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NORM MINIMIZING ESTIMATION IN THE SET-INDEXED STOCHASTIC PROCESSES

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ABSTRACT

In this article, we present the norm minimizing estimation of a set indexed stochastic process $Y = \{Y_A: A \in \mathbf{A}\}$ (by a linear and a nonlinear function of another set indexed stochastic process$X = \{X_A: A \in \mathbf{A}\}$)of the future value Y_T in terms of limits of its past X_{A_n} and Y_{A_n}. In addition, we present the orthogonality principle. We prove with some assumptions that aset indexed norm minimizing estimation of Y is X if and only if $Y - aX, X$ are orthogonal, when $a = \frac{\langle X,Y \rangle_H}{\langle X,X \rangle_H}$.

KEYWORDS:

:Set indexed stochastic process, norm minimizing estimation

1. INTRODUCTION

In this article, we present the norm minimizing estimationmethodof a set indexed stochastic processby another set indexed stochastic process, when the set index **A**is a compact set collection on a topological space T. The choice of the collection A is critical: it must be sufficiently rich in order to generate the Borel sets of$T$, but small enough to ensure the existence of a continuous process defined on **A**.

We introduce a norm minimizing estimationof a set indexed stochastic process $Y = \{Y_A: A \in \mathbf{A}\}$in terms of another set indexed stochastic process$X = \{X_A: A \in \mathbf{A}\}$by a linear and a nonlinear function of$X$. We prove with some assumptions that a set indexed of Y by linear function of set indexed process is $aX + b$when $a = \frac{\langle X,Y \rangle_H}{\langle X,X \rangle_H}$ and $b = 0$, by nonlinear function of set indexed process is $\lim_{A \nearrow T} E[Y_A | X_A]$.

In addition, we present the orthogonality principle. We prove with some assumptions that aset indexed norm minimizing estimationof $Y$is$X$if and only if $Y - aX, X$ are orthogonal, when$a = \frac{\langle X,Y \rangle_H}{\langle X,X \rangle_H}$.

Preliminaries

In the study, processes areindexed by an indexing collection **A** (see [IvMe]) of compact subsets of a locally metric and separable space$T$. We use the definition of **A** and notation from [IvMe] and all this section comefrom there:

Let (T, τ) be a non-void sigma-compact connected topologicalspace. A nonempty class **A** of compact, connected subsets of $T$is called an indexed collection if it satisfies the following:

a. $\emptyset \in \mathbf{A}$. In addition, there is an increasing sequence (B_n) of sets in $\mathbf{A}$ s.t.$T = \bigcup_{n=1}^{\infty} B_n^{\circ}$.
b. $\mathbf{A}$is closed under arbitrary intersections and if $A, B \in \mathbf{A}$ are nonempty, then $A \cap B$ is nonempty. If (A_i) is an increasing sequence in $\mathbf{A}$and if there exists $n$such that$A_i \subseteq B_n$for every$i$, then$\overline{\bigcup_i A_i} \in \mathbf{A}$.
c. $\sigma(\mathbf{A}) = \mathbf{B}$ where$\mathbf{B}$ is the collection of Borel sets of$T$.

We will require other classes of sets generated by$\mathbf{A}$. The first is$\mathbf{A}(\mathbf{u})$, which is the class of finite unions of sets in$\mathbf{A}$. We note that $\mathbf{A}(\mathbf{u})$is itself a lattice with the partial order induced by set inclusion. Let $\mathbf{C}$consists of all the subsets of $T$of the form
$C = A \setminus B, A \in \mathbf{A}, B \in \mathbf{A}(\mathbf{u})$.
A set-indexed stochastic process $X = \{X_A : A \in \mathbf{A}\}$ is additive if ithas an (almost sure) additive extension to$\mathbf{C}$:$X_\emptyset = 0$and if$C, C_1, C_2 \in \mathbf{C}$ with $C = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$ then almost surely$X_C = X_{C_1} + X_{C_2}$. In particular, if $C \in \mathbf{C}$and $C = A \setminus \bigcup_{i=1}^{n} A_i$, $A, A_1, \ldots, A_n \in \mathbf{A}$ then almost surely
$X_C = X_A - \sum_{i=1}^{n} X_{A \cap A_i} + \sum_{i<j} X_{A \cap A_i \cap A_j} - \ldots + (-1)^n X_{A \cap \bigcap_{i=1}^{n} A_i}$.
We shall always assume that our stochastic processes are additive. We note that a process with an (almost sure) additive extension to$\mathbf{C}$ also has an (almost sure) additive extension to $\mathbf{C}(\mathbf{u})$.

Norm minimizing estimation in set indexed stochastic processes

Definition 1.
(a) Let$A = \{A_n\}$be an increasing sequence in$\mathbf{A}$. We write $A_n \uparrow T$ (or, in short notation$A \uparrow T$) if $A_n \neq T$for all $n$and$\overline{\bigcup_n A_n} = T$.
(b) We write $A \nearrow T$ if $A_n \uparrow T$ for all an increasing sequence$\{A_n\}$ in $T^{\uparrow} = \{\{A_n\} : A_n \uparrow T\}$

We introduce the estimation of a set indexed stochastic process $Y = \{Y_A : A \in \mathbf{A}\}$in terms of another set indexed stochastic process$X = \{X_A : A \in \mathbf{A}\}$. Throughout this analysis, the optimality criterion will be the minimization of the norm value of the estimation.Let $X = \{X_A : A \in \mathbf{A}\}$, $Y = \{Y_A : A \in \mathbf{A}\}$be a square-integrable set indexed stochastic processes. We define the inner product:

$$\langle X, Y \rangle = \exists \lim_{A \nearrow T} Cov(X_A, Y_A)$$

(In another words, for all $\{A_n\} \in T^{\uparrow}$the limits are existing and equal). Easy to see that

$$\|X\| = \sqrt{\langle X, X \rangle}$$

is a semi-norm.

We define the equivalence relation on square-integrable set indexed stochastic processes:

$$\|X - Y\| = 0 \qquad \Leftrightarrow \qquad X \approx Y$$

We denote the quotient setby H (In another words, $H = \{[X]_\approx : X \in L^2(A)\}$ then $[X]_\approx$is an equivalence class). Now, we can define an inner product and a norm on H:

$\langle X, Y \rangle_H = \langle X, Y \rangle$and$\|X\|_H = \sqrt{\langle X, X \rangle_H}$

for all $X, Y \in H$ $(X \in [X]_\approx, Y \in [Y]_\approx)$

Definition 2.
a. Let X, Y be a random variables with finite variance. We say that estimation of Y is X if $E[(Y - X)^2]$is minimal (see [Pa]).
b. Let $X, Y \in H$. We say that set indexed norm minimizing estimation of $Y$is X if $\|X - Y\|_H^2$is minimal.

Theorem 2. Let $X, Y \in H$ and $\lim_{A\nearrow T} E[X_A] = \lim_{A\nearrow T} E[Y_A] = 0$.

a. Set indexed norm minimizing estimation of Y by constant set indexed process $X = c$ when $c = 0$.
b. Set indexed norm minimizing estimation of Y by linear function of set indexed process is $aX + b$ when $a = \frac{\langle X,Y\rangle_H}{\langle X,X\rangle_H}$ and $b = 0$.
c. Set indexed norm minimizing estimation of Y by nonlinear function of set indexed process is $\lim_{A\nearrow T} E[Y_A|X_A]$.

Proof.

a. Define $g(c) = \|Y - c\|_H^2 = \lim_{A\nearrow T} E[Y_A - c]^2$. Clearly, $g(c)$ is minimum if $g'(c) = -2\lim_{A\nearrow T} E[Y_A - c] = 0$. Then $c = 0$ for $A \nearrow T$.
b. For a given a, set indexed norm minimizing estimation of $Y - aX$ is a constant set indexed process. Then from (a) we get:
$b = 0$.
Define

$$g(a) = \|Y - aX\|_H^2 = \lim_{A\nearrow T} E[Y_A - aX_A]^2 =$$

$= \langle Y,Y\rangle_H - 2a\langle X,Y\rangle_H + \langle X,X\rangle_H a^2$.
Clearly, $g(a)$ is minimum if $g'(a) = 0$.
Then
$a = \frac{\langle X,Y\rangle_H}{\langle X,X\rangle_H}$.
c. We must find the function $g(x)$ such that
$\|Y - g(X)\|_H^2 = \lim_{A\nearrow T} E[(Y_A - g(X_A))^2]$ is minimum.
$\lim_{A\nearrow T} E[(Y_A - g(X_A))^2] = \lim_{A\nearrow T} \iint_{\Re^2} [y - g(x)]^2 \, dF_{Y_A,X_A}(x,y)$.
But
$F_{Y_A,X_A}(x,y) = F_{X_A}(x) F_{Y_A|X_A}(y)$,
then

$$\|Y - g(X)\|_H^2 = \lim_{A\nearrow T} \int_{-\infty}^{\infty} dF_{X_A}(x) \int_{-\infty}^{\infty} [y - g(x)]^2 dF_{Y_A|X_A}(y)$$

The integrands above are positive. Hence $\|Y - g(X)\|_H^2$ is minimum if the inner integral is minimum for every x. Hence it is minimum if $g(x)$ is constant. Then from (a) we get:

$g(x) = \lim_{A\nearrow T} \int_{-\infty}^{\infty} y dF_{Y_A|X_A}(y) = \lim_{A\nearrow T} E[Y_A|x]$. □

Theorem 3. (The orthogonality principle) Let $X, Y \in H$ and $\lim_{A\nearrow T} E[Y_A] = \lim_{A\nearrow T} E[X_A] = 0$. $\|Y - aX\|_H^2$ is minimal if and only if $\langle Y - aX, X\rangle_H = 0$.
(In other words, set indexed norm minimizing estimation of Y by linear function of set indexed process is aX when $a = \frac{\langle X,Y\rangle_H}{\langle X,X\rangle_H}$ and $b = 0$ if and only if $Y - aX \perp X$).

(Note: Two random variables are called orthogonal if $E[XY] = 0$. We shall use the notation $X \perp Y$ to indicate that X, Y are orthogonal).
Proof.
Based on Theorem 2(b), set indexed norm minimizing estimation of Y by linear function of set indexed process is $aX + b$ when $a = \frac{\langle X,Y\rangle_H}{\langle X,X\rangle_H}$ and $b = 0$.
If we define

$$g(a) = \|Y - aX\|_H^2 = \lim_{A\nearrow T} E[Y_A - aX_A]^2$$

then

$$g(a) = \lim_{A \nearrow T} E[Y_A - aX_A]^2 = \lim_{A \nearrow T} E[Y_A^2] - 2a\lim_{A \nearrow T} E[X_A Y_A] + a^2 \lim_{A \nearrow T} E[X_A^2]$$

Clearly, $g(a)$is minimum if $g'(a) = 0$.
Then $\lim_{A \nearrow T} E[(Y_A - aX_A)X_A] = 0$. □

Theorem 4. Let$X, Y \in H$. If the random variables X_A, Y_A are Gaussian (or X, Y are Brownian motions [BoSa, Da, Du, Fr, ReYo]) for all$A \in \mathbf{A}$ and $\lim_{A \nearrow T} E[Y_A] = \lim_{A \nearrow T} E[X_A] = 0$then

Set indexed norm minimizing estimationof Y by linear function of set indexed process is equal to set indexed norm minimizing estimationof Y by nonlinear function of set indexed process.

(In other words,$\lim_{A \nearrow T} E[Y_A|X_A] = \frac{\langle X,Y \rangle_H}{\langle X,X \rangle_H} \lim_{A \nearrow T} E[X_A]$).

Proof.

The random variablesX_A, Y_A are normal for all $A \in \mathbf{A}$and $\lim_{A \nearrow T} E[Y_A] = \lim_{A \nearrow T} E[X_A] = 0$.

From Theorem 2(b) we get that, set indexed norm minimizing estimationof Y by linear function of set indexed process is $aX + b$when $a = \frac{\langle X,Y \rangle_H}{\langle X,X \rangle_H}$ and $b = 0$. Then $X_A, Y_A - aX_A$ are uncorrelated since$\langle Y - aX, X \rangle_H = 0$. But $X_A, Y_A$are normal then$X_A, Y_A - aX_A$are independent.

Then

$$\lim_{A \nearrow T} E[Y_A - aX_A|X_A] = \lim_{A \nearrow T} E[Y_A - aX_A] = \lim_{A \nearrow T} E[Y_A] - a\lim_{A \nearrow T} E[X_A] = 0$$

and on the other hand

$$\lim_{A \nearrow T} E[Y_A - aX_A|X_A] = \lim_{A \nearrow T} E[Y_A|X_A] - a\lim_{A \nearrow T} E[X_A|X_A] = \lim_{A \nearrow T} E[Y_A|X_A] - \lim_{A \nearrow T} X_A$$

and from that we get,

$\lim_{A \nearrow T} E[Y_A|X_A] = a\lim_{A \nearrow T} E[X_A]$ when $a = \frac{\langle X,Y \rangle_H}{\langle X,X \rangle_H}$. □

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