**International Journal of Engineering, Science and Mathematics**
Vol. 12 Issue 4, April 2023
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijesm.co.in , Email: editorijmie@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

# NOTE ON $|\underline{N}, p_n^\alpha, \beta; \delta|_k$ SUMMABILITY FACTORS

**Dr. DHIRENDHRA SINGH**
Department of Mathematics
Pandit Lalit Mohan Sharma Sri Dev Suman Uttarakhand University Campus Rishikesh Dehradun
and
**Dr. PRAGATI SINHA**
Mangalmay Institute of Engineering and Technology Knowledge Park II Greater Noida-201310

**Abstract :** In this paper we have proved a theorem for $|\underline{N}, p_n^\alpha, \beta, \delta|_k$ summability. However our theorem is as follows

***THEOREM:***

Let $\{p_n\}$ be a positive non-increasing sequence such that $p_n^\alpha \to \infty$, and

$$\sum_{n=v+1}^{\infty} \left(\frac{p_n^\alpha}{P_n^\alpha}\right)^{\beta(\delta k+k-1)-k} \frac{1}{p_{n-1}^\alpha} = O\left\{\left(\frac{P_v^\alpha}{p_v^\alpha}\right)^{\beta(\delta k+k)-k} \frac{1}{P_v^\alpha}\right\}.$$

If $\sum a_n$ is $\left[\underline{N}, p_n; \delta\right]_k$, bounded and $\{\lambda_n\}$ is a sequence such of that non-negative, non-increasing,

Suchthat $\quad p \sum_n^\alpha \quad \lambda_n < \infty$

and

$$P_n^\alpha \Delta\lambda_n = O(p_n^\alpha \lambda_n),$$

then the series $\sum a_n \lambda_n$ is summable $\left|\underline{N}, p_n^\alpha, \beta, \delta\right|_k, k \geq 1$.

## *1. DEFINITIONS AND NOTATIONS*

SINGH and SHARMA [8]-Let $\sum a_n$ be a given infinite series with $s_n$ for its n-th partial sums. Define

$$p_n^{\alpha} = \sum_{v=0}^{n} \quad a_{n-v}^{\alpha-1} p_v \tag{1.1}$$

where

$$A_n^{\alpha} = (n + \alpha\ n\ ) = \frac{(\alpha+1)(\alpha+2)(\alpha+3)\dots(\alpha+n)}{n}, \alpha > -1$$

and $\{p_n\}$ a sequence of real numbers such that $p_0 > 0, p_n \geq 0, n = 1,2,3, \dots$

$$P_n^{\alpha} = \sum_{v=0}^{n} \quad p_v^{\alpha}.$$

Let $\{t_n^{\alpha}\}$ be the sequence of $(\underline{N}, p_n^{\alpha})$ mean of the sequence $\{s_n\}$,

$$t_n^{\alpha} = \frac{1}{P_n^{\alpha}} \sum_{v=0}^{n} \quad p_v^{\alpha} s_v, p_n^{\alpha} \neq 0 \tag{1.2}$$

The series $\sum a_n$ is said to be summable $|N^{-}, p_n^{\alpha}|_k, 1 \leq k$, if

BOR [1]

$$\sum_{n=1}^{\infty} \quad \left(\frac{P_n^{\alpha}}{p_n^{\alpha}}\right)^{k-1} |\Delta t_{n-1}^{\alpha}|^k < \infty, \tag{1.3}$$

and it is said to be summable $|N^{-}, p_n^{\alpha}; \delta|_k, k \geq 1$ and $\delta \geq 0$, if BOR [5]

$$\sum_{n=1}^{\infty} \quad \left(\frac{P_n^{\alpha}}{p_n^{\alpha}}\right)^{\delta k+k-1} |\Delta t_{n-1}^{\alpha}|^k < \infty, \tag{1.4}$$

where

$$\Delta t_{n-1}^{\alpha} = \frac{p_n^{\alpha}}{P_n^{\alpha} P_{n-1}^{\alpha}} \sum_{v=1}^{n} \quad P_v^{\alpha} a_v, n \geq 1 \tag{1.5}$$

In the special case when $\delta = 0$ and $\alpha = 1$, $|N^{-}, p_n^{\alpha}; \delta|_k$ summability is that same as $|N^{-}, p_n|_k$ summability.

Moreover, the series $\sum a_n$ is said to be summable $[N^{-}, p_n, \beta; \delta]_k$, $k \geq 1, \delta \geq 0$ and $\beta =$ real number, if

$$\sum_{n=1}^{\infty} \left(\frac{P_n^{\alpha}}{p_n^{\alpha}}\right)^{\beta(\delta k+k-1)} |\Delta t_{n-1}^{\alpha}|^k < \infty$$

However for $p_n = 1$, for all $n$, the above definition reduces to $[C, 1, \beta; \delta]_k$.

The series $\sum a_n$ is said to be $[N^{-}, p_n]_k, k \geq 1$, if BOR [1]

$$\sum_{v=1}^{n} p_v |s_v|^k = O(P_n), as \ n \to \infty \tag{1.6}$$

and it is said to be $\left[\underline{N}, p_n; \delta\right]_k$ bounded, $k \geq 1$, if BOR [3]

$$\sum_{v=1}^{n} \left(\frac{P_v}{p_v}\right)^{\delta k} p_v |s_v|^k = O(P_n), \ as \ \ n \to \infty \tag{1.7}$$

If we take $\delta = 0$, then $[N^{-}, p_n; \delta]_k$ boundedness is the same as $[N^{-}, p_n]_k$ boundedness.

Now, we shall define $[N^{-}, p_n^{\alpha}, \beta; \delta]_k$ as follows;

The series $\sum a_n$ is said to be $[N^{-}, p_n^{\alpha}, \beta; \delta]_k$; $(k \geq 1, \delta \geq 0$ and $\beta$ is real number) if

$$\sum_{v=1}^{n} \left(\frac{p_r}{P_r}\right)^{\beta(\delta k+k)-k} p_v^{\alpha} |s_v|^k = O(P_n^{\alpha}), \ as \ \ n \to \infty \tag{1.8}$$

A sequence $\{\lambda_n\}$ is said to be convex (ZYGMUND [9]), if $\Delta^2 \lambda_n \geq 0$ for every positive integer $n$, where $\Delta^2 \lambda_n = \Delta \lambda_n - \Delta \lambda_{n+1}$ and $\Delta \lambda_n = \lambda_n - \lambda_{n+1}$.

## 2. INTRODUCTION :

BOR [4] has proved the following theorem for $|N^{-}, p_n|_k$ summability.

***THEOREM A :***

Let $\{p_n\}$ be a positive non-increasing sequence such that $P_n \to \infty$, as $n \to \infty$ and

$$\frac{1}{p_n} = O(n) \tag{2.1}$$

If $\sum a_n$ is $[N^{-}, p_n]_k$ bounded and $\{\lambda_n\}$ is a convex sequence such that $\sum p_n \lambda_n$ is convergent, then the series $\sum a_n \lambda_n$ is summable $\left|\underline{N}, p_n\right|_k, k \geq 1$.

If we take $k = 1$ in this theorem, then we get a result of CHEN [6].

Later on Theorem A of BOR [4] was proved by SEYHAN [7] under the weaker condition. However his theorem is as follows.

***THEOREM B :***

Let $\{p_n\}$ be a positive non-increasing sequence such that $P_n \to \infty$, as $n \to \infty$. If $\sum a_n$ is $[N^{-}, p_n]_k$ bounded and $\{\lambda_n\}$ is a sequence of non-negative, non-increasing, and $\sum p_n \lambda_n$ is convergent, $P_n \Delta\lambda_n = O(p_n \lambda_n)$, then the series $\sum a_n \lambda_n$ is summable $\left|\underline{N}, p_n\right|_k, k \geq 1$.

**3** We shall prove the above theorem for $\left|\underline{N}, p_n^{\alpha}, \beta; \delta\right|_k$ summability.

***THEOREM :***

Let $\{p_n\}$ be a positive non-increasing sequence such that $P_n^{\alpha} \to \infty$, as $n \to \infty$, and

$$\sum_{n=v+1}^{\infty} \left(\frac{P_n^{\alpha}}{p_n^{\alpha}}\right)^{\beta(8k+k-1)-k} \frac{1}{P_{n-1}^{\alpha}} = O\left\{\left(\frac{P_v^{\alpha}}{p_v^{\alpha}}\right)^{\beta(8k+k)-k} \frac{1}{P_v^{\alpha}}\right\}, \tag{3.1}$$

If $\sum a_n$ is $[N^{-}, p_n; \delta]_k$, bounded and $\{\lambda_n\}$ is a sequence of non-negative, non-increasing,

$$\sum p_n^{\alpha} \lambda_n < \infty \tag{3.2}$$

and

$$P_n^{\alpha} \Delta\lambda_n = O(p_n^{\alpha} \lambda_n) \tag{3.3}$$

then the series $\sum a_n \lambda_n$ is summable $|N^{-}, p_n^{\alpha}, \beta; \delta|_k, k \geq 1$.

**4.** We need the following lemma for the proof of our theorem.

***Lemma :***

If $\{\lambda_n\}$ is a sequence such that it is non-negative, non-increasing and $\sum p_n^{\alpha} \lambda_n$ is convergent, then

$$\sum p_n^{\alpha} \lambda_n = O(1) \text{ as } n \to \infty \tag{4.1}$$

$$\sum p_n^{\alpha} \Delta\lambda_n < \infty \text{ as } n \to \infty \tag{4.2}$$

The proof of above lemma follows on the lines of BOR [2].

***5. PROOF OF THE THEOREM :***

Let $(T_n^{\alpha})$ denotes the $(N^{-}, p_n^{\alpha})$ mean of the series $\sum a_n \lambda_n$.

Then, by definition, we have

$$T_n^{\alpha} = \frac{1}{P_n^{\alpha}} \sum_{v=0}^{n} p_v^{\alpha} \sum_{r=0}^{v} a_r \lambda_r = \frac{1}{P_n^{\alpha}} \sum_{v=0}^{n} (P_n^{\alpha} - P_{v-1}^{\alpha}) a_v \lambda_v$$

then, we have

$$T_n^{\alpha} - T_{n-1}^{\alpha} = \frac{p_n^{\alpha}}{P_n^{\alpha} P_{n-1}^{\alpha}} \sum_{v=1}^{n} p_{v-1}^{\alpha} a_v \lambda_v ; n \geq 1, (P_{-1}^{\alpha} = 0)$$

By Abel's transformation we have

$$T_n^\alpha - T_{n-1}^\alpha = \frac{-p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha}\sum_{v=1}^{n-1} p_v^\alpha s_v \lambda_v + \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha}\sum_{v=1}^{n-1} P_v^\alpha s_v \Delta\lambda_v + \frac{p_n^\alpha s_n \lambda_n}{P_n^\alpha}$$

$$= T_{n,1}^\alpha + T_{n,2}^\alpha + T_{n,3}^\alpha,\ say$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\beta(8k+k-1)} \left|T_{n,r}^\alpha\right|^k < \infty,\ for \quad r = 1,2,3 \qquad (5.1)$$

Since $\lambda_n = O\left(\frac{1}{P_n^\alpha}\right) = O(1)$, by (4.1) and applying Hölder's inequality with indices $k$ and $k'$, where $\frac{1}{k} + \frac{1}{k'} = 1$ and by using (3.1), we get that

$$\sum_{n=2}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\beta(\delta k+k-1)} \left|T_{n,1}^\alpha\right|^k \leq$$

$$\leq \sum_{n=2}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\beta(\delta k+k-1)} \left|\frac{p_{n-1}^\alpha}{P_n^\alpha P_{n-1}^\alpha}\sum_{v=1}^{n-1} p_v^\alpha s_v \lambda_v\right|^k$$

$$\leq \sum_{n=2}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\beta(\delta k+k-1)-k} \frac{1}{(P_{n-1}^\alpha)^k}\left\{\sum_{v=1}^{n-1} p_v^\alpha \lambda_v |s_v|\right\}^k$$

$$\leq \sum_{n=2}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\beta(\delta k+k-1)-k} \frac{1}{P_{n-1}^\alpha}\sum_{v=1}^{n-1} p_v^\alpha (\lambda_v)^k |s_v|^k \left\{\frac{1}{P_{n-1}^\alpha}\sum_{v=1}^{n-1} p_v^\alpha\right\}^{k-1}$$

$$= O(1)\sum_{v=1}^{m} p_v^\alpha (\lambda_v)^k |s_v|^k \sum_{n=v+1}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\beta((\delta k+k-1)-k} \frac{1}{P_{n-1}^\alpha}$$

$$= O(1)\sum_{v=1}^{m} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\beta(\delta k+k)-k-1} |s_v|^k (\lambda_v)^k$$

$$= O(1)\sum_{v=1}^{m} p_n^\alpha |s_v|^k (\lambda_v)(\lambda_v)^{k-1} \left(\frac{P_v^\alpha}{p_v^\alpha}\right)^{\beta(\delta k+k)-k}$$

$$= O(1)\sum_{v=1}^{m} p_n^\alpha |s_v|^k (\lambda_v) \left(\frac{P_v^\alpha}{p_v^\alpha}\right)^{\beta(\delta k+k)-k}$$

$$= O(1)\sum_{v=1}^{m-1} \Delta\lambda_v \sum_{z=1}^{v} \left(\frac{P_z^\alpha}{p_z^\alpha}\right)^{\beta(\delta k+k)-k} p_z^\alpha |s_z|^k$$

$$+ \quad +O(1)\lambda_m \sum_{v=1}^{m} \left(\frac{P_v^\alpha}{p_v^\alpha}\right)^{\beta(\delta k+k)-k} p_v |s_v|^k$$

$$= O(1)\sum_{v=1}^{m-1} \Delta\lambda_v P_v^\alpha + O(1)\lambda_m P_m^\alpha \quad = O(1).$$

by virtue of (1.7), (4.1) and (4.2). Again, by using the condition (3.3), as in $T_{n,1}^\alpha$, we have that

$$\sum_{n=2}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\beta(\delta k+k-1)} \left|T_{n,2}^\alpha\right|^k \leq$$

$$\leq \sum_{n=2}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\beta(\delta k+k-1)-k} \frac{1}{(P_{n-1}^\alpha)^k} \left|\sum_{v=1}^{n-1} P_v^\alpha \Delta\lambda_v s_v\right|^k$$

$$= O(1)\sum_{n=2}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\beta(\delta k+k-1)-k} \frac{1}{(P_{n-1}^\alpha)^k} \left\{\sum_{v=1}^{n-1} p_v^\alpha \lambda_v |s_v|\right\}^k$$

$$= O(1). \ \ as \quad m \to \infty; \ \ as \quad in \quad T_{n,1}^\alpha$$

Finally, as in $T_{n,1}^\alpha$, we have that

$$\sum_{n=1}^{m+1} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\beta(8k+k-1)} \left|T_{n,3}^\alpha\right|^k = \sum_{n=1}^{m} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\beta(8k+k-1)} \left|\frac{p_n^\alpha s_n \lambda_n}{P_n^\alpha}\right|^k$$

$$= O(1) \sum_{n=1}^{m} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\beta(\delta k+k-1)-k} |s_n|^k (\lambda_n)^k$$

$$\leq O(1) \sum_{n=1}^{m} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\beta(\delta k+k)-k} p_n^\alpha |s_n|^k (\lambda_n)$$

$= O(1).$ as $m \to \infty$; as in $T_{n,1}^\alpha$

Therefore, we get that

$$\sum_{n=1}^{\infty} \left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\beta(\delta k+k-1)} \left|T_{n,r}^\alpha\right|^k = O(1), \ as \quad m \to \infty, \ for \quad r = 1,2,3,$$

This completes the proof of the theorem.

## *REFERENCES*


[1] BOR, H. : $\left|\underline{N}, p_n\right|_k$ summability factors of infinite series; Tamkang J. Math, Vol. 16, (1985), 13-20.

[2] BOR, H. : Commentationes Mathematicae, XXVII, (1989), 171-174.

[3] BOR, H. : Pan-Amer Math J. 2, (1992), 33-38.

[4] BOR, H. : Demonstratio Math., XXV, (1992), 187-192.

[5] BOR, H. : On the local property of $|N^{-}, p_n; \delta|_k$ summability of factored Fourier series; J. Math. Anal Appl. Vol. 179, (1993), 644-649.

[6] CHEN, M. : Math. Res. Centre nat. Taiwan Univ. (1967), 114-120.

[7] SEYHAN, H. : Pan-American Math. J., Vol. 9(4), (1999), 23-27.

[8] SINGH, N. and SHARMA, N. : On $(\underline{N}, p_n^\alpha)$ summability factors theorem of infinite series; Bull-Call. Math. Soc. 92(1), (2000), 25-38.

[9] ZYGMUND, A. : Trigonometric series, Cambridge University Press, (1979).