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FRACTIONAL INTEGRAL OPERATORS AND GENERALIZED k-MITTAG LEFFLER TYPE FUNCTION

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ABSTRACT: In the present paper we have established some results of the generalized k-Mittag Leffler function. We find images formula of generalized k-Mittag-Leffler function under the various integral operators such as Riemann Liouville fractional integral operators and Pathway fractional integral operator

KEYWORDS: k-Gamma function, k-Beta function, k-Pochhammer symbol, Generalized k-Mittag Leffler function, Fractional integral operators.

MATHEMATICS SUBJECT CLASSIFICATION: $33E12, 26A33, 44-00$

1. INTRODUCTION

In this paper we deal with the following generalized k-Mittag Leffler function defined by $Meena\ et\ al.$ [8], for $\alpha, \beta, \gamma, \delta \in C$ and $k \in R$ such that

$$E_{k,\alpha,\beta,\gamma}^{\delta,p}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{np,k}}{\Gamma_k(\alpha n + \beta)} \frac{z^n}{(\gamma)_{n,k}} \ ; \ \gamma \neq 0, -1, -2, \dots and\ \ np \in Z^+ \qquad (1.1)$$

where $Re(\alpha) > 0, Re(\beta) > 0$.

For $k = 1$ the generalized k-Mittag Leffler function (1.1), reduced into the generalized Mittag Leffler given by $Khan\ and\ Ahmed$ $[12,\ eq.(1.7),\ pp.2]$.

In 2013, $Khan\ and\ Ahmed$ [12] introduced the following form of generalized Mittag Leffler function for $\alpha, \beta, \gamma, \delta \in C$ and defined as

$$E_{\alpha,\beta,\gamma}^{\delta,p}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{np}}{\Gamma(\alpha n+\beta)} \frac{z^n}{(\gamma)_n} ; \qquad p \in (0,1) \cup N \tag{1.2}$$

where $\{Re(\alpha), Re(\beta), Re(\gamma), Re(\delta)\} > 0$ and $(\delta)_{np} = \Gamma(\delta+np)/\Gamma(\delta)$.

In 1905, $Wiman$ [1] introduced the following form of Mittag Leffler function for $\alpha, \beta \in C$ and defined as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} ; \tag{1.3}$$

where $Re(\alpha), Re(\beta) > 0$

In 1903, Swedish mathematician $G. Mittag\ Leffler$ [5] introduced the following form of Mittag Leffler function for $\alpha \in C$ and defined as

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} ; \tag{1.4}$$

where $Re(\alpha) > 0$ and $\Gamma(\alpha)$ denotes the gamma function.

2. DEFINITIONS

Definition 2.1 The k-Pochhammer symbol introduced by Diaz and Parigum [13], defined as

$$(\alpha)_{n,k} = \alpha(\alpha+k)(\alpha+2k)\ldots(\alpha+\underline{n-1}k) \tag{2.1}$$

$$(\alpha)_{(n+r)q,k} = (\alpha)_{rq,k}\ (\alpha+rqk)_{nq,k} \tag{2.2}$$

where $\alpha \in C$ and $k \in R, n \in N$

Definition 2.2 The k-Gamma function introduced by Diaz and Parigum [13], defined as

$$\Gamma_k(\alpha) = \int_0^{\infty} e^{-\frac{t^k}{k}} t^{\alpha-1} dt;\ where\ Re(\alpha) > 0, \alpha \in C\ and\ k \in R \tag{2.3}$$

$$and \qquad \Gamma_k(\alpha + k) = \alpha\, \Gamma_k(\alpha) \tag{2.4}$$

Definition 2.3 The k-Beta function introduced by Diaz and Parigum [13], defined as

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1} dt; \quad x,y \in C\ and\ k > 0 \tag{2.5}$$

$$and \qquad B_k(x,y) = \frac{\Gamma_k(x)\,\Gamma_k(y)}{\Gamma_k(x+y)}; \quad Re(x) > 0, Re(y) > 0 \tag{2.6}$$

Definition 2.4 Let $\gamma \in C$ and $k, s \in R$ then the following identity holds [6]

$$\Gamma_s(\gamma) = \left(\frac{s}{k}\right)^{\frac{\gamma}{k}-1} \Gamma_k\left(\frac{k\gamma}{s}\right) \tag{2.7}$$

$$and\ in\ particular\ case \qquad \Gamma_k(\gamma) = (k)^{\frac{\gamma}{k}-1}\Gamma\left(\frac{\gamma}{k}\right) \tag{2.8}$$

Definition 2.5 Let $\gamma \in C$ and $k, s \in R$; $n \in N$ then the following identity holds [6]

$$(\gamma)_{nq,s} = \left(\frac{s}{k}\right)^{nq} \left(\frac{k\gamma}{s}\right)_{nq,k} \tag{2.9}$$

$$and\ in\ particular\ case \qquad (\gamma)_{nq,k} = (k)^{nq}\left(\frac{\gamma}{k}\right)_{nq} \tag{2.10}$$

Definition 2.6 If μ be a real number such that $\mu > 0$, then k-Riemann Liouville fractional integral of order μ is defined as [15]

$$\left(I_k^{\mu} f\right)(x) = \frac{1}{k\,\Gamma_k(\mu)}\int_0^x (x-t)^{\frac{\mu}{k}-1} f(t) dt; \quad k \in R \tag{2.11}$$

Definition 2.7 For $f(x) \in L(a,b)$, $a > 0, Re(\eta) > 0$, $\eta \in C$ the Pathway fractional integral operator is defined as [3]

$$\left(P_{0+}^{(\eta,\mu)} f\right)(x) = x^{\eta}\int_0^{\frac{x}{a(1-\mu)}} \left[1 - \frac{a(1-\mu)t}{x}\right]^{\frac{\eta}{1-\mu}} f(t) dt; \tag{2.12}$$

where μ is the Pathway parameter which is less than 1.

3. MAIN RESULTS

Theorem 3.1 Riemann-Liouville fractional integral: If the condition (1.1) satisfied then the following image formula holds true

$$I_k^{\mu}\left[t^{\frac{\beta}{k}-1}E_{k,\alpha,\beta,\gamma}^{\delta,p}\left(w\,t^{\frac{\alpha}{k}}\right)\right](x)=x^{\frac{\beta+\mu}{k}-1}E_{k,\alpha,\beta+\mu,\gamma}^{\delta,p}\left(w\,x^{\frac{\alpha}{k}}\right) \tag{3.1}$$

where $k\in R$ and μ be a real number such that $\mu>0$.

Proof. In order to derive (3.1), we denote $L.H.S.$ of (3.1) by symbol L_1 and then expanding $E_{k,\alpha,\beta,\gamma}^{\delta,p}(t)$ by using equation (1.1)

$$L_1\equiv I_k^{\mu}\left[t^{\frac{\beta}{k}-1}E_{k,\alpha,\beta,\gamma}^{\delta,p}\left(w\,t^{\frac{\alpha}{k}}\right)\right](x)=\frac{1}{k\,\Gamma_k(\mu)}\int_0^x (x-t)^{\frac{\mu}{k}-1}t^{\frac{\beta}{k}-1}E_{k,\alpha,\beta,\gamma}^{\delta,p}\left(w\,t^{\frac{\alpha}{k}}\right)dt$$

$$L_1\equiv\frac{1}{k\,\Gamma_k(\mu)}\int_0^x (x-t)^{\frac{\mu}{k}-1}t^{\frac{\beta}{k}-1}\sum_{n=0}^{\infty}\frac{(\delta)_{np,k}}{\Gamma_k(\alpha n+\beta)}\frac{w^n}{(\gamma)_{n,k}}\,t^{\frac{\alpha n}{k}}dt$$

After interchanging the order of integration and summation, we have

$$L_1\equiv\frac{1}{k\,\Gamma_k(\mu)}\sum_{n=0}^{\infty}\frac{(\delta)_{np,k}}{\Gamma_k(\alpha n+\beta)}\frac{w^n}{(\gamma)_{n,k}}\int_0^x (x-t)^{\frac{\mu}{k}-1}t^{\frac{\alpha n+\beta}{k}-1}dt$$

Let $t=xu$, then we get

$$L_1\equiv\frac{1}{k\,\Gamma_k(\mu)}\sum_{n=0}^{\infty}\frac{(\delta)_{np,k}}{\Gamma_k(\alpha n+\beta)}\frac{w^n}{(\gamma)_{n,k}}x^{\frac{\alpha n+\beta+\mu}{k}-1}\int_0^1 (1-u)^{\frac{\mu}{k}-1}u^{\frac{\alpha n+\beta}{k}-1}du$$

Now using the $Definition$ 2.3, we arrived at

$$L_1\equiv\frac{1}{\Gamma_k(\mu)}\sum_{n=0}^{\infty}\frac{(\delta)_{np,k}}{\Gamma_k(\alpha n+\beta)}\frac{w^n}{(\gamma)_{n,k}}x^{\frac{\alpha n+\beta+\mu}{k}-1}\frac{\Gamma_k(\mu)\,\Gamma_k(\alpha n+\beta)}{\Gamma_k(\alpha n+\beta+\mu)}$$

$$L_1\equiv x^{\frac{\beta+\mu}{k}-1}\sum_{n=0}^{\infty}\frac{(\delta)_{np,k}}{\Gamma_k(\alpha n+\beta+\mu)}\frac{w^n}{(\gamma)_{n,k}}x^{\frac{\alpha n}{k}}$$

$$L_1\equiv x^{\frac{\beta+\mu}{k}-1}E_{k,\alpha,\beta+\mu,\gamma}^{\delta,p}\left(wx^{\frac{\alpha}{k}}\right)$$

Theorem 3.2 Pathway fractional integral: If the condition (1.1) satisfied then the following image formula holds true

$$P_{0+}^{(\eta,\mu)}\left[t^{\frac{\beta}{k}-1}E_{k,\alpha,\beta,\gamma}^{\delta,p}\left(w\,t^{\frac{\alpha}{k}}\right)\right](x) = x^{\eta+\frac{\beta}{k}}\ k^{\frac{\eta}{1-\mu}+1}$$
$$\times\ [a-a\mu]^{-\frac{\beta}{k}}\ \Gamma\left(\frac{\eta}{1-\mu}+1\right)\ E_{k,\alpha,\beta+k\left(\frac{\eta}{1-\mu}+1\right),\gamma}^{\delta,p}\left[w\left(\frac{x}{a-a\mu}\right)^{\frac{\alpha}{k}}\right] \qquad (3.2)$$

where $w \in R, \mu < 1$ and $Re\left(\frac{\eta}{1-\mu}\right)+1>0$.

Proof. In order to derive (3.2), we denote $L.H.S.$ of (3.2) by symbol L_2 and then expanding $E_{k,\alpha,\beta,\gamma}^{\delta,p}(t)$ by using equation (1.1)

$$L_2 \equiv x^{\eta}\int_0^{\frac{x}{a(1-\mu)}}\left[1-\frac{a(1-\mu)t}{x}\right]^{\frac{\eta}{1-\mu}} t^{\frac{\beta}{k}-1}E_{k,\alpha,\beta,\gamma}^{\delta,p}\left(wt^{\frac{\alpha}{k}}\right)dt$$

$$L_2 \equiv x^{\eta}\int_0^{\frac{x}{a(1-\mu)}}\left[1-\frac{a(1-\mu)t}{x}\right]^{\frac{\eta}{1-\mu}} t^{\frac{\beta}{k}-1}\sum_{n=0}^{\infty}\frac{(\delta)_{np,k}}{\Gamma_k(\alpha n+\beta)}\frac{w^n}{(\gamma)_{n,k}}\,t^{\frac{\alpha n}{k}}dt$$

After interchanging the order of integration and summation, we have

$$L_2 \equiv x^{\eta}\sum_{n=0}^{\infty}\frac{(\delta)_{np,k}}{\Gamma_k(\alpha n+\beta)}\frac{w^n}{(\gamma)_{n,k}}\int_0^{\frac{x}{a(1-\mu)}}\left[1-\frac{a(1-\mu)t}{x}\right]^{\frac{\eta}{1-\mu}} t^{\frac{\alpha n+\beta}{k}-1}dt$$

Let $u=\frac{a(1-\mu)t}{x}$, then we obtain

$$L_2 \equiv x^{\eta}\sum_{n=0}^{\infty}\frac{(\delta)_{np,k}}{\Gamma_k(\alpha n+\beta)}\frac{w^n}{(\gamma)_{n,k}}\left(\frac{x}{a(1-\mu)}\right)^{\frac{\alpha n+\beta}{k}}\int_0^1 [1-u]^{\frac{\eta}{1-\mu}}\,u^{\frac{\alpha n+\beta}{k}-1}du$$

Now using the $Definition$ 2.3 & 2.4, we arrived at

$$L_2 \equiv x^{\eta}\sum_{n=0}^{\infty}\frac{(\delta)_{np,k}}{\Gamma\left(\frac{\alpha n+\beta}{k}\right)\,k^{\frac{\alpha n+\beta}{k}-1}}\frac{w^n}{(\gamma)_{n,k}}\left(\frac{x}{a-a\mu}\right)^{\frac{\alpha n+\beta}{k}}\frac{\Gamma\left(\frac{\eta}{1-\mu}-1\right)\Gamma\left(\frac{\alpha n+\beta}{k}\right)}{\Gamma\left(\frac{\alpha n+\beta}{k}+\frac{\eta}{1-\mu}-1\right)}$$

$$L_2 \equiv \frac{\Gamma\left(\frac{\eta}{1-\mu}+1\right)x^{\eta+\frac{\beta}{k}}\,k^{\frac{\eta}{1-\mu}+1}}{[a-a\mu]^{\frac{\beta}{k}}}\sum_{n=0}^{\infty}\frac{(\delta)_{np,k}\ k^{-\frac{\alpha n+\beta}{k}-\frac{\eta}{1-\mu}}}{\Gamma\left(\frac{\alpha n+\beta}{k}+\frac{\eta}{1-\mu}+1\right)}\frac{1}{(\gamma)_{n,k}}\left(\frac{wx^{\frac{\alpha}{k}}}{a-a\mu}\right)^n$$

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