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TOPOLOGICALLY CLOSED POSITIVE CONE

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ABSTRACT

In this paper a vector lattice E containing subset B of E+ is studied. It is proved that if there exists a Lebesgue linear topology T on E and E+ is T-closed then minimal lattice-subspaces with T-closed positive cone exist.

Key words :Hausdorff topological space, vector lattice, Banach lattices.

1. Introduction

In 1966, Polyrakis [5] has studied, supposed that B = {Xl, x2, x } is a finite subset of C+(Q), where Q is a compact, Hausdorff topological space, the functions Xi are linearly independent and the existence.

In the present paper, the existence of minimal lattice-subspaces of a vector lattice E which contains a

subset B of E+ is studied. In the theory of Banach lattices (and in epplications) we are interested in a latticesubspace of E containing B which is as "close" as possible to the linear subspace [B! generated by B. Such a subspace is the sublatticeS(B) generated by B.

It is to be noted that lattice-subspaces have been employed in economics [2], [3].

Let E be a (partially) ordered vector space with positive cone E+ and Xa subspace of E. The cone X n E_+ will be called the induced cone of X, and the ordering defined in X by this cone the induced ordering. We will denote by X+ the induced cone ofX, i.e., X+ = x n E+. An ordered subspace of E is a subspace of E ordered by the induced cone. A lattice-subspace of E is an ordered subspace of E which is also a vector lattice (Riesz space).

Let X be a lattice-subspace of E. Then, for each x, y e X we will denote by x A y (resp. x v y) the supremum (resp. infimum) of {x, y} in X. It is clear that

x v y < x v y and x A y K x A y.

whenever x v y, x A y exist. If E is a vector lattice and x v y = x v y for any x, y e X then X is a sublattice (Riesz subspace) of E. Let E be an ordered Banach space with positive cone E+. A sequence {en} is a positive basis of E if {en} is a (Schauder) basis of E and E+ = {x = $\sum_{i=1}^{\infty} \lambda_i e_i | \lambda_i$ e R + for each i}. A positive basis {en} of E is unique (in the sense of a positive multiple). The following result (see [l] or [12]) is very important for the study of finite-dimensional lattice-subspaces.

2. Minimal Lattice-Subspaces

Let E be a vector lattice and B c E+, B # 4. Let L be the set of lattice-subspaces of E, each of which contains B. IfX e L and for any Y e L it holds .

$$Y \subseteq X \Rightarrow Y = X,$$

then we will say that X is a minimal lattice-subspace of E containing B.

If E is a vector lattice, then the sublattice generated by B is the minimum sublattice containing B.

Even if E = IRm a minimum lattice-subspace of E containing B does not always exist. So we state the following question :
Problem 1.1 Does a minimal lattice subspace of E containing B exist?

Let P be a cone of a linear space F (i.e. P is a convex subset of F. Rx e P for each x e P and e R+ and pn(-P) = {O}. Suppose thatx,y e P. If there exists z e P with the properties : z -x, z -y e P and for each M' e

p, w - x, w -y e P imply that w - z e P, then we will say that z is the supremum of {x, y} in P and we will denote

z = sup p{x, y}.

The infimum of {x, y} in P is defined analogously. If for each x, y e P, z = sup p{x, y} exists, then inf ply} also exists.

If P is a cone of a linear space F and for each x, y e P the supremum of {x, y} exisis in P, then we will say that P is a lattice cone of F.

If x = — x2 where Xl, x2 e P, then it is easy to show that sup{x, O} — sup p{Xl, x2)} - •s the supremum of {Xl, x2} in X. Therefore the following result holds.

A cone P of a vector space F is a lattice-cone if and only if the subspace X, ordered by the cone P, is a vector lattice.

In the next results of this paragraph we will suppose that E is a vector lattice equipped with a linear topology T with the properties :

(l) E+ is T-closed;

(ii) each increasing, order bounded net of E has a t-convergent subnet (i.e., ' the topology T is Lebesgue).

Property (i) implies also that Tis Hausdorff because if we suppose that x e E, x 0 and 0 e x for each open symmetric neighborhood V of zero, then 0 e -x + V; therefore x and -z belong to E+ and hence x = 0, contradiction.

If the topology T is order continuous (i.e., each decreasing net of E with infimum zero is T-convergent to zero) and E is Dedekind complete, then T satisfies (ii). If the order intervals of E are T-compact, the statement (ii) is also satisfied (for related results see [4, Theorem 10.13]). Hence, the weak star topology of a dual Banach lattice and the weak topology of a Banach lattice with order continuous norm [4, Theorem

11.9], have property (ii).

Proposition 1.2. Let (Pi)iel be a decreasing net of t-closed lattice cones of E+ (i.e., P c and i< j Pi ⊇P). Then P =   is a T-closed lattice cone of E.

Proof. P is a T-closed cone of E+. Let x, y e P. Denote by the supremum of {x, y} in Pi. For each i, j e I with i< j we have Pj c Pi c E+', therefore,

$$x, y \leq z_i \leq z_j$$

Since T has property (ii), there exists a T-convergent subnet of (Zi)iel which we will still denote by (Zi)iel. This net is also increasing, and let z = limielZi. Then for each j e I with i< j, we have: Zi, - X, -y e Pjc Pi.

Since the cone Pi is T-closed, we have that z, z - x, z -y e Pi,

for each i e I.

Therefore

$$z, z - x, z - y \in P.$$

≤

Suppose that w e P with w - x, w -y e P. Since P C Pj we have that w - zj e Pj c Pi for each j e I with i

j. Hence w - z e Pi for each i; therefore w - z e P. So we have proved that z = supp {x, y}; therefore P is a

lattice cone.

Theorem 1.3. Let P c E+ be a cone and let O(P) be the set of t-closed lattice cones of E+ each of which

contains P. Then O(P) has minimal elements.

Proof. O(P) because E+ e O(P) and O(P), ordered by the relation "D", is a partially ordered set. Suppose that F is a totally ordered subset of (Þ(P). Then by the previous result Q = A is a T-closed lattice cone of E. By Zorn's Lemma the theorem is true.

Proposition 1.4. Let (Xi)iE/ be a decreasing net of lattice-subspaces of E with T-closed positive cones. Let X = nielxi, Y = X- and = Y n E+. Then

(i) & =ni€/x:.

(ii) Y c X, Y+ and Yis a lattice-subspace of E with T-closed positive cone.

Proof. (i) X+ = x n = n rile/ x+.

(ii) Y=X-X+ c x. cxn =X+. Also $Y_+ = X_+ - \{0\}$c Y,

therefore X+ c Y+. Hence X+ = Y+. The net (XI+)IEI is a decreasing net of T-closed lattice cones of E+•, therefore Y+ is a T-closed lattice cone. Hence Y, is a lattice-subspace of E.

Theorem 1.5. Let B c E+ and I(B) = {Yc E I Yis a lattice-subspace, Y+ is T-closed and B c Y}. Then I(B) has minimal elements.

Proof. The set I(B) is nonempty because it contains E. The set I(B). ordered by the relation "2", is a partially ordered set. Let F be a totally ordered subset of I(B). By the previous proposition there exists Y e I(B) such that Yc A for each A e F. Therefore, by Zorn's Lemma I(B) has minimal elements.

Corollary 1.6. Let E be a Banach lattice with order continuous norm and B c E+. Then the set of latticesubspaces of E with (norm) closed positive cone which contains B has minimal elements.

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