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RATIONAL INEQUALITIES AND COMMON FIXED POINT THEOREMS IN MULTIPLICATIVE METRIC SPACES

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ABSTRACT

This paper is in the sequence of papers to prove the common fixed point result using rational inequalities for four mappings, in which the pair of the maps are compatible and weak compatible, also the pairs of maps are assumed to satisfy the weak commutativity in the framework of multiplicative metric spaces.

KEYWORDS:

Complete multiplicative metric space, multiplicative contraction, weak commutative mappings, and commuting mappings

AMS Subject Classification. 47H10, 54H25

1. INTRODUCTION

Since Banach [1] proved the Banach contraction principle in 1922, the existence and uniqueness of fixed and common fixed point theorems of mappings have been of great interest. Many authors have generalised the Banach contraction principle in a variety of spaces over the years, including quasi-metric spaces, fuzzy metric spaces, 2-metric spaces, cone metric spaces, partial metric spaces, and generalised metric spaces (see, for example, ([2]-[19]) and the references therein). In 2008, Bashirov et al. [20] introduced the concept of multiplicative metric spaces, studied the concept of multiplicative calculus and proved the multiplicative calculus fundamental theorem. Florack and Assen [21] demonstrated the use of multiplicative calculus in biomedical image analysis in 2012. They calculated the multiplicative distance between two non negative real numbers and two positive square matrices using the multiplicative absolute value function. This is the foundation of multiplicative metric spaces. In 2012, zavşar and evikel [23] investigated multiplicative metric spaces by highlighting their topological properties, introduced the concept of multiplicative contraction mapping, and proved some fixed point theorems for multiplicative contraction mappings on multiplicative spaces. He et al. [24] recently established a set of common fixed point theorems for four self-mappings in multiplicative metric space. Abbas et al. [25] recently proved some common fixed point results of quasi-weak commutative mappings on a closed ball using multiplicative metric spaces. Simultaneously, they investigated the necessary conditions for the existence of a common solution to the multiplicative boundary value problem. Kang et al. [26] defined and proved some common fixed point theorems for compatible mappings and their variants in multiplicative metric spaces.

Now we present some definitions and results in multiplicative metric spaces that will be required in the following section.

Definition 1.1 [20] Let X be a non-empty set. A multiplicative metric is a mapping $d: X \times X \rightarrow R^{+}$ satisfying the following axioms:

(M1) $d(x,y) \geq 1$ for all $x, y \in X$ and $d(x,y)=1 \Leftrightarrow x=y$;

(M2) $d(x,y)=d(y,x)$ for all $x, y \in X$;

(M3) $d(x,y) \leq d(x,z).d(z,y)$ for all $x, y, z \in X$ (multiplicative triangle inequality).

The pair (X,d) is called a multiplicative metric space.

Proposition 1.2 [20] Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $x_n \rightarrow x \ (n \rightarrow \infty)$ if and only if $d(x_n, x) \rightarrow 1 \ (n \rightarrow \infty)$.

Definition 1.3 [20] Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is called multiplicative Cauchy sequence if, for each $\varepsilon > 0$, there exists a positive integer $\mathrm{N} \in N$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$.

Proposition 1.4 [20] Let (X,d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X .Then $\{x_n\}$ is a called multiplicative Cauchy sequence if and only if $d(x_n, x_m) \rightarrow 1 \ (n,m \rightarrow \infty)$.

Definition 1.5 [20] A multiplicative metric space (X,d) is said to be multiplicative complete if every multiplicative Cauchy sequence in (X,d) is multiplicative convergent in X.

Proposition 1.6 [20] Let (X,d_X) and (Y,d_Y) be two multiplicative metric spaces and $f: X \rightarrow Y$ be a mapping and $\{x_n\}$ be any sequence in X. Then f is multiplicative continuous at $x \in X$ if and only if $f(x_n) \rightarrow f(x)$ for every sequence $\{x_n\}$ with $x_n \rightarrow x \ (n \rightarrow \infty)$.

1. MAIN RESULTS

In this section, we discussed the unique common fixed point of two pairs of weak commutative mappings on a complete multiplicative metric space.Our results substantially generalize and extend the results of Özavsar and Cevikel [23] and the results studied by He et al.[24].

We start our work by introducing the following concepts.

Definition 2.1 [4] The self-maps f and g of a multiplicative metric space (X,d) are said to be compatible if

$\lim_{n\to\infty} d(fgx_n gfx_n)=1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$.

Definition 2.2[3] Suppose that f and g are two self-maps of a multiplicative metric space (X,d). The pair (f,g) are called weakly compatible mappings if $fx = gx, x \in X$ implies $fgx = gfx$. That is, $d(fx, gx)=1 \Rightarrow d(fgx, gfx)=1$.

Theorem 2.1 Let (X,d) be a complete multiplicative metric space. S, T, A and B be four mappings of X into itself. Suppose that there exists $\lambda \in (0,\frac{1}{2})$ such that $S(X) \subset B(X)$, $T(X) \subset A(X)$ and

$$d(Sx,Ty) \le \max\left(d^{\lambda}(Ax,By)\left[\frac{d^{\lambda}(Ax,Sx)+d^{\lambda}(Ty,Sx)}{d^{\lambda}(By,Ty)+d^{\lambda}(By,Ax)}\right], d^{\lambda}(Ax,Ty)\left[\frac{d^{\lambda}(Sx,Ax)+d^{\lambda}(By,Ty)}{d^{\lambda}(Sx,Ty)+d^{\lambda}(Ax,By)}\right],\right.$$

$$\left. d^{\lambda}(Ty,By)\, d^{\lambda}(Ty,Sx)\left[\frac{d^{\lambda}(Ax,Sx)+d^{\lambda}(Ax,By)}{d^{\lambda}(Sx,Ty)+d^{\lambda}(Ty,By)}\right]\right) \qquad (2.1)$$

for all x, y in X. Assume one of the following conditions is satisfied:

(a) either A or S is continuous, the pair (S,A) is compatible and the pair (T,B) is weakly compatible;

(b) either B or T is continuous, the pair (T,B) is compatible and the pair (S,A) is weakly compatible;

Then S, T, A and B have a unique common fixed point.

Proof: Let $x_0 \in X$, since $S(X) \subset B(X)$ and $T(X) \subset A(X)$, there exist x_1, $x_2 \in X$ such that $y_0 = Sx_0 = Bx_1$ and $y_1 = Tx_1 = Ax_2$. By induction, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Sx_{2n} = Bx_{2n+1}, \qquad y_{2n+1} = Tx_{2n+1} = Ax_{2n+2} \qquad (2.2)$$

for all $n = 0,1,2,....$

Next, we prove that $\{y_n\}$ is a multiplicative Cauchy sequence in X. In fact, $\forall\ n \in N$, from (2.1), (2.2), we have

$$d(y_{2n}, y_{2n+1}) = d(Sx_{2n}, Tx_{2n+1})$$

$$\le \max\left(d^{\lambda}(Ax_{2n}, Bx_{2n+1})\left[\frac{d^{\lambda}(Ax_{2n}, Sx_{2n})+d^{\lambda}(Tx_{2n+1}, Sx_{2n})}{d^{\lambda}(Bx_{2n+1}, Tx_{2n+1})+d^{\lambda}(Bx_{2n+1}, Ax_{2n})}\right],\right.$$

$$d^{\lambda}(Ax_{2n}, Tx_{2n+1})\left[\frac{d^{\lambda}(Sx_{2n}, Ax_{2n})+d^{\lambda}(Bx_{2n+1}, Tx_{2n+1})}{d^{\lambda}(Sx_{2n}, Tx_{2n+1})+d^{\lambda}(Ax_{2n}, Bx_{2n+1})}\right]$$

$$d^{\lambda}(Tx_{2n+1},Bx_{2n+1})d^{\lambda}(Tx_{2n+1},Sx_{2n})\left[\frac{d^{\lambda}(Ax_{2n},Sx_{2n})+d^{\lambda}(Ax_{2n},Bx_{2n+1})}{d^{\lambda}(Sx_{2n},Tx_{2n+1})+d^{\lambda}(Tx_{2n+1},Bx_{2n+1})}\right]$$

$$d(y_{2n},y_{2n+1})\le \max\left(d^{\lambda}(y_{2n-1},y_{2n})\left[\frac{d^{\lambda}(y_{2n-1},y_{2n})+d^{\lambda}(y_{2n+1},y_{2n})}{d^{\lambda}(y_{2n},y_{2n+1})+d^{\lambda}(y_{2n},y_{2n-1})}\right],\right.$$

$$d^{\lambda}(y_{2n-1},y_{2n+1})\left[\frac{d^{\lambda}(y_{2n},y_{2n-1})+d^{\lambda}(y_{2n},y_{2n+1})}{d^{\lambda}(y_{2n},y_{2n+1})+d^{\lambda}(y_{2n-1},y_{2n})}\right],$$

$$\left.d^{\lambda}(y_{2n+1},y_{2n})d^{\lambda}(y_{2n+1},y_{2n})\left[\frac{d^{\lambda}(y_{2n-1},y_{2n})+d^{\lambda}(y_{2n-1},y_{2n})}{d^{\lambda}(y_{2n},y_{2n+1})+d^{\lambda}(y_{2n+1},y_{2n})}\right]\right)$$

$$=\max\left(d^{\lambda}(y_{2n-1},y_{2n}),d^{\lambda}(y_{2n-1},y_{2n+1}),d^{\lambda}(y_{2n+1},y_{2n})d^{\lambda}(y_{2n-1},y_{2n})\right)$$

$$d(y_{2n},y_{2n+1})\le d^{\lambda}(y_{2n+1},y_{2n})d^{\lambda}(y_{2n-1},y_{2n}) \qquad \text{[Using M(3)]}$$

$$d(y_{2n},y_{2n+1})\le d^{\frac{\lambda}{1-\lambda}}(y_{2n-1},y_{2n})=d^{h}(y_{2n-1},y_{2n})$$

(2.3)

where $h=\dfrac{\lambda}{1-\lambda}$, $h\in(0,1)$ as $\lambda\in(0,\tfrac{1}{2})$

In the similar way, we have

$$d(y_{2n+1},y_{2n+2})=d(Tx_{2n+1},Sx_{2n+2})=d(Sx_{2n+2},Tx_{2n+1})$$

$$\le \max\left(d^{\lambda}(Ax_{2n+2},Bx_{2n+1})\left[\frac{d^{\lambda}(Ax_{2n+2},Sx_{2n+2})+d^{\lambda}(Tx_{2n+1},Sx_{2n+2})}{d^{\lambda}(Bx_{2n+1},Tx_{2n+1})+d^{\lambda}(Bx_{2n+1},Ax_{2n+2})}\right],\right.$$

$$d^{\lambda}(Ax_{2n+2},Tx_{2n+1})\left[\frac{d^{\lambda}(Sx_{2n+2},Ax_{2n+2})+d^{\lambda}(Bx_{2n+1},Tx_{2n+1})}{d^{\lambda}(Sx_{2n+2},Tx_{2n+1})+d^{\lambda}(Ax_{2n+2},Bx_{2n+1})}\right]$$

$$d^{\lambda}(Tx_{2n+1},Bx_{2n+1})d^{\lambda}(Tx_{2n+1},Sx_{2n+2})\left[\frac{d^{\lambda}(Ax_{2n+2},Sx_{2n+2})+d^{\lambda}(Ax_{2n+2},Bx_{2n+1})}{d^{\lambda}(Sx_{2n+2},Tx_{2n+1})+d^{\lambda}(Tx_{2n+1},Bx_{2n+1})}\right]$$

$$d(y_{2n+2},y_{2n+1})\le \max\left[(d^{\lambda}(y_{2n+1},y_{2n}))\left[\frac{d^{\lambda}(y_{2n+1},y_{2n+2})+d^{\lambda}(y_{2n+1},y_{2n+2})}{d^{\lambda}(y_{2n},y_{2n+1})+d^{\lambda}(y_{2n},y_{2n+1})}\right],\right.$$

$$(d^{\lambda}(y_{2n+1},y_{2n+1}))\left[\frac{d^{\lambda}(y_{2n+2},y_{2n+1})+d^{\lambda}(y_{2n},y_{2n+1})}{d^{\lambda}(y_{2n+2},y_{2n+1})+d^{\lambda}(y_{2n+1},y_{2n})}\right],$$

$$(d^{\lambda}(y_{2n+1},y_{2n})d^{\lambda}(y_{2n+1},y_{2n+2}))\left[\frac{d^{\lambda}(y_{2n+1},y_{2n+2})+d^{\lambda}(y_{2n+1},y_{2n})}{d^{\lambda}(y_{2n+2},y_{2n+1})+d^{\lambda}(y_{2n+1},y_{2n})}\right]$$

$$=\max\left(d^{\lambda}(y_{2n+1},y_{2n+2}),1,d^{\lambda}(y_{2n},y_{2n+1})d^{\lambda}(y_{2n+1},y_{2n+2})\right)$$

$$=d^{\lambda}(y_{2n},y_{2n+1})d^{\lambda}(y_{2n+1},y_{2n+2})$$

$$d(y_{2n+1}, y_{2n+2}) \le d^{\frac{\lambda}{1-\lambda}}(y_{2n}, y_{2n+1}) = d^{h}(y_{2n}, y_{2n+1}) \quad (2.4)$$

where $h = \dfrac{\lambda}{1-\lambda}$, $h \in (0,1)$ as $\lambda \in (0, \frac{1}{2})$

It follows from (2.3) and (2.4) that for all $n \in N$, we have

$$d(y_n, y_{n+1}) \le d^{h}(y_{n-1}, y_n) \le d^{h^2}(y_{n-2}, y_{n-1}) \le ...... \le d^{h^n}(y_0, y_1)$$

Therefore, for all $n, m \in N$, $n < m$, by the multiplicative triangle inequality, we obtain

$$d(y_n, y_m) \le d(y_n, y_{n+1}).d(y_{n+1}, y_{n+2})..........d(y_{m-1}, y_m)$$

$$\le d^{h^n}(y_0, y_1).d^{h^{n+1}}(y_0, y_1)..........d^{h^{m-1}}(y_0, y_1)$$

$$\le d^{\frac{h^n}{1-h}}(y_0, y_1)$$

This implies that $d(y_n, y_m) \to 1\ (n, m \to \infty)$. Hence $\{y_n\}$ is a multiplicative cauchy sequence in X, By the completeness of X, there exists $z \in X$ such that $y_n \to z\ (n \to \infty)$.

Moreover, because

$\{y_{2n}\} = \{Sx_{2n}\} = \{Bx_{2n+1}\}$ and $\{y_{2n+1}\} = \{Tx_{2n+1}\} = \{Ax_{2n+2}\}$

are sub sequences of $\{y_n\}$, we obtain

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Ax_{2n+2} = z \quad (2.5)$$

Next we prove that z is a common fixed point of S, T, A and B under the condition (a).

Case 1. Suppose that A is a continuous, then $\lim_{n\to\infty} ASx_{2n} = \lim_{n\to\infty} A^2x_{2n} = Az$. Since the pair (S, A) is compatible, from (2.5) we have

$$\lim_{n\to\infty} d(SAx_{2n}, ASx_{2n}) = \lim_{n\to\infty} d(SAx_{2n}, Az) = 1$$

That is, $\lim_{n\to\infty} SAx_{2n} = Az \quad (2.6)$

By using (2.1), we have

$$d(SAx_{2n}, Tx_{2n+1}) \le \max\left(d^{\lambda}(A^2x_{2n}, Bx_{2n+1}) \left[\frac{d^{\lambda}(A^2x_{2n}, SAx_{2n}) + d^{\lambda}(Tx_{2n+1}, SAx_{2n})}{d^{\lambda}(Bx_{2n+1}, Tx_{2n+1}) + d^{\lambda}(Bx_{2n+1}, A^2x_{2n})}\right],\right.$$

$$d^{\lambda}(A^2x_{2n}, Tx_{2n+1}) \left[\frac{d^{\lambda}(SAx_{2n}, A^2x_{2n}) + d^{\lambda}(Bx_{2n+1}, Tx_{2n+1})}{d^{\lambda}(SAx_{2n}, Tx_{2n+1}) + d^{\lambda}(A^2x_{2n}, Bx_{2n+1})}\right],$$

$$\left. d^{\lambda}(Tx_{2n+1}, Bx_{2n+1}) d^{\lambda}(Tx_{2n+1}, SAx_{2n}) \left[\frac{d^{\lambda}(A^2x_{2n}, SAx_{2n}) + d^{\lambda}(A^2x_{2n}, Bx_{2n+1})}{d^{\lambda}(SAx_{2n}, Tx_{2n+1}) + d^{\lambda}(Tx_{2n+1}, Bx_{2n+1})}\right]\right)$$

Taking $n \to \infty$ on the two sides of the above inequality, using (2.5) and (2.6), we get

$$d(Az,z)\le \max\left(d^{\lambda}(Az,z)\left[\frac{d^{\lambda}(Az,Az)+d^{\lambda}(z,Az)}{d^{\lambda}(z,z)+d^{\lambda}(z,Az)}\right], d^{\lambda}(Az,z)\left[\frac{d^{\lambda}(Az,Az)+d^{\lambda}(z,z)}{d^{\lambda}(Az,z)+d^{\lambda}(Az,z)}\right],\right.$$

$$\left. d^{\lambda}(z,z)d^{\lambda}(z,Az)\left[\frac{d^{\lambda}(Az,Az)+d^{\lambda}(Az,z)}{d^{\lambda}(Az,z)+d^{\lambda}(z,z)}\right]\right)$$

$$=\max\left(d^{\lambda}(Az,z),1,d^{\lambda}(z,Az)\right)$$

$$\le d^{\lambda}(Az,z)$$

$$d(Az,z)\le d^{\lambda}(Az,z)$$

This means that $d(Az,z)=1$ since $\lambda\in\left(0,\tfrac{1}{2}\right)$, that is, $Az=z$ (2.7)

Now, we will show that $Sz=z$

Again applying (2.1), we obtain

$$d(Sz,Tx_{2n+1})\le \max\left(d^{\lambda}(Az,Bx_{2n+1})\left[\frac{d^{\lambda}(Az,Sz)+d^{\lambda}(Tx_{2n+1},Sz)}{d^{\lambda}(Bx_{2n+1},Tx_{2n+1})+d^{\lambda}(Bx_{2n+1},Az)}\right],\right.$$

$$d^{\lambda}(Az,Tx_{2n+1})\left[\frac{d^{\lambda}(Sz,Az)+d^{\lambda}(Bx_{2n+1},Tx_{2n+1})}{d^{\lambda}(Sz,Tx_{2n+1})+d^{\lambda}(Az,Bx_{2n+1})}\right],$$

$$\left. d^{\lambda}(Tx_{2n+1},Bx_{2n+1})d^{\lambda}(Tx_{2n+1},Sz)\left[\frac{d^{\lambda}(Az,Sz)+d^{\lambda}(Az,Bx_{2n+1})}{d^{\lambda}(Sz,Tx_{2n+1})+d^{\lambda}(Tx_{2n+1},Bx_{2n+1})}\right]\right)$$

Taking limit $n\to\infty$ on both sides in the above inequality, using (2.5) and (2.7), we get

$$d(Sz,z)\le \max\left(d^{\lambda}(z,z)\left[\frac{d^{\lambda}(z,Sz)+d^{\lambda}(z,Sz)}{d^{\lambda}(z,z)+d^{\lambda}(z,z)}\right], d^{\lambda}(z,z)\left[\frac{d^{\lambda}(Sz,z)+d^{\lambda}(z,z)}{d^{\lambda}(Sz,z)+d^{\lambda}(z,z)}\right],\right.$$

$$\left. d^{\lambda}(z,z)d^{\lambda}(z,Sz)\left[\frac{d^{\lambda}(z,Sz)+d^{\lambda}(z,z)}{d^{\lambda}(Sz,z)+d^{\lambda}(z,z)}\right]\right)$$

$$=\max\left\{d^{\lambda}(Sz,z),1,d^{\lambda}(z,Sz)\right\}$$

$$\le d^{\lambda}(z,Sz)$$

This implies that $d(Sz,z)=1$ since $\lambda\in\left(0,\tfrac{1}{2}\right)$, that is $Sz=z$ (2.8)

On the other hand, since $z\in SX\subset BX$, there exists $z^{*}\in X$ such that $z=Az=Sz=Bz^{*}$.

By using (2.1) and $z=Az=Sz=Bz^{*}$, we obtain

$$d(z,Tz^{*})=d(Sz,Tz^{*})\le \max\left(d^{\lambda}(Az,Bz^{*})\left[\frac{d^{\lambda}(Az,Sz)+d^{\lambda}(Tz^{*},Sz)}{d^{\lambda}(Bz^{*},Tz^{*})+d^{\lambda}(Bz^{*},Az)}\right],\right.$$

$$d^{\lambda}(Az,Tz^{*})\left[\frac{d^{\lambda}(Sz,Az)+d^{\lambda}(Bz^{*},Tz^{*})}{d^{\lambda}(Sz,Tz^{*})+d^{\lambda}(Az,Bz^{*})}\right],$$

$$d^{\lambda}\left(Tz^{*},Bz^{*}\right)d^{\lambda}\left(Tz^{*},Sz\right)\left[\frac{d^{\lambda}\left(Az,Sz\right)+d^{\lambda}\left(Az,Bz^{*}\right)}{d^{\lambda}\left(Sz,Tz^{*}\right)+d^{\lambda}\left(Tz^{*},Bz^{*}\right)}\right]\Bigg)$$

$$=\max\Bigg(d^{\lambda}(z,z)\left[\frac{d^{\lambda}(z,z)+d^{\lambda}\left(Tz^{*},z\right)}{d^{\lambda}\left(z,Tz^{*}\right)+d^{\lambda}(z,z)}\right],$$

$$d^{\lambda}\left(z,Tz^{*}\right)\left[\frac{d^{\lambda}(z,z)+d^{\lambda}\left(z,Tz^{*}\right)}{d^{\lambda}\left(z,Tz^{*}\right)+d^{\lambda}(z,z)}\right],$$

$$d^{\lambda}\left(Tz^{*},z\right)d^{\lambda}\left(Tz^{*},z\right)\left[\frac{d^{\lambda}(z,z)+d^{\lambda}(z,z)}{d^{\lambda}\left(z,Tz^{*}\right)+d^{\lambda}\left(Tz^{*},z\right)}\right]\Bigg)$$

$$=\max\left(1,d\left(z,Tz^{*}\right),d\left(Tz^{*},z\right)\right)$$

$$\leq d^{\lambda}\left(z,Tz^{*}\right)$$

This implies that $d\left(z,Tz^{*}\right)=1$, and so $Tz^{*}=z=Bz^{*}$. Since the pair (T,B) is weakly compatible, we have $Tz=TBz^{*}=BTz^{*}=Bz$ and $Az=Sz=z$ (2.9)

Now, we prove that $Tz=z$. From (2.1) and (2.9), we have

$$d(z,Tz)=d(Sz,Tz)$$

$$\leq\max\Bigg(d^{\lambda}(Az,Bz)\left[\frac{d^{\lambda}(Az,Sz)+d^{\lambda}(Tz,Sz)}{d^{\lambda}(Bz,Tz)+d^{\lambda}(Bz,Az)}\right],d^{\lambda}(Az,Tz)\left[\frac{d^{\lambda}(Sz,Az)+d^{\lambda}(Bz,Tz)}{d^{\lambda}(Sz,Tz)+d^{\lambda}(Az,Bz)}\right],$$

$$d^{\lambda}(Tz,Bz)d^{\lambda}(Tz,Sz)\left[\frac{d^{\lambda}(Az,Sz)+d^{\lambda}(Az,Bz)}{d^{\lambda}(Sz,Tz)+d^{\lambda}(Tz,Bz)}\right]\Bigg)$$

$$\leq\max\Bigg(d^{\lambda}(z,Tz)\left[\frac{d^{\lambda}(z,z)+d^{\lambda}(Tz,z)}{d^{\lambda}(Tz,Tz)+d^{\lambda}(Tz,z)}\right],d^{\lambda}(z,Tz)\left[\frac{d^{\lambda}(z,z)+d^{\lambda}(Tz,Tz)}{d^{\lambda}(z,Tz)+d^{\lambda}(z,Tz)}\right],$$

$$d^{\lambda}(Tz,Tz)d^{\lambda}(Tz,z)\left[\frac{d^{\lambda}(z,z)+d^{\lambda}(z,Tz)}{d^{\lambda}(z,Tz)+d^{\lambda}(Tz,Tz)}\right]\Bigg)$$

$$=\max\left(d^{\lambda}(z,Tz),1,d^{\lambda}(Tz,z)\right)$$

$$\leq d^{\lambda}(z,Tz)$$

This implies that $d(z,Tz)=1$, that is $Tz=z$

Hence, $z=Sz=Az=Tz=Bz$

Therefore, we obtain $z=Sz=Az=Tz=Bz$, so z is a common fixed point of S, T, A and B.

Case 2. Suppose that S is continuous, then $\lim_{n\to\infty}SAx_{2n}=\lim_{n\to\infty}S^{2}x_{2n}=Sz$. Since the pair (S,A) is compatible, from (2.5) we have

$$\lim_{n\to\infty} d(SAx_{2n}, ASx_{2n}) = \lim_{n\to\infty} d(Sz, ASx_{2n}) = 1$$

That is, $\lim_{n\to\infty} ASx_{2n} = Sz$ (2.10)

By using (2.1), we have

$$d\left(S^2x_{2n}, Tx_{2n+1}\right) \le \max\left(d^{\lambda}\left(ASx_{2n}, Bx_{2n+1}\right)\left[\frac{d^{\lambda}\left(ASx_{2n}, S^2x_{2n}\right)+d^{\lambda}\left(Tx_{2n+1}, S^2x_{2n}\right)}{d^{\lambda}\left(Bx_{2n+1}, Tx_{2n+1}\right)+d^{\lambda}\left(Bx_{2n+1}, ASx_{2n}\right)}\right],\right.$$

$$d^{\lambda}\left(ASx_{2n}, Tx_{2n+1}\right)\left[\frac{d^{\lambda}\left(S^2x_{2n}, ASx_{2n}\right)+d^{\lambda}\left(Bx_{2n+1}, Tx_{2n+1}\right)}{d^{\lambda}\left(S^2x_{2n}, Tx_{2n+1}\right)+d^{\lambda}\left(ASx_{2n}, Bx_{2n+1}\right)}\right],$$

$$\left. d^{\lambda}\left(Tx_{2n+1}, Bx_{2n+1}\right)d^{\lambda}\left(Tx_{2n+1}, S^2x_{2n}\right)\left[\frac{d^{\lambda}\left(ASx_{2n}, S^2x_{2n}\right)+d^{\lambda}\left(ASx_{2n}, Bx_{2n+1}\right)}{d^{\lambda}\left(S^2x_{2n}, Tx_{2n+1}\right)+d^{\lambda}\left(Tx_{2n+1}, Bx_{2n+1}\right)}\right]\right)$$

Taking limit $n \to \infty$ on both sides in the above inequality, using (2.5) and (2.10), we get

$$d(Sz, z) \le \max\left(d^{\lambda}(Sz, z)\left[\frac{d^{\lambda}(Sz, Sz)+d^{\lambda}(z, Sz)}{d^{\lambda}(z, z)+d^{\lambda}(z, Sz)}\right], d^{\lambda}(Sz, z)\left[\frac{d^{\lambda}(Sz, Sz)+d^{\lambda}(z, z)}{d^{\lambda}(Sz, z)+d^{\lambda}(Sz, z)}\right],\right.$$

$$\left. d^{\lambda}(z, z)d^{\lambda}(z, Sz)\left[\frac{d^{\lambda}(Sz, Sz)+d^{\lambda}(Sz, z)}{d^{\lambda}(Sz, z)+d^{\lambda}(z, z)}\right]\right)$$

$$= \max\left(d^{\lambda}(Sz, z), 1, d^{\lambda}(z, Sz)\right)$$

$$d(Sz, z) \le d^{\lambda}(Sz, z)$$

This means that $d(Sz, z) = 1$, that is, $Sz = z$ (2.11)

Since $z = Sz \in SX \subset BX$, there exists $z^* \in X$ such that $z = Sz = Bz^*$ (2.12)

Using (2.1) and (2.12), we have

$$d\left(z, Tz^*\right) = d\left(Sz, Tz^*\right)$$

$$\le \max\left(d^{\lambda}\left(Az, Bz^*\right)\left[\frac{d^{\lambda}(Az, Sz)+d^{\lambda}\left(Tz^*, Sz\right)}{d^{\lambda}\left(Bz^*, Tz^*\right)+d^{\lambda}\left(Bz^*, Az\right)}\right], d^{\lambda}\left(Az, Tz^*\right)\left[\frac{d^{\lambda}(Sz, Az)+d^{\lambda}\left(Bz^*, Tz^*\right)}{d^{\lambda}\left(Sz, Tz^*\right)+d^{\lambda}\left(Az, Bz^*\right)}\right],\right.$$

$$\left. d^{\lambda}\left(Tz^*, Bz^*\right)d^{\lambda}\left(Tz^*, Sz\right)\left[\frac{d^{\lambda}(Az, Sz)+d^{\lambda}\left(Az, Bz^*\right)}{d^{\lambda}\left(Sz, Tz^*\right)+d^{\lambda}\left(Tz^*, Bz^*\right)}\right]\right)$$

$$= \max\left(d^{\lambda}(z, z)\left[\frac{d^{\lambda}(z, z)+d^{\lambda}\left(Tz^*, z\right)}{d^{\lambda}\left(z, Tz^*\right)+d^{\lambda}(z, z)}\right], d^{\lambda}\left(z, Tz^*\right)\left[\frac{d^{\lambda}(z, z)+d^{\lambda}\left(z, Tz^*\right)}{d^{\lambda}\left(z, Tz^*\right)+d^{\lambda}(z, z)}\right],\right.$$

$$\left. d^{\lambda}\left(Tz^*, z\right)d^{\lambda}\left(Tz^*, z\right)\left[\frac{d^{\lambda}(z, z)+d^{\lambda}(z, z)}{d^{\lambda}\left(z, Tz^*\right)+d^{\lambda}\left(Tz^*, z\right)}\right]\right)$$

$$= \max\left(1, d^{\lambda}\left(z, Tz^{*}\right), d\left(z, Tz^{*}\right)\right)$$

$$\leq d\left(z, Tz^{*}\right)$$

This implies that $d\left(z, Tz^{*}\right) = 1$ and so $Tz^{*} = z = Bz^{*}$. Since the pair (T, B) is weakly compatible, we obtain $Tz = TBz^{*} = BTz^{*} = Bz$ and $Sz = Bz = Tz^{*} = z$ (2.13)

Now, we prove that $Tz = z$. Using (2.1), we have

$$d\left(Sx_{2n}, Tz\right) \leq \max\left(d^{\lambda}\left(Ax_{2n}, Bz\right)\left[\frac{d^{\lambda}\left(Ax_{2n}, Sx_{2n}\right) + d^{\lambda}\left(Tz, Sx_{2n}\right)}{d^{\lambda}\left(Bz, Tz\right) + d^{\lambda}\left(Bz, Ax_{2n}\right)}\right],\right.$$

$$d^{\lambda}\left(Ax_{2n}, Tz\right)\left[\frac{d^{\lambda}\left(Sx_{2n}, Ax_{2n}\right) + d^{\lambda}\left(Bz, Tz\right)}{d^{\lambda}\left(Sx_{2n}, Tz\right) + d^{\lambda}\left(Ax_{2n}, Bz\right)}\right],$$

$$\left. d^{\lambda}\left(Tz, Bz\right) d^{\lambda}\left(Tz, Sx_{2n}\right)\left[\frac{d^{\lambda}\left(Ax_{2n}, Sx_{2n}\right) + d^{\lambda}\left(Ax_{2n}, Bz\right)}{d^{\lambda}\left(Sx_{2n}, Tz\right) + d^{\lambda}\left(Tz, Bz\right)}\right]\right)$$

Taking limit $n \to \infty$ on both sides in the above inequality, using (2.5) and (2.13), we get

$$d\left(z, Tz\right) \leq \max\left(d^{\lambda}\left(z, Tz\right)\left[\frac{d^{\lambda}\left(z, z\right) + d^{\lambda}\left(Tz, z\right)}{d^{\lambda}\left(Tz, Tz\right) + d^{\lambda}\left(Tz, z\right)}\right],\right.$$

$$d^{\lambda}\left(z, Tz\right)\left[\frac{d^{\lambda}\left(z, z\right) + d^{\lambda}\left(Tz, Tz\right)}{d^{\lambda}\left(z, Tz\right) + d^{\lambda}\left(z, Tz\right)}\right],$$

$$\left. d^{\lambda}\left(Tz, Tz\right) d^{\lambda}\left(Tz, z\right)\left[\frac{d^{\lambda}\left(z, z\right) + d^{\lambda}\left(z, Tz\right)}{d^{\lambda}\left(z, Tz\right) + d^{\lambda}\left(Tz, Tz\right)}\right]\right)$$

$$= \max\left(d^{\lambda}\left(z, Tz\right), 1, d^{\lambda}\left(Tz, z\right)\right)$$

$$d\left(z, Tz\right) \leq d^{\lambda}\left(Tz, z\right)$$

This means that $d\left(Tz, z\right) = 1$, that is, $Tz = z$ (2.14)

On the other hand, since $z = Tz \in TX \subset AX$, there exists $z^{**} \in X$ such that $z = Tz = Az^{**}$. By using (2.1) and $Tz = Bz = z$, we obtain

$$d\left(Sz^{**}, z\right) = d\left(Sz^{**}, Tz\right)$$

$$d\left(Sz^{**}, Tz\right) \leq \max\left(d^{\lambda}\left(Az^{**}, Bz\right)\left[\frac{d^{\lambda}\left(Az^{**}, Sz^{**}\right) + d^{\lambda}\left(Tz, Sz^{**}\right)}{d^{\lambda}\left(Bz, Tz\right) + d^{\lambda}\left(Bz, Az^{**}\right)}\right], d^{\lambda}\left(Az^{**}, Tz\right)\left[\frac{d^{\lambda}\left(Sz^{**}, Az^{**}\right) + d^{\lambda}\left(Bz, Tz\right)}{d^{\lambda}\left(Sz^{**}, Tz\right) + d^{\lambda}\left(Az^{**}, Bz\right)}\right],\right.$$

$$\left. d^{\lambda}\left(Tz, Bz\right) d^{\lambda}\left(Tz, Sz^{**}\right)\left[\frac{d^{\lambda}\left(Az^{**}, Sz^{**}\right) + d^{\lambda}\left(Az^{**}, Bz\right)}{d^{\lambda}\left(Sz^{**}, Tz\right) + d^{\lambda}\left(Tz, Bz\right)}\right]\right)$$

$$d\left(Sz^{**},z\right)\le \max\left(d^{\lambda}(z,z)\left[\frac{d^{\lambda}\left(z,Sz^{**}\right)+d^{\lambda}\left(z,Sz^{**}\right)}{d^{\lambda}(z,z)+d^{\lambda}(z,z)}\right], d^{\lambda}(z,z)\left[\frac{d^{\lambda}\left(Sz^{**},z\right)+d^{\lambda}(z,z)}{d^{\lambda}\left(Sz^{**},z\right)+d^{\lambda}(z,z)}\right],\right.$$

$$\left. d^{\lambda}(z,z)d^{\lambda}\left(z,Sz^{**}\right)\left[\frac{d^{\lambda}\left(z,Sz^{**}\right)+d^{\lambda}(z,z)}{d^{\lambda}\left(Sz^{**},z\right)+d^{\lambda}(z,z)}\right]\right)$$

$$d\left(Sz^{**},z\right)\le \max\left(\left(d^{\lambda}\left(z,Sz^{**}\right),1,d^{\lambda}\left(Sz^{**},z\right)\right)\right.$$

$$\le d^{\lambda}\left(Sz^{**},z\right)$$

This implies that $d^{\lambda}\left(Sz^{**},z\right)=1$ and so $Sz^{**}=z=Az^{**}$. Since the pair (S,A) is compatible, $d(Sz,Az)=d\left(SAz^{**},ASz^{**}\right)=d(z,z)=1$.

So $Az=Sz$. Hence $z=Sz=Az=Tz=Bz$.

Next, we prove that S, T, A and B have a unique common fixed point. Suppose that $w\in X$ is also a common fixed point of S, T, A and B, then

$$d(z,w)=d(Sz,Tw)$$

$$\le \max\left(d^{\lambda}(Az,Bw)\left[\frac{d^{\lambda}(Az,Sz)+d^{\lambda}(Tw,Sz)}{d^{\lambda}(Bw,Tw)+d^{\lambda}(Bw,Az)}\right], d^{\lambda}(Az,Tw)\left[\frac{d^{\lambda}(Sz,Az)+d^{\lambda}(Bw,Tw)}{d^{\lambda}(Sz,Tw)+d^{\lambda}(Az,Bw)}\right],\right.$$

$$\left. d^{\lambda}(Tw,Bw)d^{\lambda}(Tw,Sz)\left[\frac{d^{\lambda}(Az,Sz)+d^{\lambda}(Aw,Bz)}{d^{\lambda}(Sw,Tz)+d^{\lambda}(Tw,Bw)}\right]\right)$$

$$d(z,w)\le \max\left(d^{\lambda}(z,w)\left[\frac{d^{\lambda}(z,z)+d^{\lambda}(w,z)}{d^{\lambda}(w,w)+d^{\lambda}(w,z)}\right], d^{\lambda}(z,w)\left[\frac{d^{\lambda}(z,z)+d^{\lambda}(w,w)}{d^{\lambda}(z,w)+d^{\lambda}(z,w)}\right]\right),$$

$$\left. d^{\lambda}(w,w)d^{\lambda}(w,z)\left[\frac{d^{\lambda}(z,z)+d^{\lambda}(z,w)}{d^{\lambda}(z,w)+d^{\lambda}(w,w)}\right]\right)$$

$$d(z,w)\le \max\left(d^{\lambda}(z,w),1,d^{\lambda}(w,z)\right)$$

$$\le d^{\lambda}(z,w)$$

This implies that $d(z,w)=1$, so $w=z$. Therefore, z is a unique common fixed point of S, T, A and B.

Finally, if condition (b) holds, then the argument is similar to that above, so we delete it.

This completes the proof.

Theorem 2.2 Let (X,d) be a complete multiplicative metric space, S, T, A and B be four mappings X into itself. Suppose that there exists $\lambda\in\left(0,\frac{1}{2}\right)$ and $p,q\in Z^{+}$

such that $S(X)\subset B(X)$, $T(X)\subset A(X)$ and

$$d\left(S^p x, T^q y\right) \le \max\left(d^\lambda(Ax,By)\left[\frac{d^\lambda\left(Ax,S^p x\right)+d^\lambda\left(T^q y,S^p x\right)}{d^\lambda\left(By,T^q y\right)+d^\lambda(By,Ax)}\right], d^\lambda\left(Ax,T^q y\right)\left[\frac{d^\lambda\left(S^p x,Ax\right)+d^\lambda\left(By,T^q y\right)}{d^\lambda\left(S^p x,T^q y\right)+d^\lambda(Ax,By)}\right],\right.$$

$$\left. d^\lambda\left(T^q y,By\right)d^\lambda\left(T^q y,S^p x\right)\left[\frac{d^\lambda\left(Ax,S^p x\right)+d^\lambda(Ax,By)}{d^\lambda\left(S^p x,T^q y\right)+d^\lambda\left(T^q y,By\right)}\right]\right) \qquad (2.15)$$

for all x, $y \in X$. Assume the following conditions are satisfied:

(a) The pairs (S, A) and (T, B) are commutative mappings;

(b) one of S, T, A and B is continuous.

Then S, T, A and B have a unique common fixed point.

Proof: From $S(X)\subset B(X)$, $T(X)\subset A(X)$ we have

$$S^p X \subset S^{p-1}X \subset \dots \subset S^2 X \subset S^1 X \subset BX$$

and $$T^q X \subset T^{q-1}X \subset \dots \subset T^2 X \subset T^1 X \subset AX$$

since the pairs (S, A) and (T, B) are commutative mappings,

$$S^p A = S^{p-1}SA = S^{p-1}AS = S^{p-2}(SA)S = S^{p-2}AS^2 = \dots\dots = AS^p$$

and

$$T^q B = T^{q-1}TB = T^{q-1}BT = T^{q-2}(TB)T = T^{q-2}BT^2 = \dots\dots = BT^q$$

That is to say, $S^p A = AS^p$ and $T^q B = BT^q$.

We know that the compatible pairs (S^p, A) and (T^q, B) are also weakly compatible. Therefore, by Theorem 2.1, we can find that S^p, T^q, A and B have a unique common fixed point z.

In addition, we prove that S, T, A and B have a unique common fixed point.

From (2.15), we have

$$d(Sz,z) = d\left(S^p(Sz), T^q z\right)$$

$$d(Sz,z) \le \max\left(d^\lambda(ASz,Bz)\left[\frac{d^\lambda\left(ASz,S^p Sz\right)+d^\lambda\left(T^q z,S^p Sz\right)}{d^\lambda\left(Bz,T^q z\right)+d^\lambda(Bz,ASz)}\right],\right.$$

$$d^\lambda\left(ASz,T^q z\right)\left[\frac{d^\lambda\left(S^p Sz,ASz\right)+d^\lambda\left(Bz,T^q z\right)}{d^\lambda\left(S^p Sz,T^q z\right)+d^\lambda(ASz,Bz)}\right],$$

$$\left. d^\lambda\left(Bz,T^q z\right)d^\lambda\left(T^q z,S^p Sz\right)\left[\frac{d^\lambda\left(ASz,S^p Sz\right)+d^\lambda(ASz,Bz)}{d^\lambda\left(S^p Sz,T^q z\right)+d^\lambda\left(T^q z,Bz\right)}\right]\right)$$

$$= \max\left(d^{\lambda}(Sz,z)\left[\frac{d^{\lambda}(Sz,Sz)+d^{\lambda}(z,Sz)}{d^{\lambda}(z,z)+d^{\lambda}(z,Sz)}\right], d^{\lambda}(Sz,z)\left[\frac{d^{\lambda}(Sz,Sz)+d^{\lambda}(z,z)}{d^{\lambda}(Sz,z)+d^{\lambda}(Sz,z)}\right],\right.$$

$$\left. d^{\lambda}(z,z)d^{\lambda}(z,Sz)\left[\frac{d^{\lambda}(Sz,Sz)+d^{\lambda}(Sz,z)}{d^{\lambda}(Sz,z)+d^{\lambda}(z,z)}\right]\right)$$

$$= \max\left(d^{\lambda}(z,Sz), 1, d^{\lambda}(z,Sz)\right)$$

$$\leq d^{\lambda}(Sz,z)$$

This implies that $d(Sz,z)=1$, so $Sz=z$.

On the other hand, using (2.15) we have

$$d(z,Tz)=d\left(S^{p}z,T^{q}(Tz)\right)$$

$$d(z,Tz)\leq \max\left(d^{\lambda}(Az,BTz)\left[\frac{d^{\lambda}(Az,S^{p}z)+d^{\lambda}(T^{q}Tz,S^{p}z)}{d^{\lambda}(BTz,T^{q}Tz)+d^{\lambda}(BTz,Az)}\right],\right.$$

$$d^{\lambda}(Az,T^{q}Tz)\left[\frac{d^{\lambda}(S^{p}z,Az)+d^{\lambda}(BTz,T^{q}Tz)}{d^{\lambda}(S^{p}z,T^{q}Tz)+d^{\lambda}(Az,BTz)}\right],$$

$$\left. d^{\lambda}(BTz,T^{q}Tz)d^{\lambda}(T^{q}Tz,S^{p}z)\left[\frac{d^{\lambda}(Az,S^{p}z)+d^{\lambda}(Az,BTz)}{d^{\lambda}(S^{p}z,T^{q}Tz)+d^{\lambda}(T^{q}Tz,BTz)}\right]\right)$$

$$= \max\left(d^{\lambda}(z,Tz)\left[\frac{d^{\lambda}(z,z)+d^{\lambda}(Tz,z)}{d^{\lambda}(Tz,Tz)+d^{\lambda}(Tz,z)}\right], d^{\lambda}(z,Tz)\left[\frac{d^{\lambda}(z,z)+d^{\lambda}(Tz,Tz)}{d^{\lambda}(z,Tz)+d^{\lambda}(z,Tz)}\right],\right.$$

$$\left. d^{\lambda}(Tz,Tz)d^{\lambda}(Tz,z)\left[\frac{d^{\lambda}(z,z)+d^{\lambda}(z,Tz)}{d^{\lambda}(z,Tz)+d^{\lambda}(Tz,Tz)}\right]\right)$$

$$= \max\left(d^{\lambda}(z,Tz), 1, d^{\lambda}(Tz,z)\right)$$

$$\leq d^{\lambda}(z,Tz)$$

This implies that $d^{\lambda}(z,Tz)=1$, so $Tz=z$.

Therefore, we obtain $Sz=Tz=Az=Bz=z$, so z is a common fixed point of S, T, A and B.

Finally, we prove that S, T, A and B have a unique common fixed point.

Suppose that $w\in X$ is also a common fixed point of S, T, A and B, then

$$d(z,w)=d\left(S^{p}z,T^{q}w\right)$$

$$d\left(S^{p}z,T^{q}w\right)\leq \max\left(d^{\lambda}(Az,Bw)\left[\frac{d^{\lambda}(Az,S^{p}z)+d^{\lambda}(T^{q}w,S^{p}z)}{d^{\lambda}(Bw,T^{q}w)+d^{\lambda}(Bw,Az)}\right], d^{\lambda}(Az,T^{q}w)\left[\frac{d^{\lambda}(S^{p}z,Az)+d^{\lambda}(Bw,T^{q}w)}{d^{\lambda}(S^{p}z,T^{q}w)+d^{\lambda}(Az,Bw)}\right],\right.$$

$$d^{\lambda}\left(Bw,T^{q}w\right)d^{\lambda}\left(T^{q}w,S^{p}z\right)\left[\frac{d^{\lambda}\left(Az,S^{p}z\right)+d^{\lambda}\left(Az,Bw\right)}{d^{\lambda}\left(S^{p}z,T^{q}w\right)+d^{\lambda}\left(T^{q}w,Bw\right)}\right]\Bigg)$$

$$d(z,w)\le \max\left(d^{\lambda}(z,w)\left[\frac{d^{\lambda}(z,z)+d^{\lambda}(w,z)}{d^{\lambda}(w,w)+d^{\lambda}(w,z)}\right],d^{\lambda}(z,w)\left[\frac{d^{\lambda}(z,z)+d^{\lambda}(w,w)}{d^{\lambda}(z,w)+d^{\lambda}(z,w)}\right]\right),$$

$$d^{\lambda}(w,w)d^{\lambda}(w,z)\left[\frac{d^{\lambda}(z,z)+d^{\lambda}(z,w)}{d^{\lambda}(z,w)+d^{\lambda}(w,w)}\right]\Bigg)$$

$$=\max\left(d^{\lambda}(z,w),1,d^{\lambda}(w,z)\right)$$

$$\le d^{\lambda}(z,w)$$

This implies that $d(z,w)=1$, so $w=z$.

Therefore, z is a unique common fixed point of S, T, A and B.

Corollary 2.1: Let (X,d) be a complete multiplicative metric space, S and T be two mappings of X into itself. Suppose that there exists $\lambda\in(0,\frac{1}{2})$ such that

$$d(Sx,Ty)\le \max\left(d^{\lambda}(x,y)\left[\frac{d^{\lambda}(x,Sx)+d^{\lambda}(Ty,Sx)}{d^{\lambda}(y,Ty)+d^{\lambda}(y,x)}\right],d^{\lambda}(x,Ty)\left[\frac{d^{\lambda}(Sx,x)+d^{\lambda}(y,Ty)}{d^{\lambda}(Sx,Ty)+d^{\lambda}(x,y)}\right],\right.$$

$$d^{\lambda}(Ty,y)d^{\lambda}(Ty,Sx)\left[\frac{d^{\lambda}(x,Sx)+d^{\lambda}(x,y)}{d^{\lambda}(Sy,Ty)+d^{\lambda}(Ty,y)}\right]\Bigg)$$

for all x, $y\in X$. Then S and T have a unique common fixed point.

By taking $A=B=I$ in Theorem 2.1, we can prove the above result.

Corollary 2.2: Let (X,d) be a complete multiplicative metric space, S and T be two mappings of X into itself. Suppose that there exists $\lambda\in(0,\frac{1}{2})$ and $p,q\in Z^{+}$ such that

$$d\left(S^{p}x,T^{q}y\right)\le \max\left(d^{\lambda}(x,y)\left[\frac{d^{\lambda}\left(x,S^{p}x\right)+d^{\lambda}\left(T^{q}y,S^{p}x\right)}{d^{\lambda}\left(y,T^{q}y\right)+d^{\lambda}(y,x)}\right],d^{\lambda}\left(x,T^{q}y\right)\left[\frac{d^{\lambda}\left(S^{p}x,x\right)+d^{\lambda}\left(y,T^{q}y\right)}{d^{\lambda}\left(S^{p}x,T^{q}y\right)+d^{\lambda}(x,y)}\right],\right.$$

$$d^{\lambda}\left(y,T^{q}y\right)d^{\lambda}\left(T^{q}y,S^{p}x\right)\left[\frac{d^{\lambda}\left(x,S^{p}x\right)+d^{\lambda}(x,y)}{d^{\lambda}\left(S^{p}x,T^{q}y\right)+d^{\lambda}\left(T^{q}y,y\right)}\right]\Bigg)$$

for all x, $y\in X$. Then S and T have a unique common fixed point.

By taking $A=B=I$ in Theorem 2.2, we can prove the above result.

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