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Exponential Integration and Examination of Related Special Functions for general q-exponentials and missing gamma functions

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Abstract
It is important that we continue with specific work involving probability, statistics and combinations and be able to offer some products. It can be derived from the general gamm a property that is close to the normal gamma characteristic with a given value of q, for example q = 0.9, the larger the q value changes.

Ein(z) = $\int_0^z \frac{1-e^{-t}}{t}$dt = $\sum_{n=1}^{\infty} \frac{(-)^{n-1}z^n}{nn!}$ (z ∈ C)(1)

And it's a whole function. Its link to the conventional exponential integral ε_1(z)=$\int_z^{\infty} t^{-1}e^{-1}$dt, valid in the cut plane | arg z| < π, [1]

Ein(z) = log z + γ +ε_1(z), (2)

γ = **0.5772156. .. yog** Euler-Mascheroni **tas mus li.**

Mainardi and Masina recently **proposed an extension** to Ein(z) by **replacing** the exponent ial function (1) with **a** Mittag-Leffling **argument.**

E_α(z) = $\sum_{n=0}^{\infty} \frac{z^n}{T\ (\alpha n+1)}$(z ∈ C, α> 0),

generalizing the exponential feature e^z. The function for every α> 0 was introduced in the cutting plane | arg z| < π

Ein_α(z) = = $\int_0^z \frac{1-E_\alpha(-t^\alpha)}{t^\alpha}$dt = $\sum_{n=0}^{\infty} \frac{(-)^n z^{\alpha n+1}}{(\alpha n+1)T\ (\alpha n+\alpha+1)}$, (3)

This simplifies the **Ein** function when α = 1 (z). **With** 0 ≤ α ≤ **1, this function** can be used physically when **examining** the properties of **a** linear viscoelastic creep model. He **did** sim ilar **work for** sine and cosine integrals. **The graph** of all these functions α ∈ [0, 1] **is** given **as:**

$E_{\alpha,\beta}$(z) = $\sum_{n=0}^{\infty} \frac{z^n}{T\ (\alpha n+\beta)}$(z ∈ C, α> 0),

where β **will** be **accepted as true. We'll then explore** the **next step in the evaluation pro cess.**

$$Ein_{\alpha,\beta}(z) = \int_0^z \frac{1-E_{\alpha,\beta}(-t^{\alpha})}{t^{\alpha}}dt = \sum_{n=1}^{\infty} \frac{(-)^{n-1}}{T\ (\alpha n+\beta)} \int_0^z t^{\alpha n-\alpha}\ dt$$

$$= z \sum_{n=0}^{\infty} \frac{(-)^n z^{\alpha n}}{(\alpha n+1)T\ (\alpha n+\alpha+\beta)} \quad (4)$$

When n−1 in the final **sum** is replaced by n. If β = **1,** this **reduces** to (3)

Einα,1(z) = Einα(z). **Section**

This **can** be **done using** the **concept** of hypergeometric **integral** functions **described** in **ref .** 〖Ein〗_(α,β)(x) for x → +∞ when α ∈ (0, 1] **The expansion is a** logarithmic **expression e ach result**

$\boldsymbol{\alpha} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\dots\dots.$

LITERATURE REVIEW

KwaraNantomah (2021) HAL is an open, multidisciplinary repository for the storage and s haring of scientific information, whether published or not. Participation can come from Fre nch or foreign universities and research or from public research centers or businesses. The HAL Open Multidisciplinary Archive aims to preserve and publish scientific and unpublis hed research documents from French and foreign universities and research, public and/or p rivate clinics.

Michael Milgram (2020) The two notations combine to form a Riemann combination, the f unction ξ(s) and hence the inverse ζ(s). The equation has several principles, the most impor tant of which is the value of flux ζ(s) anywhere on a straight line in the complex plane on t he specified strip.Both of these are descriptive ζ (σ + it), asymptotic everywhere (t → ∞), c ritical scores and approximate solutions can be obtained in the Riemann Hypothesis, which is true. This solution promises a simple but powerful connection between the real and ima ginary equations μ and t.

In this article, Francesco Mainardi and Enrico Masina (2018) examine the Schelkunoff tran sform properties and expand them using the Mittag-

Leffler function. We obtain a new property that may be important for linear viscoelasticity because of its excellent monotony. We will also review general sine and cosine functions. Francesco Mainardi (2018) We propose a new rheological model in terms of unreal ν∈[0,1] that reduces Maxwell's body to ν=0 and Becker's body to ν=1. The relevant creep laws are given and the exponential function of the Becker model is replaced and extended by the Mittag-Leffler order function. To see that the transition from Maxwell body to Becker body is a function of time, the interaction with detonation and absence of velocity is then investigated for¨difference. In addition, we can estimate the relaxation function by numerically solving the quadratic Volterra equation according to classical linear viscoelasticity theory.

IvanoColombaro, (2017) In this paper, we explore various linear viscoelastic models expressed in the Laplace domain using properly scaled continuous Bessel forces.
The Dirichlet series demonstrates these functions over time. The remaining modules and their combinations lead to endless, invisible delays and rest times. In fact, we get the viscoelastic class as the argument ν>1. Such models have rheological properties similar to the Maxwell fractional model (of the order of 1/2) in the short run and similar to the Maxwell model in the long run.

THE EXPONENTIAL INTEGRAL AND ITS FUNCTION

We can start directly with the mathematical structure of the exponential integral function.

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t}\, dt \quad x \in \mathrm{W}^1$$

Then we can see that the exponential integral is defined as a combination of a certain expression and the parametric integral.
You can use the Risch algorithm to prove that this factor is not a prime function, that is, there is no Ei(x) factor in the basis function. Such a function has a pole at t = 0, so we define this integral as Cauchy's critical value:

$$\mathrm{Ei}(x) = \lim_{\alpha \to 0} \left[\int_{-\infty}^{\alpha} \frac{e^t}{t}\, dt + \int_{\alpha}^{x} \frac{e^t}{t}\, dt \right]$$

We can better define $\mathrm{Ei}(x)$ in a parity transformation

$$\begin{cases} t \to -t \\ x \to -x \end{cases}$$

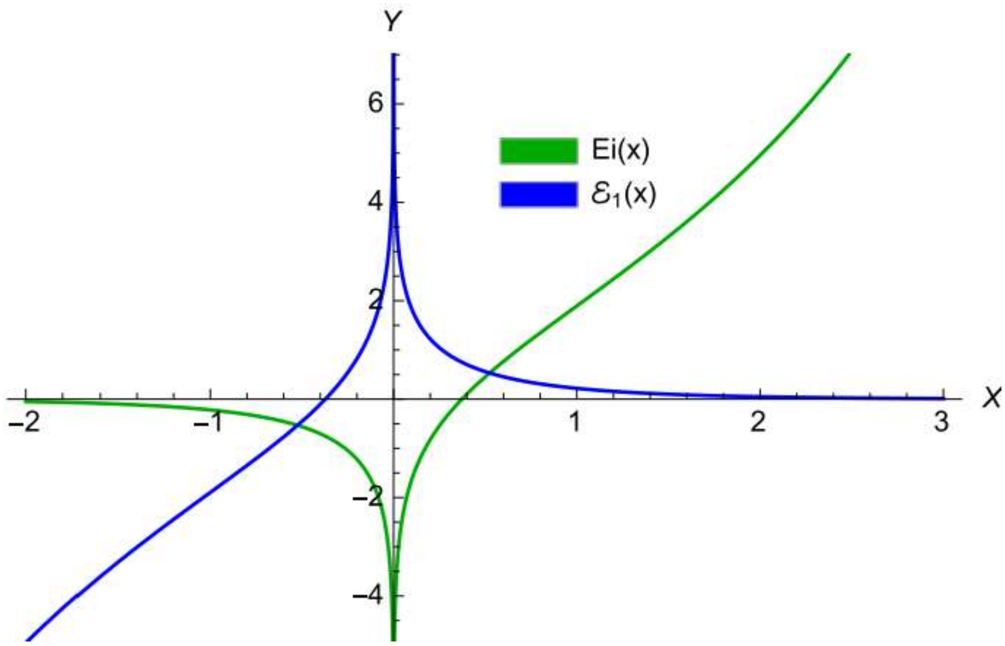
one gets

$$\mathcal{E}_1(x) = -\mathrm{Ei}(-x) = \int_x^{+\infty} \frac{e^{-t}}{t}\, dt$$

If the argument accepts complex values, the definition will be confusing as the branch is be tween 0 and ∞. By expressing the difference in the variable z = x + y, we can adjust the ex ponential terms in the complex plane using the following expression:

$\mathcal{E}_1(z) = \int_z^{+\infty} \frac{e^{-t}}{t}\, dt$ | arg(z) < π

The **diagram below will help** to understand the graphical behavior of **these** two functions.



We can immediately write down some useful known values:

Ei(0) = −∞ Ei(−∞) = 0 Ei(+∞) = +∞

$\mathcal{E}_1(0) = +\infty$ $\mathcal{E}_1(+\infty) = 0$ $\mathcal{E}_1(-\infty) = -\infty$

It's actually simple to find the values of $\mathcal{E}_1(x)$ from Ei(x) (and vice versa) by using the previously written relation:

$\mathcal{E}_1(x) = -\mathrm{Ei}(-x)$

We notice that the function $\mathcal{E}_1(x)$ is a monotonically decreasing function in the range (0, ∞). The function $\mathcal{E}_1(z)$ is actually nothing but the so-called Incomplete Gamma Function:

$$\mathcal{E}_1(z) \equiv \mathrm{T}(0, \mathrm{z})$$

Where

Γ (s, z) = = $\int_z^{+\infty} t^{s-1} e^{-t}\ dt$

Indeed, by putting s = 0 we immediately find $\mathcal{E}_1(z)$.

By introducing the small Incomplete Gamma Function

$$\gamma(s\,, z) = \int_0^z t^{s-1} e^{-t}\ dt$$

We can put down a very clear, sometimes helpful and direct relationship between the three Gamma functions:

T (s, 0) = γ (s, z) + T (s, z)

Now let's return to the Exponential Integral.

Let's make a naïve variable change

t → z u dt = z du

Step by step we get:

$$\int_z^{+\infty} \frac{e^{-t}}{t}\ dt \ \rightarrow \int_1^{+\infty} \frac{e^{-zu}}{zu} z\ du = \int_1^{+\infty} \frac{e^{-zu}}{u} du$$

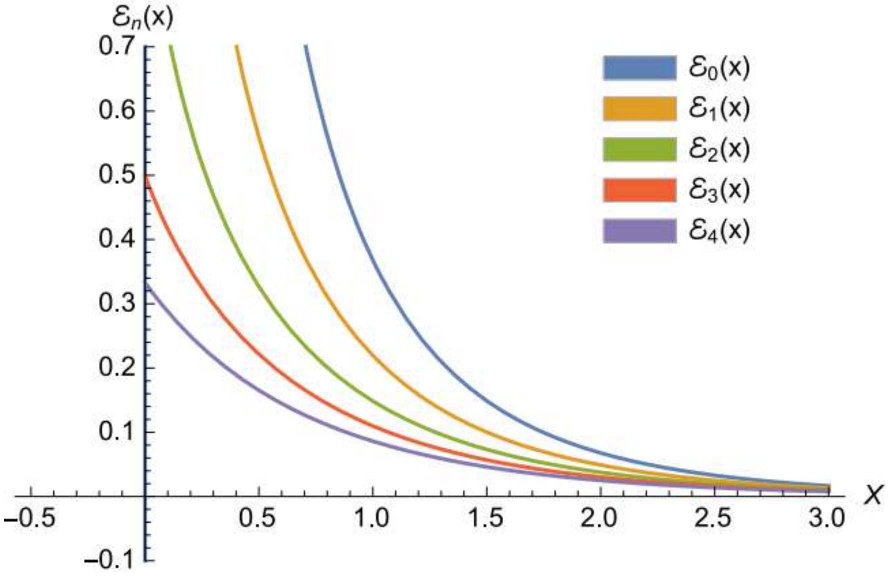
In this way we define the General Exponential Integral:

$\mathcal{E}_n(z) = \int_1^{+\infty} \frac{e^{-zu}}{u^n} du$n ∈R

with the particular value

$\mathcal{E}_n(0) = \frac{1}{n-1}$

We can show the behavior of the first five functions $\mathcal{E}_n(x)$, namely for n = 0,... , 5.



GENERALIZED GAMMA FUNCTION

Definition of the generalized gamma function by integral, by means of the q-exponential distribution

$\Gamma q(z+1) = \int_0^{\infty} x^z e_q{}^x \, dx,$ **(5)**

Where q∈(0,1] and z∈ℂ and$\Re e\{z\}>0$. In the limit q→1, we have

Γq(z)=Γ1(z)=Γ(z) and Γ(n)=(n−1)!

For n∈ℕ and Γ(z+1)=zΓ(z) for z∈C.

Since$e_q{}^{x-1} \ll x^{z-1}$,when we can write x is positive and x is(0,1)

$$\left|\int_0^1 e_q{}^x x^{z-1} \, dx\right| < \left|\int_{\epsilon}^1 x^{z-1} \, dx\right| = \frac{1}{z} - \frac{\epsilon^z}{z} \quad (6)$$

and the integral for x>0 for 1/x is restricted.

By fixing and reducing x the integral value grows monotonously, i.e.

$$\int_0^1 e_q{}^x x^{z-1} \, dx = \lim_{\epsilon \to 0} \int_{\epsilon}^1 e_q{}^x x^{z-1} \, dx, \exists \forall_x > 0 \quad (7)$$

${e_q}^{ix}$ Presents the properties$\left[{e_q}^{ix}\right] = {e_q}^{-ix}$,q-exponential functions are deformed by means of a real parameter q in the conventional exponential function

$$ {e_q}^{x} = \begin{cases} [1+(q-1)x]^{\frac{1}{q-1}}, & -\infty < x \leq 0, \\ [1+(1-q)x]^{\frac{1}{1-q}}, & 0 \leq x < \infty, \end{cases} \quad (8) $$

The opposite of the q-exponential functions is the lnq(x) function, defined as the q-logarithm

$$ \mathrm{Inq}(x) = \begin{cases} \frac{x^{q-1}-1}{q-1}, & 0 < x \leq 1, \\ \frac{x^{1-q}-1}{1-q}, & 1 \leq x, < \infty. \end{cases} \quad (9) $$

The formulation of equation (8) can only be used for q∈(0,1) and in this term x and q are se parated mathematically. There are two equivalent ways to derive the exact term: one is to r eplace all terms using the range q ∈ (0,1). As in the model above, consider for a moment th e expression and deformation parameter changes.

$$ {e_q}^{x} = [1+(1-q)x]^{\frac{1}{q-1}}, \begin{cases} -\infty < x \leq 0, & q \in [1,2), \\ 0 \leq x < \infty, & q \in (0,1]. \end{cases} $$

$$ \mathrm{Lnq}(x) = \frac{x^{1-q}-1}{1-q} \begin{cases} 0 < x \leq 1, & q \in [1,2), \\ 1 \leq x < \infty, & q \in (0,1]. \end{cases} \quad (10) $$

The parameter q is the non-additive degree. Therefore we get a generalized gamma function in the integral equation (5) and utilizing the concept of q-exponential

$$ \Gamma q(p+1) \quad (11) $$

$$ = \frac{p(p-1)(p-2)(p-3)\times\cdots\times[p-(p-1)]}{(2-q)(3-2q)(4-3q)(5-4q)\times\cdots\times[p+2-(p+1)q]} \int_0^{\infty} ({e_q}^{-x})^{(p+2)(1-q)+q}\, dx, $$

Where Γ(p+1)=p!For p∈N and Γ (z+1)=zΓ(z), z∈C. Thus, we followed the recurrence relation for the generalized gamma function provided by the standard factor function

$$ T_{q}(z+1) = \frac{z\Gamma(z)}{\prod_{j=1}^{p}[j+2-(j+1)q]}, \quad (12) $$

As a result, we may get the q-factorial expression, [p]q!

$$[p_q]! = \frac{p!}{\prod_{j=1}^{p}[j+2-(j+1)q]} \qquad (13)$$

Where p∈N.

We also get the incomplete gamma functions

$$T_{q}\ (a\ ,x) = \int_{x}^{\infty} z^{a-1} {e_q}^{z}\ dz. \qquad (14)$$

$$\gamma_{q}\ \ (a\ ,x) = \int_{0}^{x} z^{a-1} {e_q}^{z}\ dz, \qquad (15)$$

With $\Re$e(a)>0, where

$$T_{q}\ (a\ ,x) + \gamma_{q}\ \ (a\ ,x) = T_{q}\ (a). \qquad (16)$$

We have the following generalized features, if the unfulfilled gamma function is involved.

$$\mathrm{erfc}_q(x) = \frac{1}{\sqrt{\pi}}\ \gamma q\ (1/2, x^2) \qquad (17)$$

This is the generalized additional error function

$$E_{qn}\ (x) = \int_{1}^{\infty} \frac{{e_q}^{-xt}}{t^n}\ dt, \qquad (18)$$

If the exponential integral function is defined generalizedEq1(x)=−Eqi(−x) as

$$E_{q1}(x) = \int_{-\infty}^{x} \frac{{e_q}^{t}}{t}\ dt. \qquad \textbf{(19)}$$

Figure 1 shows the general gamma function function q for q = 0.9 and the standard gamma function
function corresponding to the case q = 1. Q(z) different plots with q value, as shown below
.

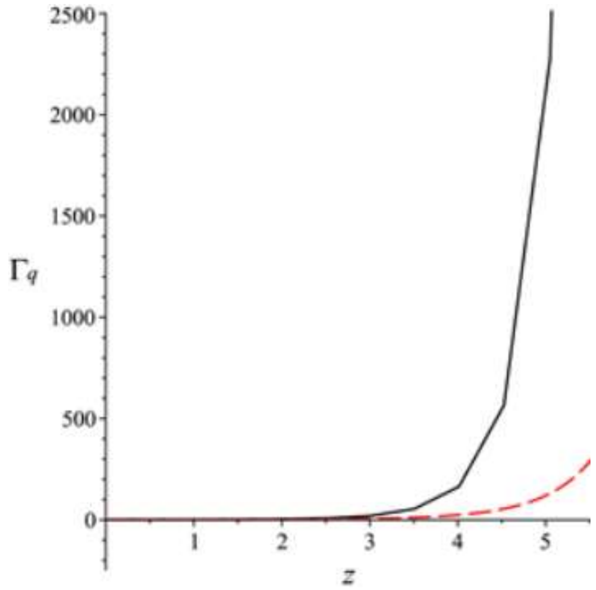


Figure 1**. Plot of the generalized gamma function Γq(z)**

In illustration 2. This is because the q-exponential function represents the function family (one for each q inside the interval (0,1) while the q=1(ex) situation only corresponds to one exponential function of the type of q. The q-gamma function indicates an approach nearer than q-exponentials to ordinary exponential for various q values.

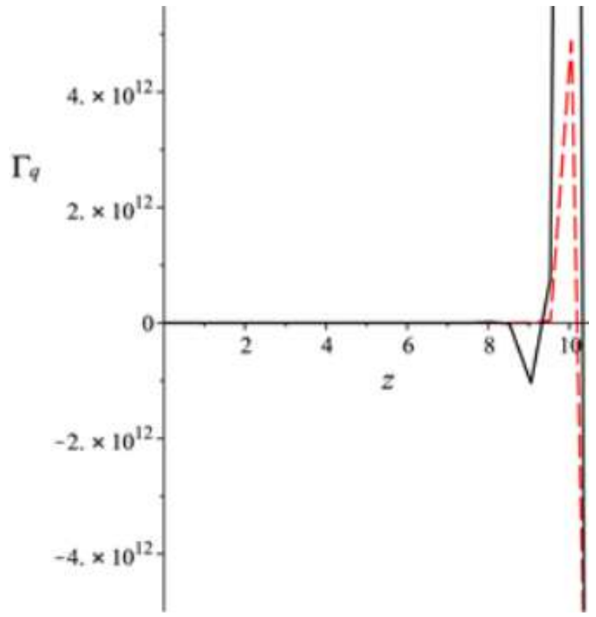


Figure 2**. Plot of the generalized gamma function Γq(z) for multiple q values**

The generalized incomplete q- gamma function is provided as a result of the q-exponential definition.

$$T_{q\ (a\ ,x)}= \frac{x^{a-1}}{2-q}\left(e_q{}^{-x}\right)^{2-q} + \frac{a-t}{2-q}\ Tq\ (a-1,x) \qquad \textbf{(20)}$$

And

$$\gamma_{q}\ (a\ ,x) = \frac{x^{a-1}}{q-2}\left(e_q{}^{-x}\right)^{2-q} + \frac{a-t}{2-q}\gamma q\ (a-1,x). \qquad \textbf{(21)}$$

Furthermore, the exponential integral function is also available generally

$$E_{q1}(x) = \frac{1}{2-q}\left(e_q{}^{x}\right)^{\frac{2-q}{1-q}}. \qquad \textbf{(22)}$$

Finally, we have a widespread integrated logarithm

$li_q\ (x) = E_q i\ (In_q\ (t))$, given by

$$li_{q1}\ (x) = \int_0^x \frac{dt}{In_q\ (t)}. \qquad \textbf{(23)}$$

Consequently, we get $li_q\ (x)$ given as

$$li_{q1}\ (x) = \int_0^x \frac{dt}{In_q\ (t)} = (1-q)\int_0^x \frac{dt}{t^1 - q - 1}$$

$$= -\int_0^x dt \sum_{n=0}^{\infty} (t^1 - q)^n$$

$$= -\sum_{n=0}^{\infty} \frac{x^{n(1-q)}}{n(1-q)} \qquad (24)$$

where $|x|<1$.

CONCLUSION

A seemingly new expansion for the exponential integral E1 in the gamma function is given and more expansions are shown. This second extension is presented here as "the multiplic ation theorem". The additional gamma function has an additional undetected effect and can be used to create a continuous link E1 with multiple parameters. A general procedure is de scribed for converting a power series to a gamma expansion.

REFERENCE

1. Olver, F.W.J.; Lozier, D.W.; Boisvert, R.F.; Clark, C.W. (Eds.) NIST Handbook of Mathematical Functions; Cambridge University Press: Cambridge, UK, 2010.
2. Mainardi, F.; Masina, E. On modifications of the exponential integral with the Mittag-Leffler function. Fract. Calc. Appl. Anal. 2018, 21, 1156–1169.
3. Mainardi, F.; Masina, E.; Spada, G. Generalization of the Becker model in linear viscoelasticity: Creep, relaxation and internal friction. Moss. Time dependent. Mater. 2019, 23, 283–294.
4. Paris, R.B.; Kaminski, D. Asymptotics and Mellin-Barnes Integrals; Cambridge University Press: Cambridge, UK, 2001.
5. Abramowitz, Milton; Irene Stegun (1964). Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables. Abramowitz and Stegun. New York: Dover. ISBN 978-0-486-61272-0.,
6. KwaraNantomah. Harmonic mean inequality for the exponential integral function. 2021. ffhal03220402f
7. Michael Milgram, "An Integral Equation for Riemann's Zeta Function and its Approximate Solution", Abstract and Applied Analysis, Vol. 2020, Article ID 1832982, 29 pages, 2020. https://doi.org/10.1155/2020/1832982
8. Mainardi, Francesco and Masina, Enrico. "On Modifications of the Exponential Integral Using the Mittag-Leffler Function: " Fractional Calculus and Applied Analysis, Vol. 21, No. 5, 2018, pp. 1156-1169. https://doi.org/10.1515/fca-2018-0063
9. F. Mainardi, E. Masina and G. Spada, Generalization of the Becker model in linear viscoelasticity: Creep, relaxation and internal friction. Mechanics of time-dependent materials, Publ. online 2/02/2018, p. 12

10. Colombaro, I., Giusti, A. & Mainardi, F. A class of linear viscoelastic models based on Bessel functions. Mechanica 52, 825–832 (2017).

11. Umarov, S., Tsallis, C. and Steinberg, S. (2008) Milan Journal of Mathematics, 76, 307-328.

12. Oikonomou, T. and Gagci, G.B. (2010) Physics Letters A, 374, 2225-2229.

13. Tsallis, C. (1988) Journal of Statistical Physics, 52, 479-487.

14. Jauregui, M. and Tsallis, C. (2011) Physics Letters A, 375, 2085-2088.

15. Komarov, M.A. (2018) (2019) Journal of Approximation Theory, 240, 126-128.