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ON THE DEGREE OF APPROXIMATION OF FUNCTIONS

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ABSTRACT

In this paper we have generalised a known theorem on the degree of approximation of its belonging to the class $Lip\{\alpha, q\}$ with a new method by using non – negative and non – increasing sequence,on the degree of approximation of function belonging to the class $Lip\{\psi(t), p\}$ for $p > 1$with period 2π .

[1] DEFINITIONS AND NOTATIONS

Let $f(x)$ be a 2π - periodic function integrable $L^p\ (p>1)$ and

Let
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$f(x) \equiv \sum_{n=0}^{\infty} A_n(x) \tag{1.1}$$

Be its fourier series

We say that $f(x) \in Lip(\psi(t), p)$, $p>1$

If

$$\left\{\int_0^{2\pi} |f(x+t) - f(x)|^p\, dt\right\}^{\frac{1}{p}} = o(\psi(t)) \tag{1.2}$$

Where $\psi(t)$ is a positive increasing function

We define the norm $\|f\|_p$ as

$$\|f\|_p = \left\{\int_0^{2\pi} |f(x)|^p\, dx\right\}^{\frac{1}{p}} \qquad p \geq 1 \tag{1.3}$$

And let the degree of approximation $E_n(f)$ is given by

$$E_n(f) = \underbrace{\min}_{T_n} \|f - T_n\|_p \tag{1.4}$$

Where $T_n(x)$ is some n^{th} degree trigonometrical polynomial.

[2] Definition (LORENTZ [2]). A sequence $\{S_n(x)\}$ is said to be almost convergent to a limit S, if

$$\lim_{n\to\infty} \frac{1}{(n+1)} \sum_{K=\rho}^{n+\rho} S_k = S$$

Uniformly with respect to k.

An almost convergrnce is ageneralization of cordinary convergence, recently SHARMA DIXIT AND SHUKLA [7], have defined almost Borel summability.

[3] Definition(SHARMA AND QURESHI [6]). A series $\sum u_n$ with the sequence of partial sums $\{S_n\}$ is said to be almost Riesz summableto S, provided

$$t_{n,p} = \frac{1}{p_n} \sum_{k=0}^{n} p_k \, s_{k,p} \to s, \quad as \quad n \to \infty,$$

Uniformly with respect to p, where

$$s_{k,p} = \frac{1}{(k+1)} \sum_{\mu=p}^{k+p} S_\mu$$

And $\{p_n\}$ be a sequence of non-negative constants, such that $p_g > 0$ and

$$P_n = p_0 + p_1 + p_2 + p_3 + \dots\dots\dots + p_{n.}$$

[4] Definition (QURESHI [3]). If $\langle a_{n,k} \rangle \langle n = 0,1,2,3,\dots..k = 0,1,2,\dots\dots..n \rangle$, $a_{n,0} = 1$ be a triangular matrix with real or complex elements , then a series $\sum_{n=0}^{\infty} u_n$ with the sequence of partial sums $\{S_n\}$ is said to be almost triangular matrix summable to S, provided

$$\sigma_{n,p} = \sum_{k=0}^{n} a_{n,k} \, s_{k,p} \to s \ as \ n \to \infty$$

Uniformly with respect to p.

INTRODUCTION

[5] In 1972 SAHNEY, GOPALAND RAO [5] proved the following theorem A and KHAN [1] proved theorem B.

THEOREM A : If $f(x)$ is a periodic and belonging to the class $Lip(\alpha, p)$ $0 \le \alpha \le 1$, and the non- negative and non-increasing generating sequence $\{p_n\}$ be defined as

$$P_n = P_{\langle n\rangle} = p_0 + p_1 + p_2 + p_3 + \dots\dots\dots p_n \to \infty \text{ as } n \to \infty$$

And if

$$\left(\int_1^n \frac{(P(y))^q}{y^{2\alpha+2-q}}\,dy\right)^{\frac{1}{q}} = o\left(\frac{P_{\langle n\rangle}}{n^{\alpha+\frac{1}{q}-1}}\right)$$

Then

$$E_n(f) = \|f - N_n\|_p = o\left(\frac{1}{n^{\alpha-\frac{1}{p}}}\right)$$

Where $N_n(x)$ is the (N, p_n) - mean of (1.1).

<u>THEOREM B</u> : If $f(x)$ is a periodic function and belongs to the class $Lip(\alpha, p)$ for $o < \alpha < 1$ and if $R(y)/y^{\alpha}$ is non- decreasing then

$$E_n(f) = \min\left\|f - t_n^{p,q}\right\|_p = o\left(\frac{1}{n^{\alpha-\frac{1}{p}}}\right)$$

Where $\{p_n\}$ *and* $\{q_n\}$ be non - negative and non – increasing generating sequence defined for the generalized *Nörlund* (N, p_n, q_n) method such that

$$P_n = p_0 + p_1 + p_2 + p_3 + \dots\dots\dots p_n \to \infty \text{ as } n \to \infty$$
$$Q_n = q_0 + q_1 + q_2 + q_3 + \dots\dots\dots q_n \to \infty \text{ as } n \to \infty$$
$$R_n = p_0 q_n + p_1 q_{n-1} + p_2 q_{n-2} + p_3 q_{n-3} + \dots\dots\dots p_n q_0 \to \infty \text{ as } n \to \infty$$

And $t_n^{p,q}$ is the generalized *Nörlund* mean of (1.1). QURESHI [4] proved the following theorem.

<u>THEOREM C</u> : If $\{a_{n,k}\}_{k=0}^{n}$ non - negative and non – increasing sequence with respect to k, then the degree of approximation of a periodic function f with period 2π and belonging to the class $Lip\,\alpha$ by almost triangular matrix means is given by

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$$\underbrace{\max}_{0\le x\le 2\pi}\left|f(x)-\sigma_{n,p}(x)\right| = \begin{cases} o\left\{\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\left(\frac{1}{n}\right)^{\alpha-1}\right\}, & 0<\alpha<1 \\ o\left\{\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\log n\right\}, \quad for & \alpha=1 \end{cases}$$

Where $\sigma_{n,p}(x)$ is the almost triangular matrix means of the partial sum (1.1).

In 1984 QURESHI [3] prove the following theorem.

THEOREM D : If $\{a_{n,k}\}_{k=0}^{n}$ is a non - negative and non – increasing sequence with respect to k, then the degree of approximation of a periodic function f with period 2π and belonging to the class $Lip(\alpha,q)$, for $0<\alpha\le 1$, $q>1$, then

$$\left\|\sigma_{n,p}-f\right\|=0\left\{\left(\frac{1}{n}\right)^{\alpha-1-\frac{1}{q}}\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\right\}$$

Where $\sigma_{n,p}$ is the almost triangular matrix means of the partial sums of (1.1).

MAIN RESULS

We have generalized the above theorem in the following form :

[6] THEOREM : If $\{a_{n,k}\}_{k=0}^{n}$ is a non - negative and non – increasing sequence with respect to k, a function $f(x)$ with period - 2π and belongs to the class $Lip(\psi(t),p)$, for $p>1$, then

$$\left\|\sigma_{n,p}-f(x)\right\|=0\left\{\psi\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^{-1-\frac{1}{p}}\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\right\}$$

Where $\psi(t)$ is a positive increasing function and follows :

$$\left\{\int_0^{\pi/n}\left(\frac{\psi(t)}{t^{1/p}}\right)dt\right\}^{\frac{1}{p}} = 0\left(\psi\left(\frac{1}{n}\right)\right) \qquad (6.1)$$

$$\left\{\int_{\pi/n}^{\pi}\left(\frac{\psi(t)}{t^{1/p+1}}\right)^p dt\right\}^{\frac{1}{p}} = 0\left(\psi\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^{-1}\right) \tag{6.2}$$

PROOF OF THE THEOREM

[7] Following [7] we write –

$$s_{k,p}(x)-f(x)=\frac{1}{2\pi(k+1)}\int_0^{\pi}\phi(t)\frac{\left[\cos pt-\cos(k+p+1)\right]}{\sin^2\frac{t}{2}}dt$$

Where $\phi(t)=f(x+t)+f(x-t)-2f(x)$

We have

$$\sigma_{n,p}-f(x)=\sum_{k=0}^{n}a_{n,k}\left\{s_{k,p}(x)-f(x)\right\}$$

$$=\frac{1}{2\pi}\int_0^{\pi}\phi(t)\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\frac{2\sin(k+2p+1)\frac{t}{2}\sin(k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}dt$$

$$=\frac{4}{\pi}\left[\int_0^{\frac{\pi}{n}}+\int_{\frac{\pi}{n}}^{\pi}\right]\frac{\phi(t)}{t^2}\sum_{k=0}^{n}\frac{a_{n.k}}{k+1}\sin(k+2p+1)\frac{t}{2}\sin(k+1)\frac{t}{2}dt+o(1)$$

$$=I_1+I_2+O(1) \qquad say$$

Now

$$I_1=\frac{4}{\pi}\int_0^{\frac{\pi}{n}}\frac{\phi(t)}{t^2}\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\sin(k+2p+1)\frac{t}{2}\times\sin(k+1)\frac{t}{2}\times dt$$

Applying *Hölder*'s inequality and the fact that –

$$\phi(t)\in Lip(\psi(t),p)$$

we have

$$I_1 \le \frac{4}{\pi}\left\{\int_0^{\frac{\pi}{n}} |\phi(t)|^p\, dt\right\}^{\frac{1}{p}} \left\{\int_0^{\frac{\pi}{n}} \left|\frac{1}{t^2}\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\sin(k+2p+1)\frac{t}{2}\times\sin(k+1)\frac{t}{2}\right|^q dt\right\}^{\frac{1}{q}}$$

$$= o(1)\left\{\int_0^{\frac{\pi}{n}} \left(\frac{\psi(t)}{t^{1/p}}\right)^p dt\right\}^{\frac{1}{p}} \left\{\int_0^{\frac{\pi}{n}} \left|\frac{1}{t^2}\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\sin(k+2p+1)\frac{t}{2}\times\sin(k+1)\frac{t}{2}\right|^q dt\right\}^{\frac{1}{q}}$$

$$= o\psi\left(\frac{1}{n}\right) o\left\{\sum_{k=0}^{n} a_{n,k}\left(\int_0^{\frac{\pi}{n}}\left(\frac{1}{t}\right)^q dt\right)\right\}^{\frac{1}{q}}$$

$$= o\psi\left(\frac{1}{n}\right) o\left\{\sum_{k=0}^{n} a_{n,k}\left(\frac{1}{n}\right)\right\}^{\frac{1}{p}}$$

$$= o\left\{\psi\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^{-\frac{1}{p}}\sum_{k=0}^{n} a_{n,k}\right\}$$

$$= o\left\{\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\psi\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^{-1-\frac{1}{p}}\right\}$$

Where $\frac{1}{p}+\frac{1}{q}=1$, such that , $1 \le q \le \infty$, since

$$\sum_{k=0}^{n}\psi\left(\frac{1}{n}\right)a_{n,k}\frac{1}{n}\left(\frac{1}{n}\right)^{-1-\frac{1}{P}} < \sum_{k=0}^{n}\psi\left(\frac{1}{n}\right)\frac{a_{n,k}}{k+1}\left(\frac{1}{n}\right)^{-1-\frac{1}{p}}$$

We have

$$I_1 = O\left(\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\left(\psi\left(\frac{1}{n}\right)\right)\left(\frac{1}{n}\right)^{-1-\frac{1}{p}}\right)$$

Similarly,

$$I_2 = \frac{4}{\pi}\int_{\frac{\pi}{n}}^{\pi}\frac{\phi(t)}{t^2}\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\sin(k+2p+1)\frac{t}{2}\times\sin(k+1)\frac{t}{2}\times dt$$

Similarly as above, we have

$$I_2 = o\left[\left\{\int_{\frac{\pi}{n}}^{\pi}\left|\frac{\phi(t)}{t}\right|^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{\pi}{n}}^{\pi}\left|t\sum_{k=0}^{n}a_{n,k}\frac{\sin(k+2p+1)\frac{t}{2}\times\sin(k+1)\frac{t}{2}}{t^2}\right|^q dt\right\}^{\frac{1}{q}}\right]$$

$$= o(1)\left[\left\{\int_{\frac{\pi}{n}}^{\pi}\left(\frac{\psi(t)}{t^{1/p+1}}\right)^p dt\right\}^{\frac{1}{p}}\left\{\left|\frac{1}{t}\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\right|^q\right\}^{\frac{1}{q}}\right]$$

$$= o\left(\psi\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^{-1}\right)o\left\{\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\left(\int_{\frac{\pi}{n}}^{\pi}\frac{1}{t^q}dt\right)^{\frac{1}{q}}\right\}$$

$$= o\left(\psi\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^{-1}\right)o\left\{\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\left(\int_{1}^{n}\frac{1}{y^{2-q}}dy\right)^{\frac{1}{q}}\right\}$$

$$= o\left(\psi\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^{-1}\right)o\left\{\left(\frac{1}{n}\right)^{-\frac{1}{p}}\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\right\}$$

$$= o\left(\psi\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^{-1-\frac{1}{p}}\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\right)$$

Hence

$$\left|a_{n,p}(x) - f(x)\right| = o\left\{\psi\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^{-1-\frac{1}{p}}\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\right\}$$

Uniformally for x therefore –

$$\left\|a_{n,p} - f\right\| = \underbrace{\sup}_{0\le x\le 2\pi}\left|a_{n,p}(x) - f(x)\right|$$

$$= o\left\{\psi\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^{-1-\frac{1}{p}}\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\right\}$$

THIS IS COMLETE PROOF OF OUR THEOREMS

REFERENCES

[1]. KHAN, KUZOOR. H. "On the degree of approximation of function belonging is the class $Lip(\alpha, q)$"Ind. Jaur. Pure and appl. Math. 5(1974) No. 2, P. 132.

[2]. LORENTZ, G.G. "Acontribution to the theory of divergent series" Acta Math. 80(1948) 167-198.

[3]. QURESHI, K. "On the degree of approximation of a function belonging to the class $Lip(\alpha, q)$"Tankang J. Math. Vol.15 No. 1 (1984).

[4]. QURESHI, K. "On the degree of approximation at periodic function f by almost triangular matrix mean"communicated for publication.

[5]. SAHNEY, B. N., GOPAL, V. AND RAO, V. Error bands in the approximation of function Bull. Austral, Math. Soe. 6(1972) 16 – 18.

[6]. SHARMA, P.L. AND QURESHI, K. "On the degree of approximation at periodic function f by the almost Riesz means". Ranchi Univ. Math. J. Vol. 11(1980).

[7]. SHARMA, P.L., Dixit, S.S. AND SHUKLA, V. B. "An almost Borel Summability of fourier series".Tankang J. Math. Vol.1 (1972) 11-18.

[8]. SIDDQUI, A. H. DISSERTATION, Department of Mathematics, A.M.U. Aligarh, India (1967).