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HISTORICAL DEVELOPMENT AND GROWTH OF SUMMABILITY & ITS VARIOUS METHODS

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Abstract

A series which is not convergent is called divergent. The association of a sum with a given infinite series in this manner, naturally, opened up possibilities of the so-called 'summability' of infinite series in senses other than the classical sense of Cauchy. several distinguished mathematicians such as Cesáro, Hölder and Frobenius, various methods of associating finite 'sum' with oscillating divergent series were developed and before the thirties of the ninteenth century, 'summability' had become an important aspect of modern analysis. The fundamental idea underlying these 'summability methods' was to replace the notion of convergence in Cauchy's sense by some more general process based on a suitable generalization of the sequence of partial sums.

Keywords: Summability, Cauchy, History etc.

1. BRIEF HISTORY OF SUMMABILITY

AUGUSTIN – LOUIS CAUCHY and N.H. ABELwere among the pioneers in introducing a rigorous foundation for the algebra of infinite series [1].

The concept of the convergence of infinite series was first rigorously formulated by CAUCHY in 1820 in his "Course d' analyse de l'ecole polytechnique," Part I : "Analyse algébrique." Convergence was defined by Cauchy as follows [2]:

Given an infinite series of real or complex terms$\sum_{n=0}^{\infty} a_n$, let { S_n} denotes the sequence of its 'partial sums', i.e [3].

$$s_n = \sum_{k=0}^{n} a_k \qquad (\mathrm{n} = 0, 1, 2, \ldots)$$

Suppose that there is a number s, such that $\{S_n\}$ converges to s, i.e., given any $\in > 0, |s_n - s| < \in$ for all sufficient large n. Then Cauchy defined the 'sum' of $\sum a_n$ as the unique number s. This definition of the 'sum' of an infinite series is 'unique'. Any infinite series which has a 'sum' in Cauchy's sense is said to be 'convergent' [4].

2. MODERN ORIGIN OF SUMMABILITY

Broadly speaking, commonly used methods of summability fall into one or other of the two categories, viz. T-methods based upon the formation of a sequence of auxiliary means defined by the sequence-to-sequence transformation [5].

$$t_m = \sum_{m} C_{m,n} s_n \qquad (m = 0, 1, 2 \ldots) \tag{2.1}$$

s being the n-th partial sum of the series $\sum a_n$ and $C_{m,n}$, being the element of the m-th row and n-th column of the matrix of $\|T\| = (m, n)$ the matrix of summability, I -methods based upon the formation of a functional transformation [6].

$$t(x) = \sum_{n} \phi_n(x) s_n \tag{2.2}$$

or, by the function-to-function transformation

$$t(x) = \int_0^\infty \phi(x, y) s(y) dy \tag{2.3}$$

where x is a continuous parameter, and $\emptyset_n(x)$ or $\emptyset(x, y)$ is defined over an appropriate interval of x or (x and y).

A series $\sum a_n$ or the sequence of partial sums $\{S_n\}$ is said to be summable to a finite number s by T -method or ϕ -method according as the sequence $\{t_m\}$ or the function t(x) tends to s as m tends to infinite or x tends to the appropriate limit depending upon the method.

A series $\sum a_n$ is said to be absolutely convergent, if $\sum|a_n| < \infty$ that is, if

$$\sum | s_n - s_{n-1} | < \infty. \qquad 2.4$$

The interpretation of the phenomenon (2.4) as the bounded variation of the sequence $\{S_n\}$, laid the foundation of the structure of absolute summability. More precisely, a series $\sum a_n$ or the sequence of its partial sums $\{S_n\}$ is said to be absolutely summable to sum s by a T-method or φ -method according as the sequence {tm} or the function t (x) is of the bounded variation as a sequence or as a function over the relevant interval of x respectively and further the $t_m \to s$ as m→ t(x)→s, as x tends to a suitable limit [7].

It may be remarked that absolute convergence implies convergence.

3. TRANSFORMATION OF SEQUENCES

The sequence-to-sequence transformation (2.1) is said to be conservative (absolutely conservative) if the convergence (absolute convergence) of the sequence $\{S_n\}$ implies that of the sequence $\{t_m\}$ in each case, and is said to be regular (absolute regular), if further

$$\lim_{n\to\infty} s_n = s$$

implies

$$\lim_{m\to\infty} t_m = s .$$

It has shown that an absolutely conservative transformation is not necessarily conservative.The necessary and sufficient conditions that the transformation given above be conservative one; are

$$\lim_{m\to\infty} C_{m,n} = \delta_n \qquad (n = 0,1,2\ldots)$$

$$\lim_{m\to\infty} \sum_{n=0}^{\infty} C_{m,n} = \delta$$

$$\sum_{n=0}^{\infty} |C_{m,n}| \leq k \qquad (m = 0,1,2,\ldots)$$

3.1, 3.2, 3.3

where k is a constant independent of m.If, in addition,δ_n= 0 for each n, and δ_n = 1 then (3.1) gives the necessary and sufficient conditions for the transformation to be regular.

The necessary and sufficient conditions that the transformation (2.1)limit based transformation should be absolutely conservative are [8]:

$$\sum_{n=0}^{\infty} C_{m,n} \text{ converges for each } m$$

And

$$\sum_{m=0}^{\infty} \left| \sum_{n=p}^{\infty} (C_{m,n} - C_{m-1,n}) \right| \leq k$$

where k is a constant independent of p. Also (3.2) implies the existence of the limits:

$$\lim_{m\to\infty} \sum_{n=0}^{\infty} C_{m,n} = \alpha$$

$$\lim_{m\to\infty} C_{m,n} = \alpha_n \quad (n = 0, 1, 2, \ldots.)$$

The transformation (2.1) is absolutely regular if in addition

$$\alpha = 1,\ \alpha_n = 0 \qquad (n = 0,1,2,\ldots)$$

4. VARIOUS SUMMABILITY METHODS

This is the occasion to discuss several summability methods. Here a number of definitions and concepts relevant to the body work of our paper are stated [9].

4.1 CESÁRO SUMMABILITY

Cesáro summability introduced by FEKET in 1911 for nonnegative integral orders. The method was given a comprehensive treatment in considerable details by KOGBETLIANTZ in 1925 [10].

The n-th Cesáro sum of order $\alpha (\alpha > -1)$of a given series $\sum a_n$ with the sequence of partial sums {Sn} is defined by the identity:

$$s_n^{\alpha} = \sum_{i=0}^{n} A_{n-i}^{\alpha-1} s_i$$

WhereA_n^{α} is given by

$$\sum_{n=0}^{\infty} A_n^{\alpha} x^n = (1-x)^{-\alpha-1}; \qquad (|x| < 1)$$

We define σ_n^{α}be the n-th Cesáro mean of order α of $\{S_n\}$ by

$$\sigma_n^{\alpha} = \frac{s_n^{\alpha}}{A_n^{\alpha}}$$

The series $\sum a_n$is said to be summable$(C,\alpha), \alpha > -1$ to sum s, if$\lim_{n\to\infty} \sigma_n^{\alpha} = s$ of and the series is said to be absolutely summable (C,α) or sumamble $|C,\alpha|, \alpha > -1$, if the sequence

$$\sigma_n^\alpha \in BV.$$

It is clear from definition that summability (C,0) and absolute summability (C,0) are same as convergence and absolute convergence respectively.

Although$|C,\alpha| \Rightarrow (C,\alpha)$ was shown by KOGBETLIANTZ, by means of negative example and that$|C,\alpha| \Rightarrow |C,\beta|$ for$\beta > \alpha > -1$, is termed as consistency theorem for absolute Cesáro summability and is due to its general form to KOGBETLIANTZ. He also proved that if $\beta > \alpha > -1$ and $\sum a_n$is summable $|C,\beta|$ then the (C,α) transforms series of $\sum a_n$, viz. $\sum(\sigma_n^\alpha - \sigma_{n-1}^\alpha)$ is summable$|C,\beta-\alpha|$ and conversely. Shorter proof of these results for$\alpha > -1, \beta > -1, \beta - \alpha > -1$, and also of the consistency theorems for absolute Cesáro summability have been subsequently supplied.

Recently, FLETT extended the definition of absolute Cesáro summability by introducing a number of parameters. Thus according to him a series $\sum a_n$is said to be summable $|C,\alpha,r|_k$ $k \geq 1, \alpha > -1$, and r is a positive real number, if$\sum n^{k+kr-1}|\sigma_n^\alpha - \sigma_{n-1}^\alpha| < \infty$. On the other hand, if r = 0 and k = 1, we have summability $|C,\alpha|$which is the same as the summability $|C,\alpha|$. It follows from certain results of FLETT that summability $|C,\alpha|_k$ $k > 1$ and summability $|C,\alpha|$are independent of each other.

4.2 NÖRLUND SUMMABILITY

Nörlund summability, though originally initiated in 1902 by WORONOI and having remain unknown till pointed out by TAMARKIN in 1932, was independently introduced by NÖRLUND in 1919 and it has now become customary to associate it with his name [11].

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{S_n\}$. Let $\{p_n\}$ be a sequence of positive constants, real or complex and let us write,

$$P_n = p_0 + p_1 + \cdots + p_n : P_{-m} = p_{-m} = 0, (m \geq 1)$$

And

$$\left(\sum p_n x^n\right)^{-1} = \sum C_n x^n$$

whenever it holds.

The sequence-to-sequence transformation.

$$t_n = \frac{1}{P_n} \sum_{i=0}^{n} p_{n-i} s_i \qquad (P_n \neq 0)$$

defines the sequence $\{t_n\}$ of the Nörlund mean of $\{S_n\}$ generated by the coefficients $\{p_n\}$.

The series $\sum a_n$ or the sequence $\{S_n\}$ is defined to be summable by the Nörlund method or summable $(N,P_n)$to s (finite), if$\lim_{n\to\infty} t_n \to s$ and it is said to be absolutely summable (N,p_n) or$|N,\ p_n|$ if ,$t_n \in BV$ that is

$$\sum |t_n - t_{n-1}| < \infty.$$

Necessary and sufficient conditions for the regularity of the Nörlund mean are:

$$p_n = O(|P_n|), \qquad \text{as } n \to \infty$$

$$\sum_{i=0}^{n} |p_i| = O(|P_n|), \qquad \text{as } n \to \infty$$

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Necessary and sufficient conditions that the method (N,p_n) should be absolutely regular are:

$$\sum_{n=1}^{\infty}\left|\frac{P_{n-i}}{P_n}-\frac{P_{n-i-1}}{P_{n-1}}\right|<\infty$$

for each *i*.

4.3 WEIGHTED MEAN OR (N,p_n) –SUMMABILITY

Let $\sum a_n$ be a given infinite series with partial sums $\{S_n\}$. Let $\{p_n\}$ be a sequence of constants such that

$$P_n = p_0 + p_1 + p_2 + \cdots + p_n, P_{-m} = p_{-m} = 0 \quad (m \geq 1)$$

$$= \sum_{k=0}^{n} p_k.$$

Then (N,p_n) mean of $\{n_s\}$ is given by

$$t'_n = \frac{1}{P_n}\sum_{k=0}^{n} p_k s_k \quad (P_n \neq 0)$$

The series $\sum a_n$ or the sequence $\{S_n\}$ is said to be summable (N,p_n) to s (finite), if $\lim_{n\to\infty} t_n' = s$ and it is said to be summable $|\bar{N}, P_n|$ if $t_n' \in BV$.

If $P_n = \frac{1}{n+1}(n \geq 0)$,$P_n \sim \log(n+1)$ then the $(\bar{N}, P_n)$mean or equivalently (R,logn,1) mean is called the Riesz 'logarithmic mean' of order 1.

It is worth noting that when $P_n > 0\ (n \geq 0)$and$P_n \to \infty$, the $(\bar{N}, P_n)$method is regular and absolute regular, then the series $\sum a_n$is said to be summable $|\bar{N}, P_n|_k k \geq 1$, if [12]

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|t'_n - t'_{n-1}|^k < \infty$$

when k=1, $|\overline{N}, P_n|_k$ summability is the same as $|\overline{N}, P_n|$ and when p_n=1 for all values of n, and k=1, $|\overline{N}, P_n|_k$ summability is same as $|C, 1|$-summability.

4.4 MATRIX SUMMABILITY

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{S_n\}$ and let A=(a_n,k) be an infinite matrix with real or complex elements then the transformation of {Sn}, given by the matrix multiplication

$$t_n = \sum_{k=0}^{n} a_{n,k} s_k$$

(assuming 't_n' exists, for every n 1,2,...), defines the matrix transform of the sequence $\{S_n\}$, or the $\sum u_n$ generated by the elements of the matrix A. If $\lim_{n\to\infty} t_n = s$ of the sequence $\{S_n\}$, or the series $\sum u_n$ is said to be summable,(an,k) or simply summable |A|, if the corresponding auxiliary sequence $\{t_n\}$ is of bounded variation that is to say

$$\sum_{n=1}^{\infty}|t_n - t_{n-1}| < \infty.$$

The necessary and sufficient conditions for the A-method to be regular (that is $\lim_{n\to\infty} s_n = s = \lim_{n\to\infty} t_n = s$) are

$$\sum_{k=0}^{\infty}|a_{n,k}| < K, \text{ for every } n,$$

$$\text{for every } K, \lim_{n\to\infty} a_{n,k} = 0$$

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} = 1$$

where K is a constant.

If the matrix element a_n,k=0, for every k>n, then the matrix is called the lower triangular matrix.

In particular, if

$$a_{nk} = \begin{cases} \dfrac{p_{n-k}}{P_n} & (k \le n) \\ 0 & (k > n) \end{cases}$$

Where$p_n = \sum_{k=0}^{n} p_k \neq 0$ then t_n, is the same as Nörlund mean, generated by the sequence of coefficients$\{p_n\}$.

Similarly, if

$$a_{n,k} = \begin{cases} \dfrac{\binom{n-k+\alpha}{\alpha-1}}{\binom{n+\alpha}{\alpha}}, & \alpha > 0, \quad \text{for } k \le n \\ 0 \qquad , & \qquad \text{for } k > n \end{cases}$$

then t_n mean is the same as (C,α) mean i.e. the familiar Cesáro mean of order α.

5. CONCLUSION

So it is concluded that the effect of knowledge of summability in the development of "Theory of Functions" is well known. Convergence of infinite series being simply a particular case of summability, the importance need not be further emphasized. One may also refer to the idea of mean convergence and summability factors and absolute summability, which have served to extend and generalize a number of classical concepts in the theory of functions

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