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SOLVING DIOPHANTINE EQUATIONS WITH ADVANCED TECHNIQUES IN NUMBER THEORY: A COMPREHENSIVE STUDY OF PELL'S EQUATION AND ITS GENERALIZATIONS

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Abstract

There are some quadratic Diophantine equations that can be transformed into Pell's equation and that offer an endless number of essential solutions, yet not every one of them can be. Lagrange was the first to demonstrate that Pell's equation has a limitless reach in the event that d is a positive number that is not an ideal square. These solutions have a non-insignificant number of possible outcomes. It is a descriptive study wherein the theorems and a number hypothetical method are used to demonstrate the assertion. This essay seeks to research and discover answers to the summed up Pell's equation. To decide the solvability of the summed up Pell's equation, the study's procedure comprises a survey and discussion of previously published materials. There are several methods for solving Pell's equations, but the ones we viewed as most powerful in the summed up Pell's equation solutions were the continuous part strategy, PQa technique, LMM technique, and savage power search technique.

Keywords: *Diophantine, Techniques, Pell's Equation, Generalizations*

1. INTRODUCTION

As per mathematician Carl Friedrich Gauss, number hypothesis is the sovereign of mathematics and mathematics itself. The subject of number hypothesis, a subfield of unadulterated mathematics, concerns the idea of integers and, all the more specifically, the characteristics of regular numbers. It very well might be used in an expansive assortment of numerical and scientific contexts. Number hypothesis has so various subfields. The study of Diophantine equations by the Greek mathematician Diaphanous 250 of Alexandria during the third century is one of the conventional and significant branches of number hypothesis, and we think about necessary solutions. The two most notable antiquated number theorists were Euclid and Diaphanous of Alexandria. Euclid's commitment consists of thirteen volumes, three of which are about number hypothesis of the positive integers. Be that as it may, everything is written in a mathematical language. Euclid's contributions incorporate the characteristics of number divisibility, including the idea of odd and even numbers, as well as a strategy for deciding the greatest normal divisor of two numbers. He fostered the recipe for the sum of a limited mathematical progression and for every Pythagorean triple. He also proposed the possibility of an indivisible number and demonstrated that it must gap something like one of the two numbers in an indivisible number that divides the result of two integers.

The study of the characteristics of entire number hypothesis is a significant and captivating field of mathematics known as number hypothesis. Fantastic introductions to number hypothesis might be tracked down in several works. Gauss, who was viewed as the "Ruler of Mathematics," alluded to number hypothesis as the "Sovereign of Mathematics," and mathematics as the "Sovereign of the Sciences"

Diophantine was a third-century Hellenistic mathematician. One of the first to use arithmetical symbols was the mathematician Diaphanous of Alexandria, who researched Diophantine equations. A numerical assessment of Diaphanous issues was made by Diaphanous. A Diophantine equation is one for which basic or intelligent solutions are investigated. Logarithmic equations with at least one variables are the ones in particular that use polynomial expressions. Diophantine equations could have a single solution or a limitless number of them. Among the most remarkable illustrations of Diophantine equations are the Pythagorean and Pell's equations. A quadratic Diophantine equation is a

$$\boldsymbol{x^2 + pxy + qy^2 + rx + sy + t = 0} \qquad \textbf{(1)}$$

where the essential coefficients are p, q, r, s, and t. Mathematicians have been interested in whole number solutions to Diophantine equations The objective of studying a Diophantine equation is to decide whether a solution exists and, provided that this is true, the number of. As a result, equation (1.1's) Diophantine equation is a Cartesian conic with number solutions.

Let $\Delta = b^2 - 4ac$be (1.1)'s discriminator

Theorem 1.1 Assume that (Owings, 1970) the coefficients in (1.1) are objective integers with Then, at that point, if equation (1.1) has a single whole number solution, there are a limitless number of possible solutions, with the accompanying special case: • If equation (1.1) represents two basically silly straight lines, there is just space for a single whole number solution.

- Just whole number solutions are possible if equation (1.1) is a hyperbola with essentially judicious asymptotes.
- As a result, the conic of (1.1) turns into a hyperbola when 0. As a rule, the equation (1.1) can be condensed to the equation of Pell.

$$x^2 - dy^2 = N \qquad (2)$$

Sans square if x, y Z, N is a non-zero number, and d > 1. In the eighteenth hundred years, John Pell and were inspired to track down the number solutions to these equations

The number solution (x, y) has a limitless number of number solutions if x, y 6= 0. Assume that (x, y) = (u0, v0) is the answer to issue (1.2). As a result, (x, y) = (un, vn), where N, solves issue (1.2)

$$u_n = \frac{1}{2}\left[\left(u_0 + V_0\sqrt{d}\right)\left(p_0 + q_0\sqrt{d}\right)^{n-1} + \left(u_0 - V_0\sqrt{d}\right)\left(p_0 + q_0\sqrt{d}\right)^{n-1}\right] \in Z \qquad (3)$$

These answers can be discovered by applying the binomial hypothesis and building repeat relations. It is notable that, despite the fact that if equation (1.2) can be solved in terms of integers x and y, finding the basic answer may challenge.

Theorem 1.2 There exists an endless number of indispensable solutions if the equation x 2 dy2 = N contains one. Pell's equation, which simplifies the type of the Diophantine equation

$$x^2 - dy^2 = 1 \qquad (4)$$

The Pell's equation is known for N = 1 in (1.2) the first researchers of this equation were Brahmagupta (598-670) and Bhaskara (1114-1185)

Not Pell is credited with fostering the first hypothesis. On the off chance that d is a positive, flawed square as per Lagrange; Pell's equation has an endless number of solutions

1.1 Objectives of the Study

In the study, the solutions to summed up Pell's equations are investigated using the proceeding with portion, PQa, LMM, and Bruth-force search methods.

- To assess the summed up Pell's equations' solvability;
- To focus on the summed up Pell's equations' applications in commonsense contexts.

1.2 Signification of the Study

The judicious estimate to the square root, the sum of consecutive numbers, the subsequent Heronian triangles, the smooth number, the brilliant proportion, and PC encryption systems are a couple of instances of how proceeded with fractions are used in Pell's equation.

2. REVIEW OF LITREATURE

Most individuals consider the improvement of Euclid's strategy, which finds the largest normal variable of two integers, to be the start of continuous portion. The technique divides until there could be as of now not any residue.

Rafael Bombelli (Olds, 1963) had a go at using boundless continuous fractions to work out square roots. Rafael Bombelli demonstrated that the square base of 13 might be expressed as a continuous division in the sixteenth and seventeenth centuries. A couple of years after the fact, PietroCataldi accomplished the indistinguishable result with the square base of 18. In the eighteenth and nineteenth centuries, Euler, Lambert, Lagrange, and others established the hypothesis of continuous fractions. In his study of nonexclusive homogeneous second-degree Diophantine equations with two unknowns, Lagrange used proceeding with fractions. Gauss made the expansive hypothesis of quadratic forms to resolve several Diophantine problems. The present number hypothesis Diophantine approximations still utilize proceeded with fractions.

A "Diophantine" is a term used to describe the mathematician Diopphantus of Alexandria (Cooke, 1997). He studied these equations and was perhaps the earliest mathematician of the third hundred years to bring symbols into polynomial math (Austin, 1981). Diophantine Analysis is the study of Diophantine issues according to a numerical perspective. The improvement of nonexclusive theories of Diophantine equations was a significant improvement of the twentieth 100 years.

Mathematicians in middle age India fostered the first systematic research of methods for distinguishing basic solutions to Diophantine equations. Since the hour of Aryabhata (499 Promotion), systematic methods for finding whole number solutions to Diophantine equations have been established in Indian writing (Ansari, 1977).

Diophantusanalyzes issues connected with Pell's equation, and we might simplify Archimedes, which is a dairy cattle issue, and 8 to solve Pell's equation (Mahoney and Sean, 1994), despite the fact that there is no proof that Archimedes understood this connection. Brahmagupta solved more troublesome Diophantine equations in (Pingree, 2000). He inspected Pell's equation and offered a strategy for resolving second-request Diophantine equations like x 2 61y 2 = 1 in his Samasabhavana. The French mathematician Pierre de Organization offered this equation as an issue in 1657 because these techniques were obscure in the west (Mahoney and Sean, 1994).

Brahmagupta truly made the first commitment at or around Pell's equation. Fermat still tried to solve the Diophantine equation x 2 61y 2 = 1. This issue was at first solved by

Euler in the seventeenth hundred years. Other Diophantine equations were also solved by him.

Meanwhile, Bhaskara II established the answer to this issue some years prior using a changed variety of Brahmagupta's strategy (Pingree, 2000). He fostered the cyclic procedure, or chakravala in the language of the Indians, a methodology for solving Pell's equation x 2 dy2 = 1 start from any close pair (a, b) with a 2 db2 = 1. Bhaskara II is informed that the technique will end after a specific number of stages. This happens when the equation x 2 dy2 = N, where N is one of the integers 1, 2, or 4, is established. Bhaskara II gives instances in Bijaganita, one of which is x 2 61y 2 = 1. The following expansion to Pell's equation came from Narayana, who in the fourteenth century created a discourse on Bhaskara II's Bijaganita.

Equation (1.2) might be resolved using whole number solutions for a whole number N = n 2. It will change the worth of the variable (a 0, b0) = ((r/n),(s/n)). It is possible to compose the summed up Pell's equation as the Pell's equation with number answers, (a 0 k, b0 k). As a result, (ak, bk) = (nr 0 k, ns 0 k) are the whole number solutions to the summed up Pell's equation.

Lagrange's strategy (Pletser, 2013) of proceeded with fractions has been discussed by various authors (Lagrange, 1867; Matthews, 2000; Roberstson, 2004) with different modifications to track down the basic solution of the summed up Pell's equation as well as how to get extra solutions from the basic solution.

The glorious ideas of Brahmagupta, Bhaskara II, or Narayana seemed unfamiliar to seventeenth-century European mathematicians. At the point when Fermat tested mathematicians in Europe and Britain in 1657, this at first provoked their curiosity. The continuous fractions technique was made by Euler and refined by Lagrange to solve Pell's concern. 9 Lagrange released improvements to Euler's Elements of Variable based math in 1771, and one of those adjustments was a formalization of Euler's technique for deciding x 2 dy2 = 1 by using the continuous division strategy. There are an endless number of possible answers, and every one depends on the continuous portion expansion of d.

The specific Pell's equations had solutions discovered by Wallis and Brounker. In 1759, Euler presented the first universal solution (Burton, 1998). Hypothesis: All solutions to Pell's equation X2 P Y 2 = 1 might be found among the focalized division expansions of P, when P is prime, was first evolved by Lagrange (Burton, 1998). Lagrange demonstrated that each quadratic unreasonable continuous division expansion is intermittent over a specific point.

3. PELL'S EQUATION

In a 2009 article named "Proceeded with Part and Their Applications to Solving Pell's Equation (3.1)," Peter shown that the equation is always solvable and that its basic solution is associated with the convergents of the proceeding with portion expansion of d. They also reach the conclusion that some numerical issues with applications to true scenarios can be solved using Pell's equation.

Gregor Dick, an alternate number theorist, expounded on an alternate general type of Pell's equation in the structure (4.4), discussing it and the Lagrange technique for tracking down its solutions. He also presented an alternate way to deal with the issue that depended on the proceeded with portion expansion of d, which he does demonstrate is identical to the Lagrange technique.

3.1 Pell's Equation

L.J. Mordell [39] is the essential source for Diophantine equations. The so-called Pell's equation is the most significant of these equations. The type of the Diophantine equation

$$x^2 - dy^2 = 1 \qquad \textbf{(5)}$$

The Pell's equation, which is habitually alluded to as such and was at first researched by Brahmagupta and Bhaskara, is an equation where x, y are integers and d > 1, somewhat flawed square. It was given John Pell's name as a result of Euler's erroneous reference. It should have been called Fermat instead, as verified by L.E. Dickson in his History of the Hypothesis of Numbers. Fermat is a notable French number theorist. Be that as it may, as the advancement of mathematics demonstrates, several theorems about this equation were discovered well before Fermat's time by the prestigious Indian number theorist Brahmagupta, who was succeeded by Bhaskara II. Greek and Hindu mathematicians also inspected Pell's equation, as indicated by B.L.van der Waerden. Also, L.E. Dickson has referenced Brahmagupta's commitment. Number hypothesis was a flourishing field in India from 500 to 1400 A.D. Brahmagupta in the seventh 100 years, Bhaskara II in the twelfth hundred years, and NarayanaPanditha in the fourteenth century all made significant contributions. Through Arabs, the Brahmagupta and Bhaskara equation numerical ideas were brought to Europe. This equation was first alluded to as Pell's equation by Leonhard Euler, and since then, it has consistently been alluded to by that name in numerical writing. Numerous issues might be addressed using Pell's equation since it always has the simple answer (x, y) = (1, 0) and has limitless solutions. For resolving Pell's equations, Brahmagumpta and Bhaskara, two Indian mathematicians, created techniques. The C hakravala Strategy, presented by Brahmgupta, is a solution to Pell's equation. These equations were utilized to estimate the square foundation of 2 during Pythagoras' time

In general, if x2 − dy2 = 1, then

$$\frac{x^2}{y^2} = d + \frac{1}{y^2} \qquad (6)$$

As a result, for y enormous, x y is a close guess to d, which is known to Archimedes. Indian mathematician Baudhayana saw that x = 577 and y = 408 are solutions of the equation x 2 2y 2 = 1, and he used the part 577 408 to estimation 2. The Pell's equation, x 2 91y 2 = 1, had solutions in the seventh hundred years, with the lowest answer being x = 1151 and y = 120. The least-positive answer to the Pell's equation x 2 61y 2 = 1 was discovered by the Hindu mathematician Bhaskara in the twelfth 100 years.

x = 1776319049 and y = 2261590.

Numerous Indian mathematicians have inspected the Diophantine equation x 2 dy2 = 1, where d is a non-square positive whole number. From an equation's answer (x, y)

$$x^2 - dy^2 = \in = (\pm 1, \pm 2, \pm 4) \qquad (7)$$

For the scenario where = 1, Brahmagupta tracked down a solution (x0, y0) with x0 > x and inferred a limitless number of solutions for this case. The equation x 2 dy2 = 1 has no whole number solutions, as indicated by Bhaskara, unless d can be expressed as the sum of two squares [15]. Joseph-Louis Lagrange demonstrated the boundless number of solutions (x, y) for each d and gave a proper version of the continuous fractions technique to solve an equation (3.1). The 24 following hypothesis was established by Lagrange in 1768, who also demonstrated that there are a limitless number of solutions to the equation (3.1).

Without giving any proof, Fermat said in 1657 that Pell's equation had a limitless number of solutions if d > 0, not an ideal square.

For instance, in the event that the answer to equation (3.1) is (x, y), we acquire

$$1^2 = (x^2 - dy^2)^2 = (x^2 + dy^2) - (2xy)^2 d \qquad (8)$$

As a result, equation (3.1) also has a solution in (x 2 + dy, 2xy). In like manner, on the off chance that Pell's equation has a solution, it has a limitless number of solutions.

Fermat tested John Wallis and William Bouncer in 1657 to discover necessary answers to the equations x 2 151y 2 = 1 and x 2 313y 2 = 1. He advised them not to submit consistent answers since any degree of mathematician could think of them.

Wallis answered by

(1728148040, 140634693)

As a response to the first equation. Euler was seeking positive numbers m and n in 1770 such that

$$\frac{n(n+1)}{2} = m^2 \qquad (9)$$

He did this by adding 1 and increasing the two sides of the later equation by 8.

$$(2n+1)^2 = 8m^2 + 1 \qquad (10)$$

Assuming x = 2n + 1 and y = 2m, he determines that x 2 2y 2 = 1. This Pell's equation has solutions that result in square-three-sided numbers. because we have

$$\frac{\left(\frac{(x-1)\left(\frac{x-1}{2}+1\right)}{2} = \frac{y^2}{2}\right)}{2} \qquad (11)$$

At the end of the day, the (y 2)th square number equals the (x 1 2)th three-sided number.

Tracking down the Pell's equation (3.1) for a positive whole number d > 1 that is not an ideal square should be possible in various ways. Nonetheless, the most viable methodology for finding the basic answer is one that is based on the continuous portion increase of d. Similar to this, Williams and Tekcan also investigate specific whole number solutions to Pell's equation.

3.2 Solvability of Pell’s Equation with some Theorems, Algorithms and Examples

In the eighteenth hundred years, Lagrange gave the first exhaustive demonstration that Pell's equation (3.1) is always solvable in integers x and y with y 6= 0 by using the continuous part expansion of d.

Hypothesis 2.6's expansion of d into a proceeded with part, alongside the kth joined pkqk of the proceeding with portion expansion, forms the establishment for what we are going to describe.

$$\sqrt{d = [a_{0,} a_{1,} a_2 \dots a_{r,} - 1, a_{r,} 2_{a0}]} \qquad (12)$$

The repeat relations from Hypothesis 3.1 might be used to find the focalized points pk and qk.

Theorem 3.1 Let the continuous part of the reasonable number be [a0; a1; a2; a3; , an]. We specify

$$p_{-1}, p_{-2} = 0, q_{-1} = 0, q_{-2} = 1 \qquad (13)$$

Then

$$pk = a_k p_k - 1 + p_{k-2}, qk = akqk - 1 + qk - 2 \qquad (14)$$

Where p0, p1,...,pn are the numerators of the focalized of the proceeding with division and q0, q1,..., qn are the coefficients of the merged.

Confirmation

This hypothesis is shown through enlistment on k.

k = 0 results in;

$$C_0 = \frac{p_0}{q_0} = a_0 = \frac{a_0.1+0}{a_0.0+1} = \frac{a_0.p_{-1}+p-2}{a_0.p_{-1}+q-2} \tag{15}$$

When k=1 we have

$$C_1 = \frac{p_1}{q_1} = a_0 + \frac{1}{a_1} = \frac{a_0.a_1+1}{a_1.1+0} = \frac{a_1 p_0.0p-1}{a_1.q_0+q_{-1}} \tag{16}$$

$$\therefore p_1 = a_1.p_0 + p - 1, \quad q_1 = a_1.q_0 + q_{-1} \tag{17}$$

Hence

$$p_k = a_k p_{k-1} + p_{k-2,} q_k = a_k q_{k-1} + q_{k-2} \tag{18}$$

For k = 2, 3, , r, where i n, we presum that this theorem holds.

Then

$$C_k = \frac{p_k}{q^k} = \frac{a_k p_k - 1 + p_k - 2}{a_k p_k - 1 + q_k - 2} \tag{19}$$

4. GENERALIZED PELL'S EQUATION

4.1 Hilbert's Tenth Problem

The solvability of all Diophantine problems was the 10th of David Hilbert's renowned problems when it was suggested in 1900 [66]. The 10th on the list, sometimes known as Hilbert's 10th Issue, asked for a nonexclusive method for assessing whether Diophantine equations are solvable or unsolvable in integers. Make a strategy by which it very well not entirely settled by a predetermined number of operations on the off chance that the equation is solvable in integers. Hilbert's 10th Issue is a Diophantine equation with quite a few obscure quantities and fundamental mathematical coefficients.

The 10th test posed by Hilbert is to give a calculation that indicates whether a specific Diophantine equation has a solution.

4.2 Diophantine Equation

The expression "Diophantine" relates to the Hellenistic mathematician Diaphanous of Alexandria, who studied such equations and was quite possibly the earliest mathematician to integrate symbols into variable based math, and lived in the third century [6]. Diaphanous started the numerical study of Diophantine problems, which is known as Diophantine analysis. Finding numbers that simultaneously answer the equations is all expected for Diophantine problems, which have fewer equations than unknowns. judicious coefficient arithmetical equations that are searching for number or level-headed number solutions. In Diophantine equations, it is normally accepted that there are a larger number of unknowns than there are equations. They are also alluded to as uncertain equations as a result. The possibility of a Diophantine equation is also used in current mathematics to describe logarithmic equations whose solutions are sought in the arithmetical integers of specific arithmetical extensions of the field Q of reasonable numbers of the field of p-adic numbers.

The study of Diophantine equations lies between logarithmic math and number hypothesis. One of the earliest numerical issues is sorting out some way to solve equations with number answers. Diophantine equations foresaw several approaches to studying equations of the second and third degrees that wouldn't be totally evolved until the nineteenth century [60]. Diophantine equations are notoriously difficult to study.

Also, polynomials can be specified.

$$\boldsymbol{F(x, y_1, \dots . y_n)} \qquad \textbf{(20)}$$

With whole number coefficients such that no technique exists to decide whether the equation holds for some random worth of x.

$$\boldsymbol{F(x, y_1, \dots . y_n) = 0} \qquad \textbf{(21)}$$

Mathematicians in archaic India were the first to deliberately research methods for deciding necessary solutions of Diophantine equations, and they proceeded to study Diophantine equations exhaustively after that. Since the hour of Aryabhata, systematic techniques for finding number solutions to Diophantine equations have been reported in Indian writings [4].

The terms "Straight Diophantine Equation" and "Non-direct Diophantine Equation" allude to two distinct types of these equations. The straight Diophantine equation's conventional vital solution's first express description

$$\boldsymbol{ay + bx = c} \qquad \textbf{(22)}$$

Takes place in his work Aryabhatiya. One of Aryabhata's most significant contributions to the field of unadulterated mathematics is this calculation. Aryabhata used the strategy to give indispensable answers to simultaneous first-degree Diophantine equations, a test with significant astronomical implications.

An alternate illustration of a Diophantine equation is

$$x^2 + y^2 = z^2 \qquad (23)$$

The lengths of the small sides x, y, and the hypotenuse z of right-calculated triangles with vital side lengths are represented by positive essential solutions to this equation; these values are alluded to as Pythagorean numbers. The formulae give all triples of reasonably prime Pythagorean numbers.

$$x = m^2 - n^2, y = 2mn, z = m^2 + n^2 \qquad (24)$$

Where (m > n > 0) m and n are somewhat indivisible numbers.

In his Aritmetika, Diaphanous discusses the pursuit of sensible answers to specific classes of Diophantine equations.

5. CONCLUSION

The first researchers to analyze quadratic Diophantine equations were Lagrange, Legendre, and Gauss. They focused on classifying and solving the quadratic Diophantine equations using the correspondence rule and discriminate. The law of quadratic correspondence,discriminate, and other tantamount algorithms, in any case, are not sufficient for appropriately grasping the basic principles fundamental these issues because of the assortment and intricacy of problems of this sort. Arithmetical number hypothesis uncovered further normal patterns in these issues based on the masterpieces created by Gauss and Dirichlet.

In this work, the quadratic Diophantine equations' solutions were purposefully investigated, for certain findings coming from theorems and examples coming from a number hypothetical methodology. As a rule, it very well might be necessary to get new conclusions using mathematical, logical, topological, and number hypothetical approaches. Based on a survey and discussion of the previously accessible papers, the essential result of quadratic Diophantine equations is transformed into the pell's equation, which establishes the solutions in integers.

By using the continuous part of d, Euler demonstrated a more reasonable technique for solving Pellian equations in 1765; nonetheless, Lagrange was the first to demonstrate that solutions exist. He gave a straightforward way to deal with finding fundamental solutions to equation (4.4) in his notable memoirs. Lagrange's decrease is one more name for this procedure. The LMM calculation is a changed variety of this procedure. Numerous areas of mathematics have used the answer to Pell's equation. In its simplest structure, the series of fractions xi yi generally approximates d, where (xi, yi) is the I-th solution for a given not square regular d. There might be uses for Pell's equation in both estimation hypothesis and cryptography. Software programs are frequently utilized to streamline the check method.

REFERENCES

1. C. Baltus. Continued fraction and the Pell Equation: The work of Euler and Lagrange. Comm. Anal. Theory continued fraction, 1994.
2. D.M. Burton. Elementary Number Theory. Allyn and Bacon, Inc., 1980.
3. Dubickas and J. Steuding. The polynomial Pell equation. Ele. Mathe., 59:133–143, 2004.
4. Dutta and A.N. Singh. History of Hindu Mathematics: A Source Book. Asia Publishing House, Bombay, India, 1962.
5. E.J. Barbeau. Pell’s Equationh, Problem Books in Mathematics. Springer-Verlag, New York, 2003.
6. L.E. Dickson. History of the Theory of Numbers, volume III. Chelsea Publishing Co., New York, 1952. 83
7. L.E. Dickson. Introduction to the theory of numbers. J. Theor. Nr. Bordx., 14:257–270, 1957.
8. M.M. Austin. The Hellenistic World from Alexander to the Roman Conquest: A Selection of Ancient Sources in Translation. Cambridge University Press, 1981
9. S. Chowla. On the class number of real quadratic field. USA: Proceedings of the National Academy of Sciences, 1982.
10. S.M.R. Ansari. Aryabhata I, his life and his contributions. Bull. Astro. Soc., 5(1):10–18, 1977.
11. S.P. Arya. On the brahmagupta-bhaskara equation. Math. Ed., 8(1):23–27, 1991.
12. T. Andreescu and D. Andrica. 360 Problems for Mathematical Contests. GIL Publishing House, Zalau, 2003.
13. T. Andreescu and D. Andrica. Note on a general negative Pell’s equation. Octo. Math., 10(2):703–705, 2002.

14. T. Andreescu, D. Andrica, and I. Cucurezeanu. An Introduction to Diophantine Equation: A Problem-Based Approach. Springer, New York, 2010.
15. Y.T. Cheng. Continued Fractions. Cornell University, AMS Classification, 11A55, 2007.