEQUALITIES FOR THE MAXIMUM MODULUS OF A POLYNOMIAL

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Abstract :- In this paper, Inequalities for the maximum modulus of a polynomial and its derivative studied some of the properties of polynomials with derivative from a given continuous function. The use of inequalities of this kind is a fundamental method in proof of inverse problems of approximation theory.

Keywords:- Continuous functions, Inequalities, Rational Polynomial.

1. INTRODUCTION AND STATEMENT OF RESULTS

The concept of best approximation was introduced into mathematical analysis mainly by the work of the famous Russian mathematician Pi. Chebyshev (1821-1894), who studied some of the properties of polynomials with deviation from a given continuous function. According to Telyakovskiy [1191. "Among those that are fundamental in Approximation Theory are the extremal problems connected with inequalities for the derivative of polynomials. The use of inequalities of this kind is a fundamental method in proofs of inverse problems of approximation theory " In a prize winning essay as a problem of best approximation, Bernstein [22] proved and made considerable use of an inequality concerning the derivative of polynomials. It is interesting that the First result in this area was connected with some investigations of the well- known chemist Mendeleev [95], who asked: if the bound of a rational polynomial over a given interval is known, how large may its derivative be in this interval? Actually Mendeleev was interested only in a special case in which the polynomial is of degree two. He himself was able to prove that if p(x)is a quadratic polynomial and

$p(x)$ ⎹ ⎹ on [-1,1], then $p(x)$ □ 4 on the same interval. This a best possible result as is shown by the polynomial.

inequalities (1.1.7) and (1.1.9) were further improved by Aziz and Dawood [101 under the same hypothesis by proving the following interesting results:

THEOREM J_1: If p(z) is a polynomial of degree n which does not vanish in the disk z □ 1, then

| |

$$\max_{|z|=1} |p'(z)| \le \frac{n}{2}\left[\max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)|\right] \quad (1.1.10)$$

The result is sharp and equality holds for the polynomial

$p(z)$ □ □ □ □ z^n where | | | | □ □ □

z □ 1, then

THEOREM K_1. if P(z) is a polynomial of degree n which

does not vanish in the disk | |

$p(z)$ □ □ z^n □ where □ □ □

$$\max_{|z|=R>1} |p(z)| \le \left(\frac{R^n +}{2}\right.$$ □

The result is best possible and equality in (1.1.11) holds | | | |

for the polynomial

As a generalization of Theorem GI, Malik [891 proved the

following:

no zeros in

THEOREM L_I: if p(z) is a polynomial of degree n having

$z \square K, K \square 1$, then

The result is best possible and equality in (1.1.12) holds for p(z) $(z+k)^n$.

As an improvement of Theorem L, we have the following result due to Govil [58];

THEOREM M_I. If p(z) is a polynomial of degree n having

no zeros in the disk $|z| \square K, K \square 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k}\left[\max_{|z|=1}|p(z)| - \min_{|z|=k}|p(z)|\right] \qquad (1.1.13)$$

The result is best possible and equality holds for the polynomial p(z) = $(z+k)^n$.

Turan [123] considered the class of polynomials having all

zeros in $|z| \square 1$ and proved the following:

THEOREM N_I If p(z) is a polynomial of degree n having

The result is sharp and equality in (1.1,14) holds for the

polynomial having all its zeros on $|z| = 1$.

The following refinement of Theorem N was proved by Aziz and Dawood [10].

THEOREM O_1: If p(z) is a polynomial of degree n haying

all its zeros in $|z| \leq 1$,then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2}\left[\max_{|z|=1} |p(z)| + \min_{|z|=1} |p(z)|\right] \qquad (1.1.15)$$

The result is best possible and equality in (1.1.15) holds for

$p(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta|$

By applying Theorem L_1 to the polynomial $z^n p(1/z)$ Malik

[89] obtained the following generalization of (1114);

THEOREM P_I: If p(z) is a polynomial of degree n haying

all its zeros in $z|\square| 1$ where $k \square 1$ then

$$\max_{|z|=1} | p'(z) | \geq \frac{n}{1+k} \max_{|z|=1} | p(z) | \qquad (1.1.16)$$

The result is sharp and the extremal polynomial is p(z) =

$(z+ k)^n$

Goyil [60] considered the case $K \square 1$ and proved the following results:

THEOREM Q_1. If p(z) is a polynomial of degree n haying

all its zeros in $z|\square| 1$ where $k \square 1$ then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |p(z)| \qquad (1.1.17)$$

The result is sharp and equality in (1.1.17) holds for p (z)

$= z^n + k^n$

Aziz [61 obtained the following refinement of Theorem Q_1 by involving the moduli of the zeros of the polynomial p(z):

TIIEOREM R_1. If all the zeros of the polynomial p(z) = (z

- z,) of degree n lie in $|z| \leq 1$ where $k \geq 1$ then

$$\max_{|z|=1} |p'(z)| \geq \frac{2}{1+k^n} \sum_{j=1}^{n} \frac{k}{k+|z_j|} \max_{|z|=1} |p(z)| \qquad (1.1.18)$$

The result is sharp and equality in (1.1.18) holds for p(z) =

$z^n + k^n$

As a refinement and generalization of Theorem RI, Govil

[59] proved

THEOREM S_1. Let $p(z) = \sum_{v=0} a_v z^v = a_n \prod_{j=1} (z - z_j)$; $a_n \neq 0$, be a polynomial of

degree $n \geq 2$, $|z_j| \leq k_j$, $1 \leq j \leq n$ and let $k = \max(k_1, k_2, \ldots\ldots k_n) \geq 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|$$

$$+ \frac{2|a_{n-1}|}{(1+k^n)} \sum_{v=1}^{n} \left(\frac{1}{k+k_v}\right)\left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right) + |a_1| \left(1 - 1/k^2\right), \text{ if } n > 2 \qquad (1.1.19)$$

and

$$\max_{|z|=1}|p'(z)| > \frac{2}{1+k^n}\left(\sum_{v=1}^{n}\frac{k}{k+k_v}\right)\max_{|z|=1}|p(z)| + \frac{(k-1)^n|a_1|}{1+k^n}\sum_{v=1}^{n}\left(\frac{1}{k+k_v}\right)+(1-1/k)|a_1|$$

if $n-2$ (1.1.20)

In (1.1.19) and (1_1_20), equality holds for p(z) = z^n+k^n ".

Since klk+k_v≥ 1/2 for 1 ≤ v ≤ n, the above Theorem gives in particular the following improvement of Theorem Q_1.

THEOREM T_1: If $p(z) = a_n\prod_{v=1}^{n}(z-z_v)$; $a_n \neq 0$, is a polynomial of degree $n \geq 2$,

having all its zeros in $|z| \leq k$, where k≥1, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |p(z)| + \frac{n|a_{n-1}|}{k(1+k^n)}\left[\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right] + |a_1|\left(1-1/k^2\right)$$

$$\text{if } n > 2 \qquad (1.1.21)$$

and

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |p(z)| + \frac{(k-1)^n |a_1|}{k(1+k^n)} + (1-1/k)|a_1|,$$

$$\text{if } n = 2 \qquad (1.1.22)$$

In (1.1 .21) and (1.1.22), equality holds for $p(z) = z^n+z^k$

For quite some time it was believed that if p(z) # 0 in $|z| \square k$, k □ 1, then the inequality analogous to (1,1,12) should be

$$\max_{|z|=1} |p'(z)| \le \frac{n}{1+k^n} \max_{|z|=1} |p(z)|, \qquad (1.1.23)$$

till Professor E.B. Saff gave the example p(z) = (z-1/2) (z+1/3) to counter this belief.

For polynomials of degree n, having no zeros in |z| □ k, k □ 1, Govil [61] proved the following:

THEOREM U_1. If p(z) is a polynomial of degree n, having all its zeros

on |z| □ k where k □ 1 then

$$\max_{|z|=1} |p'(z)| \le \frac{n}{k^{n-1}+k^n} \max_{|z|=1} |p(z)|. \qquad (1.1.24)$$

Govil [62] obtained inequality (1.1.23) with an extra condition by proving the following:

THEOREM V_1. If p(z) is a polynomial of degree n such that p(z) # 0 in

$z \square k, |k| \square 1$

$$\max_{|z|=1} |p'(z)| \le \frac{n}{1+k^n} \max_{|z|=1} |p(z)| \qquad (1.1.25)$$

provided $|p'z|$ and $|q'z|$ become maximum at the same point on the circle $z \square k$ where k □ 1, then

$|z| \square 1$, where q(z)= $z^n p(1/z)$ .Equality in

(1.1.25) holds for p(z) = $z^n + k^n$.

It was shown by Aziz and Mohammad [121 that

if p(z) is a polynomial

of degree n which does not vanish in | |

$$M(P,R) \le \left(\frac{R+k}{1+k}\right)^n M(P,1) \text{ for } 1 \le R \le k^2. \qquad (1.1.26)$$

By applying inequality (1.1.26) to the polynomial z^n p(1/ z) , Jain [82] was able to prove the following result:

THEOREM W_1. If p(z) is a polynomial of degree n having all its zeros

in $z \square k, |k| \square 1$, then

Equality holds in (1.1.27) for $p(z) = (z+k)^n$

For polynomials not vanishing in $z|\square|1$ Rivlin [112] proved the following:

THEOREM X_1. If $p(z)=\sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, $p(z)\neq 0$ in

$|z| < 1$, then

$$M(p,r) \geq \left(\frac{1+r}{2}\right)^n M(p,1) \text{ for } r \leq 1 \qquad (1.1.28)$$

The result is best possible and the external polynomial is

$$p(z) - \left(\frac{\alpha+\beta z}{2}\right)^n, |\alpha|=|\beta|.$$

Extensions of Theorem X_1 where considered by Govil [63]. After rioting that (1.1.28) can be replaced by some what more general inequality

$$M(p,r) \geq \left(\frac{1+r}{1+R}\right)^n M(P,R) \text{ for } 0 \leq r \leq R \leq 1, \qquad (1.1.29)$$

he proved:

THEOREM Y_1 Let p(z) be a polynomial of degree n, and p(z) □ 0 in

$|z| \square 1$ if p′(0)=0, then for $0 \square r \square 1$, we have

$$M(p,r) \geq \left(\frac{1+r}{1+R}\right)^n \left[1 \Big/ \left\{ 1 - \frac{(1-R)(R-r)n}{4} \left(\frac{1+r}{1+R}\right)^{n-1} \right\} \right] M(p,R) \qquad (1.1.30)$$

For the class of polynomials having no zeros in $|z| \square k$, k □ 1 , inequality

(1.1.29) was generalized by Dewan and Bidkham 1411 by proving the following result:

THEOREM Z_1. If p(z) = $\sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n such that it

has no zeros in $|z| \square k$, k □ 1 then for $0 \square r \square \square \square 1$

$$M(p,r) \geq \left(\frac{r+k}{\lambda+k}\right) M(p,\lambda) \qquad (1.1.31)$$

The result is best possible and equality holds for the polynomial

$$p(z) = \left(\frac{z+k}{\lambda+k}\right)^n$$

For the class of polynomials satisfying

$p(z) \equiv \lambda z^n p(1/z)$, $|\lambda| = 1$, the following result is proved by Govil [60],

Mara and Rodriguez [100] and Saff and Sheil-Small [114] independently of each other:

THEOREM 2A1. If p(z) = $\sum_{v=0}^{n} a_v z^v$ is a self-inversive polynomial of degree n then

$$\max_{|z|=1} |p'(z)| = \frac{n}{2} \max_{|z|=1} |p(z)| \qquad (1.1.32)$$

It was proposed by Q.1. Rahman to study the class of

polynomials satisfying $p(z) \equiv z^n p(1/z)$ and obtain inequalities analogous

to (1.1.6) and Govil, Jain and Labelle [66] proved the following result in this direction:

THEOREM Z111. If p(z) is a polynomial of degree n satisfying

$p(z) \square z^n p(1/z)$ and has all its zeros either in the left half-plane or in the

night half plane, then

$$\max_{|z|=1} |p'(z)| \le \frac{n}{\sqrt{2}} \max_{|z|=1} |p(z)| \qquad (1.1.33)$$

$$\max_{|z|=R>1} |p(z)| \le \frac{R^n + \sqrt{2} - 1}{\sqrt{2}} \max_{|z|=1} |p(z)| \qquad (1.1.34)$$

and

$$\max_{|z|=1} |p'(z)| \ge \frac{n}{2} \max_{|z|=1} |p(z)| \qquad (1.1.35)$$

The inequality (1,1.35) is sham and equality holds for $p(z) \square \left(\frac{z \square 1}{\square \; 2} \right)$

Dewan and Govil [44] proved the inequality (1.1.35) by removing the condition that the zeros of the polynomial p(z) lie either in the left half-plane or in the right half-plane. This was later proved independently by Frappier and Rahman [52] and Aziz [6].

THEOREM ZC_I If p(z) =

$p(z) \equiv z^n p(1/z)$, then

$\sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n satisfying

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)| \qquad (1.1.36)$$

The result is best possible and equality holds for the polynomial $P(z) = 1+z^n$.

Other types of extremal problems have been considered among others by Rahman [108], Giroux and Rah.man [56], Frappier [51], ErdOs [49], lbragimov and Momedov [75], Mamedhamov [90], Newman [99], Dewan and Bidkham [41], Giroux, Rahman and Schmeisser [57], Govil [64], Abi-Khuzam [I], Ducan [47] and Aziz and Rather[15].

References:-

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