International Journal of Engineering, Science and Mathematics
Vol. 6 Issue 2, June 2017,
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijesm.co.in, Email: ijesmj@gmail.com

ON TRANSFORMATION OF CERTAIN BILATERAL BASIC HYPERGEOMETRIC SERIES AND THEIR APPLICATIONS

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Abstract. In this paper, we obtain new transformation and summation formulae for bilateral basic hypergeometric series and give a simple proof of Jacobi's triple product identity. Further, we present applications of the main results.

Keywords. Bilateral basic hypergeometric series; Jacobi's triple product identity; mock theta functions; theta functions.

Introduction

Euler was the first to introduce the concept of basic hypergeometric series in 1748. But Heine made this into a systematic theory by generalizing Gauss' 2F1 series. Heine's transformation formula, Jacobi's triple product identity, and Cauchy's q-binomial theorem are perhaps the foundation stones for further development of the theory of basic hypergeometric series through the summation and transformation formulae. Since then a number of mathematicians namely Dougall [6], Jackson [9,10], Ramanujan [11], Bailey [3,4], Slater [13], Andrews [1], Ismail [8] and Schlosser [12] have used techniques such as residue calculus and analytic continuation and contributed to the summations and transformations of basic and bilateral basic hypergeometric series identities. For more details, one may refer the books by Gasper and Rahman [7] and Slater [13]. In 1970, Andrews [2] obtained a transformation formula for bilateral basic hypergeometric series by using an easily proved transformation lemma given below.

Lemma 1.1. Subject to suitable convergence conditions, if $c_n = \sum_{m=0}^{\infty} a_{m+n} b_m$, then

$$\sum_{m=0}^{\infty} b_m \sum_{n=-\infty}^{\infty} a_n = \sum_{n=-\infty}^{\infty} c_n .$$

The main objective of this paper is to obtain new transformations and summation formulae for bilateral basic hypergeometric series and to give a simple proof for Jacobi's triple product identity using (1.1). Further, we find applications of these identities. In section 2, we present some standard definitions and identities. In section 3, we prove our main results. In sections 4 and 5, we present applications of our main results.

Some standard definitions and identities

In this section, we present some standard definitions and identities which will be used in the later part of the paper. Throughout this paper, we assume that |q| < 1. The q-shifted factorial is defined by

$$(a)_\infty := (a;\ q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n) \ ,$$

$$(a)_n := (a;\ q)_n := \frac{(a)_\infty}{(aqn)_\infty} \quad , n, \text{an integer.}$$

We use the notation

$(a_1, a_2,\ a_3, \ldots,\ a_m)_n = (a_1)_n(\ a_2)_n(\ a_3)_n \ldots (\ a_m)_n$, n,an integer or ∞ and an identity

$$(a)_{m+n} = (a)_m(aq^m)_n\ ,$$

where n is a non-negative integer. The generalized basic hypergeometric series is defined by

$$r\phi s\begin{pmatrix} a_1, a_2, \ldots,\ a_r \\ b_1,\ b_2, \ldots,\ b_s \end{pmatrix}; q; z\Big) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \ldots (a_r)_n}{(q)_n(b_1)_n(b_2)_n \ldots (b_s)_n} z^n,$$

where $r = s + 1$ *and* $|z| < 1$. The bilateral hypergeometric series is defined by

$$r\psi r\begin{pmatrix} a_1, a_2, \ldots,\ a_r \\ b_1,\ b_2, \ldots,\ b_r \end{pmatrix}; q; z\Big) := \sum_{n=-\infty}^{\infty} \frac{(a_1)_n(a_2)_n \ldots (a_r)_n}{(b_1)_n(b_2)_n \ldots (b_s)_n} z^n,$$

where$\left| \frac{b_1\, b_2 \ldots\, b_r}{a_1 a_2 \ldots\, a_r} \right| < |z| < 1$

Ramanujan's general theta function $f\ (a, b)$ [5,11] is given by

$$f\ (a, b) := 1 + \sum_{n=1}^{\infty}(ab)^{n(n-1)/2}(a^n + b^n)$$

$$= \sum_{-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \quad , |ab| < 1.$$

The special cases of $f\ (a,b)$ are given by

$$\varphi(q) := f\ (q,q) = 1 + 2\sum_{k=1}^{\infty} q^{k^2} = \frac{(-q;\ q^2)_\infty (q^2;\ q^2)_\infty}{(q;\ q^2)_\infty (-q^2;\ q^2)_\infty}\ , \tag{2.2}$$

$$\psi(q) := f\ (q,q^3) = \sum_{k=1}^{\infty} q^{k(k+1)/2} = \frac{(q^2;\ q^2)_\infty}{(q;\ q^2)_\infty}\ , \tag{2.3}$$

$$f(-q) := f\ (-q,-q^2) = \sum_{k=1}^{\infty} (-1)^k q^{k(k+1)/2} = (q;\ q)_\infty\ , \tag{2.4}$$

$$x(q) \coloneqq (-q,q^2)_\infty (2.5)$$

The q-binomial theorem [7, equation (II.3), p. 354] is given by

$$\sum_{n=0}^{\infty} \frac{(a;\ q)_n}{(q;\ q)_n} z^n = \frac{(az;\ q)_\infty}{(z;\ q)_\infty}. \tag{2.6}$$

The two interesting special cases [7, equations (II.1), (II.2), p. 354] which are due to Euler are given by

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;\ q)_n} = \frac{1}{(z;\ q)_\infty}. \tag{2.7}$$

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} z^n}{(q;\ q)_n} = (-z;\ q)_\infty. \tag{2.8}$$

The sum of a $1\phi1$ series [7, equation (II.5), p. 354] is given by

$$\sum_{n=0}^{\infty} \frac{(a;\ q)_n}{(q,c;\ q)_n} (-c/a)^n\ q^{\binom{n}{2}} = \frac{(c/a;\ q)_\infty}{(c;\ q)_\infty}\ . \tag{2.9}$$

The Sears' transformations of $3\phi2$ series [7, equations (III.9), (III.10) p. 359] are given by

$$\sum_{n=0}^{\infty} \frac{(a,b,c;\ q)_n}{(q,d,e;\ q)_n} (de/abc)^n$$

$$= \frac{(e/a, de/bc;\ q)_\infty}{(e, de/abc;\ q)_\infty} \sum_{n=0}^{\infty} \frac{(a, d/b, d/c;\ q)_n}{(q, d, de/bc)_n} (e/a)^n \ , \qquad (2.10)$$

$$\sum_{n=0}^{\infty} \frac{(a, b, c;\ q)_n}{(q, d, e;\ q)_n} (de/abc)^n$$

$$= \frac{(b, de/ab, de/bc;\ q)_\infty}{(d, e, de/abc;\ q)_\infty} \sum_{n=0}^{\infty} \frac{(d/b\ , e/b, de/abc;\ q)_n}{(q, de\ /\ ab, de/bc)_n} (e/a)^n \ , \qquad (2.11)$$

Result and Discussion:

In this section, we prove our main results.

Theorem 3.1. We have

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n \ = \frac{(b/a;\ q)_\infty}{(b, b/az;\ q)_\infty} \sum_{n=-\infty}^{\infty} (a)_n \, z^n, \qquad (3.1)$$

where $\{max|\frac{b}{a}|, |\frac{1}{a}\ |\} \ < \ |z| \ < \ 1.$

Proof. Let

$$a_n \ = \frac{(a;\ q)_n}{(b;\ q)_n} z^n, \qquad b_n \ = \frac{(-b/az)^n q^{n(n-1)/2}}{(q;\ q)_n} \ .$$

Then

$$c_n \ = \sum_{m=0}^{\infty} a_{m+n} b_m \ = \sum_{m=0}^{\infty} \frac{(a;\ q)_{m+n}}{(b;\ q)_{m+n}} z^{\ m+n} \frac{(-b/az)^m q^{m(m-1)/2}}{(q;\ q)_m}$$

$$= \frac{(b/a;\ q)_\infty}{(b;\ q)_\infty} (a;\ q)_n z^n,$$

on using (2.9). Hence by the transformation lemma 1.1, we obtain

$$\sum_{m=0}^{\infty} \frac{(-b/az)^m q^{m(m-1)/2}}{(q;\,q)_m} \sum_{n=-\infty}^{\infty} \frac{(a;\,q)_n}{(b;\,q)_n} z^n = \sum_{n=-\infty}^{\infty} \frac{(b/a;\,q)_\infty}{(b;\,q)_\infty} (a;\,q)_n z^n.$$

The equality holds when $max\left\{\left|\frac{b}{a}\right|, \left|\frac{1}{a}\right|\right\} < |z| < 1$. Now, using (2.8) on the left-hand side of the above identity, we obtain (3.1) due to some simplifications.

Theorem 3.2. We have

$$\sum_{n=-\infty}^{\infty} (a)_n z^n = \frac{(q, q/az, az;\, q)_\infty}{(q/a, z;\, q)_\infty} \,, \tag{3.2}$$

where $\left|\frac{1}{a}\right| < |z| < 1$.

Proof. Let

$$a_n = (a)_n\, z^n, \qquad bn = \frac{(q/az)^n}{(q)_{\,n}} \,.$$

Then

$$c_n = \sum_{m=0}^{\infty} a_{m+n} b_m = \sum_{m=0}^{\infty} (a;\, q)_{m+n}\, z^{m+n} \frac{(q/az)^m}{(q;\, q)_m}$$

$$= \frac{(q;\, q)_\infty}{(q/a;\, q)_\infty} \frac{(a)_n z^n}{(q;\, q)_n} \,,$$

on using (2.6). Hence by the transformation lemma 1.1, we obtain

$$\sum_{m=0}^{\infty} \frac{(q\,/\,az)^m}{(q)_m} \sum_{n=-\infty}^{\infty} (a)_n z^n = \sum_{n=-\infty}^{\infty} \frac{(q;\, q)_\infty}{(q/a;\, q)_\infty} \frac{(a)_n z^n}{(q;\, q)_n} \,,$$

The equality holds when $\left|\frac{1}{a}\right| < |z| < 1$. Now, using (2.7) on the left-hand side and (2.6) on the right-hand side of the above identity, we obtain (3.2), due to some simplifications.

Remark. Substituting (3.2) in (3.1), we obtain the well-known Ramanujan's 1ψ1 summation formula [11]:

$$\sum_{n=-\infty}^{\infty} \frac{(a;\, q)_n}{(b;\, q)_n} z^n = \frac{(q, b/a, az, q/az;\, q)_\infty}{(b, q/a, z, b/az;\, q)_\infty} \,.$$

Theorem 3.3 (Jacobi). We have

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = (q^2, -qz, -q/z;\ q^2)_\infty, z \neq 0. \tag{3.3}$$

Proof. Let

$$a_n = q^{n^2} z^n, \qquad b_n = \frac{(-q/z)^n}{(q^2;\ q^2)_n} .$$

Then

$$c_n = \sum_{m=0}^{\infty} a_{m+n} b_m = \sum_{m=0}^{\infty} q^{n^2} z^n \frac{(q/az)^{m+n}}{(q;\ q)_{m+n}} \frac{(-q/z)^m}{(q^2;\ q^2)_m}$$

$$= q^{n^2} z^n \frac{(q^2;\ q^2)_\infty}{(q^2;\ q^2)_n} ,$$

on using (2.8). Hence by the transformation lemma 1.1, we obtain

$$\sum_{m=0}^{\infty} \frac{(-q/z)^m}{(q^2;\ q^2)_m} \sum_{n=-\infty}^{\infty} q^{n^2} z^n = \sum_{n=-\infty}^{\infty} q^{n^2} z^n \frac{(q^2;\ q^2)_\infty}{(q^2;\ q^2)_n}$$

which holds for $z \neq 0$. Now, using (2.7) on the above identity, we obtain (3.3) due to some simplifications.

Theorem 3.4. We have

$$\sum_{n=-\infty}^{\infty} \frac{(b,c)_n}{(d,e)_n} z^n = \frac{(de/bc)_\infty}{(e, de/bcz)_\infty} \sum_{m=0}^{\infty} \frac{(d/b, d/c)_m}{(q, de/bc, d)_m} (-1)^m q^{\binom{m}{2}} e^m$$

$$\sum_{n=-\infty}^{\infty} \frac{(b,c)_n}{(0, dq^m)_n} (q^n)^m , \tag{3.4}$$

where $|\frac{de}{bc}| < |z| < 1$.

Proof. Let

$$a_n = \frac{(b,c;\ q)_n}{(d,e;\ q)_n} z^n, \qquad b_n = \frac{(-1)^n q^{\binom{n}{2}} (de/bcz)^n}{(q;\ q)_n}\ .$$

Then

$$c_n = \sum_{m=0}^{\infty} a_{m+n} b_m = \sum_{m=0}^{\infty} \frac{(b,c;\ q)_{m+n}}{(d,e;\ q)_{m+n}} z^{m+n} \frac{(-1)^m q^{\binom{n}{2}} (de/bcz)^m}{(q;\ q)_m}$$

$$= \frac{(b,c;\ q)_n}{(d,e;\ q)_n} z^n \sum_{m=0}^{\infty} \frac{(bq^n, cq^n;\ q)_m}{(q, dq^n, eq^n;\ q)_m} (-1)^m q^{\binom{n}{2}} (de/bc)^m$$

$$= (b,c)_n z^n \frac{(de/bc)_\infty}{(e)_\infty} \sum_{m=0}^{\infty} \frac{(d/b, d/c)_m}{(q, de/bc)_m (d)_{m+n}} (-1)^m q^{\binom{n}{2}} (eq^n)^m\ ,$$

on using the identity obtained by letting $a \to \infty$ in (2.10). Hence by the transformation lemma 1.1, we obtain (3.4) due to some simplifications.

Theorem 3.5. We have

$$\sum_{n=-\infty}^{\infty} \frac{(b,c)_n}{(d,e)_n} z^n$$

$$= \frac{(b, de/bc)_\infty}{(d,e,de/bcz)_\infty} \sum_{m=0}^{\infty} \frac{(d/b, e/b)_m}{(q, de/bc)_m} b^m \sum_{n=-\infty}^{\infty} (c)_n (zq^m)^n\ , \qquad (3.5)$$

where $\{max\, |\frac{de}{bc}|, |\frac{1}{c}|\} < |z| < 1\ and\ |b| < 1.$

Proof. Let

$$a_n = \frac{(b,c;\ q)_n}{(d,e;\ q)_n} z^n, \qquad b_n = \frac{(-1)^n q^{\binom{n}{2}} (de/bcz)^n}{(q;\ q)_n}.$$

Then

$$c_n = \sum_{m=0}^{\infty} a_{m+n} b_m = \sum_{m=0}^{\infty} \frac{(b,c;\ q)_{m+n}}{(d,e;\ q)_{m+n}} z^{m+n} \frac{(-1)^m q^{\binom{n}{2}} (de/bcz)^m}{(q;\ q)_m}$$

$$= \frac{(b,c;\ q)_n}{(d,e;\ q)_n} z^n \frac{(bq^n, cq^n;\ q)_m}{(q, dq^n, eq^n;\ q)_m} (-1)^m q^{\binom{n}{2}} (de/bc)^m$$

$$= (c)_n z^n \frac{(b, de/bc)_\infty}{(d,e)_\infty} \sum_{m=0}^{\infty} \frac{(d/b, e/b)_m}{(q, de/bc)_m} (bq^n)^m$$

on using the identity obtained by letting $a \to \infty$ in (2.11). Hence by the transformation lemma 1.1, we obtain

$$\sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}} (de/bcz)^m}{(q;\ q)_m} \sum_{n=-\infty}^{\infty} \frac{(b,c;\ q)_n}{(d,e;\ q)_n} z^n$$

$$= \sum_{n=-\infty}^{\infty} (c)_n z^n \frac{(b, de/bc)_\infty}{(d,e)_\infty} \sum_{m=0}^{\infty} \frac{(d/b, e/b)_m}{(q, de/bc)_m} (bq^n)^m$$

The equality holds when $max\left\{\left|\frac{de}{bc}\right|, \left|\frac{1}{c}\right|\right\} < |z| < 1\ and\ |b| < 1$. Now, using (2.8) on the above identity, we obtain (3.5) due to some simplifications.

Theorem 3.6. We have

$$\sum_{n=-\infty}^{\infty} q^{n(n+1)} (-zq^{2n+2};\ q^2)_\infty = \frac{(q^2, -q^2, -1;\ q^2)_\infty}{(z;\ q^2)_\infty}\ , |z| < 1. \tag{3.6}$$

Proof. Let

$$a_n = q^{n(n+1)}, b_n = \frac{z^n}{(q^2;\ q^2)_n}\ .$$

Then

$$c_n = \sum_{m=0}^{\infty} a_{m+n} b_m = \sum_{m=0}^{\infty} q^{(m+n)(m+n+1)} \frac{z^m}{(q^2;\ q^2)_m}$$

$$= q^{n(n+1)} (-zq^{2n+2}; q^2)_\infty$$

on using (2.8). Hence by the transformation lemma 1.1, we obtain

$$\sum_{m=0}^{\infty} \frac{z^m}{(q^2;\, q^2)_m} \sum_{n=-\infty}^{\infty} q^{n(n+1)} = \sum_{n=-\infty}^{\infty} q^{n(n+1)}(-zq^{2n+2};\, q^2)_\infty$$

The equality holds when $|z| < 1$. Now, using (2.7) we obtain (3.6), due to some simplifications.

Theorem 3.7. If

$$h(z) = (zq;\, q^2)_\infty \left\{ \sum_{n=0}^{\infty} \frac{z^{2n} q^{4n^2}}{(zq;\, q^2)_{2n}} + \sum_{n=1}^{\infty} (q/z; q^2)_{2n} \right\},$$

then

$$h(z) + zqh(zq^2) = \frac{(q^2, -qz, -q/z;\, q^2)_\infty}{(-1, qz;\, q^2)_\infty} \quad . \tag{3.7}$$

Proof. Let

$$a_n = z^n q^{n^2}, \qquad b_n = \frac{(-1)}{(q^2;\, q^2)_n} \quad .$$

Then

$$c_n = \sum_{m=0}^{\infty} a_{m+n} b_m = \sum_{m=0}^{\infty} z^{(m+n)(m+n)} \frac{(-1)^m}{(q^2;\, q^2)_m}$$

$$z^n q^{n^2}(zq^{2n+1}; q^2)_\infty$$

on using(2.8).

Now, using the transformation lemma 1.1, (2.7) and (3.3), we obtain due to some simplifications,

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(q^2;\, q^2)_m} \sum_{n=-\infty}^{\infty} z^n q^{n^2} = \sum_{n=-\infty}^{\infty} z^n q^{n^2}(zq^{2n+1}; q^2)_\infty$$

This, on simplification and by using (3.3) and (2.7), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{z^n q^{n^2}}{(zq;\ q^2)_n} = \frac{(q^2, -qz, -q/z; q^2)_\infty}{(-1, qz;\ q^2)_\infty} \ .$$

Finally, by breaking $\sum_{n=-\infty}^{\infty} c_n$ into even and odd terms, we obtain

$$h(z)\ +\ zqh(zq^2)\ =\ \sum_{n=-\infty}^{\infty} c_n$$

which yields (3.7).

1. Applications of the main results

In this section, we use our main results to obtain identities involving theta and mock theta functions.

COROLLARY 4.1

If we define the complete (bilateral) mock theta function $\gamma_c(q)$ of third order by

$$\gamma_c(q) \coloneqq \sum_{n=-\infty}^{\infty} \frac{q^{n(n+1)}}{(-q;\ q^2)_{n+1}}$$

Then

$$\gamma_c(q) = 2\psi(q^2)\frac{f(-q^4)}{f(-q^2)} \tag{4.1}$$

Proof. We have, by (3.6),

$$\sum_{n=-\infty}^{\infty} \frac{q^{n(n+1)}}{(-zq^2;\ q^2)_n} = \frac{(q^2, -q^2, -1;\ q^2)_\infty}{(z, -zq^2;\ q^2)_\infty} \ . \tag{4.2}$$

Changing z to q in (4.2), we obtain, after some simplifications,

$$\sum_{n=-\infty}^{\infty} \frac{q^{n(n+1)}}{(-q;\ q^2)_{n+1}} = \frac{(q^2, -q^2, -1;\ q^2)_\infty}{(q, -q;\ q^2)_\infty} \ . \tag{4.3}$$

Using (2.3) and (2.4) on the right-hand side of (4.3), we obtain (4.1).

COROLLARY 4.2

We have

$$\sum_{n=-\infty}^{\infty} \frac{q^{n(n+1)/2}}{(-q;\ q)_{n+1}} = \frac{f(-q^2)}{f(-q)} \,. \tag{4.4}$$

Proof. Changing $z\ to\ q^2$ in (4.2) and then changing $q\ to\ q^{1/2}$ in the resulting identity, we obtain, after some simplifications,

$$\sum_{n=-\infty}^{\infty} \frac{q^{n(n+1)/2}}{(-q;\ q)_{n+1}} = 2(-q;q)_\infty \,. \tag{4.5}$$

Using (2.4) on the left-hand side of (4.5), we obtain (4.4).

COROLLARY 4.3

We have

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;\ q)_{n+1}} = \frac{f(-q^2)}{f(-q)} \,. \tag{4.6}$$

Proof. Changing z to $-q^2$ in (4.2) and then changing q to $q^{1/2}$ in the resulting identity, we obtain, after some simplifications,

$$\sum_{n=-\infty}^{\infty} \frac{q^{n(n+1)/2}}{(q;\ q)_{n+1}} = 2\frac{(q^2;\ q^2)_\infty}{(q;\ q)_\infty} \,.$$

which can be written as

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;\ q)_{n+1}} = 2\frac{(q^2;\ q^2)_\infty}{(q;\ q)_\infty} \,. \tag{4.7}$$

Using (2.4) on the left-hand side of (4.7), we obtain (4.6).

2. Some special cases

Changing a to −1/a and z to −bq in (3.1), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{(-1/a)_n}{(b)_n}(-bq)^n = \frac{(-ab)_\infty}{(b,a/q)_\infty} \sum_{n=-\infty}^{\infty} (-1/a)_n(-bq)^n \quad ,|a|,|b| < 1$$

Changing a to q^2 and b to -1 in (5.1), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^n}{(1+q^{n-2})(1+q^{n-1})} = \frac{1}{2(-1/q^2;\ q)_\infty} \sum_{n=-\infty}^{\infty} (-1/q^2;\ q)_n q^n. \tag{5.2}$$

Changing a to q^2 and b to $-q^3$in (5.1), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{4n}}{(1+q^{n-2})(1+q^{n-1})(1+q^n)(1+q^{n+1})(1+q^{n+2})}$$

$$= \frac{(q^5;q)_\infty}{(-1/q^2;\ q;q)_\infty} \frac{1}{2(-1/q^2;\ q)} \sum_{n=-\infty}^{\infty} (-1/q^2;\ q)_n q^n \tag{5.3}$$

Changing a to q^2 and b to $-q$ in (5.1), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{(1+q^{n-2})(1+q^{n-1})(1+q^n)}$$

$$= \frac{(q^3;q)_\infty}{(-1/q^2;\ q;q)_\infty} \sum_{n=-\infty}^{\infty} (-1/q^2;\ q)_n q^n \tag{5.4}$$

Changing a to q^2 and b to $-q^2$in (5.1), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{3n}}{(1+q^{n-2})(1+q^{n-1})(1+q^n)(1+q^{n+1})}$$

$$= \frac{(q^4;q)_\infty}{(-1/q^2;\ q;q)_\infty} \sum_{n=-\infty}^{\infty} (-1/q^2;\ q)_n q^{3n} \tag{5.5}$$

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