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Generalization and sharpening of some inequalities for Polynomials

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Abstract

The goal of this paper is to establish some results for the polar derivative of a polynomial in the plane that are inspired by a classical result of Turán that relates the sup-norm of the derivative on the unit circle to that of the polynomial itself under some conditions. The obtained results sharpen as well as generalize some known estimates that relate the sup-norm of the polar derivative and the polynomial. Moreover, some concrete numerical examples are presented, showing that in some situations, the bounds obtained by our results can be considerably sharper than the previous ones known in very rich literature on this subject.

Keywords: Polynomials, Inequalities, Zeros.

Mathematics Subject Classification: Primary 30A10, 30C10, Secondary 30D15.

1. INTRODUCTION

The study of extremal problems of functions and the results where some approaches to obtaining polynomial inequalities for various norms and with various constraints on using different methods of the geometric function theory is a classical topic in analysis. The Erdös-Lax and Turán-type inequalities relating the norm of the derivative and the polynomial itself as well as generalizing the classical polynomial inequalities play a key role in the literature for proving the inverse theorems in approximation theory, and of course have their own intrinsic interest. These inequalities for constrained polynomials have been the subject of many research papers which is witnessed by many recent articles (for example, see [4], [5], [6], [8], [9] and [10])

According to well-known inequality of Bernstein [2] on the derivative of a polynomial $P(z)$ of degree n, we have

$$\max_{|z|=1}|P'(z)| \le n\max_{|z|=1}|P(z)|. \qquad (1.1)$$

The result is best possible and equality holds for a polynomial having all its zeros at the origin.

The above inequality (1.1) can be sharpened, if the zeros of the polynomial are restricted. In this direction, Erdös conjectured and later Lax [7] proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then

$$(1.2)\max_{|z|=1}|P'(z)| \le \frac{n}{2}\max_{|z|=1}|P(z)|$$

The inequality (1.2) is best possible and equality holds for $P(z) = a + bz^n$, where $|a| = |b|$. As an extension of (1.2), Malik [10] proved that if $P(z) \neq 0$in $|z| < k, k \ge 1$, then

$$(1.3)\max_{|z|=1}|P'(z)| \le \frac{n}{1+k}\max_{|z|=1}|P(z)|$$

For the class of polynomials not vanishing in $|z| < k, k \le 1$, the precise estimate of maximum $|P'(z)|$ on $|z| = 1$ is not easily obtainable. For quite some time it was believed that if $P(z) \neq 0$in $|z| < k, k \le 1$, then the inequality analogous to (1.3) should be

$$(1.4)\max_{|z|=1}|P'(z)| \le \frac{n}{1+k^n}\max_{|z|=1}|P(z)|.$$

However, Professor Saff gave the example $P(z) = \left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)$ to counter this belief. In 1980, it was shown by Govil [4] that (1.4) holds with an additional hypothesis and proved the following result.

Theorem 1.1. Let $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ be a polynomial of degree n having no zero in $|z| < k, k \le 1$and $Q(z) = z^n\overline{P(1/\bar{z})}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then

$$(1.5)\max_{|z|=1}|P'(z)| \le \frac{n}{1+k^n}\max_{|z|=1}|P(z)|$$

The result is best possible and equality holds in (1.5) for $P(z) = z^n + k^n$.

Using Dubinin lemma [2], Singh and Chanam [14] have proved the following refinement of inequality (1.5)

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Theorem 1.2. Let $P(z)=\sum_{\nu=0}^{n}c_\nu z^\nu$ be a polynomial of degree n having no zero in $|z|<k, k\leq 1$and $Q(z)=z^n\overline{P(1/\bar{z})}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z|=1$, then

$$(1.6)\max_{|z|=1}|P'(z)|\leq\frac{1}{1+k^n}\left(n-\frac{(\sqrt{|c_0|}-k^{n/2}\sqrt{|c_n|})k^n}{\sqrt{|c_0|}}\right)\max_{|z|=1}|P(z)|$$

The result is best possible and equality holds in (1.6) for $P(z)=z^n+k^n$.

On the other hand, in 1939, P. Turán [12] provided a lower bound estimate of the derivative to the size of the polynomial by restricting its zeros, and proved that if a polynomial $P(z)$ of degree n has all its zeros in $|z|\leq 1$, then

$$(1.7)\max_{|z|=1}|P'(z)|\geq\frac{n}{2}\max_{|z|=1}|P(z)|$$

Also using lemma of Dubinin [3], Singh and Chanam [14] have proved the following refinement of Turán's inequality Theorem 1.3. If $P(z)=\sum_{\nu=0}^{n}c_\nu z^\nu$ is a polynomial of degree n which has all its zeros in the disk $|z|\leq k, k\geq 1$, then

$$(1.8)\max_{|z|=1}|P'(z)|\geq\frac{1}{1+k^n}\left(n+\frac{k^{n/2}\sqrt{|c_n|}-\sqrt{|c_0|}}{k^{n/2}\sqrt{|c_n|}}\right)\max_{|z|=1}|P(z)|$$

The result is best possible and equality holds in (1.8) for $P(z)=z^n+k^n$.

Recently Authors [11] have obtained following refinement of Theorem 1.2

Theorem 1.4. Let $P(z)=\sum_{\nu=0}^{n}c_\nu z^\nu$ be a polynomial of degree n having no zero in $|z|<k, k\leq 1$and $Q(z)=z^n\overline{P(1/\bar{z})}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z|=1$, then for $0\leq t\leq 1$, we have

$$(1.9)\begin{aligned}\max_{|z|=1}|P'(z)| &\leq\left[n-\frac{k^n}{1+k^n}\left(n+\frac{\sqrt{|c_0|}-k^{n/2}\sqrt{|c_n|+tm}}{\sqrt{|c_0|}}\right)\right]\max_{|z|=1}|P(z)|\\ &-\frac{k^n}{1+k^n}\left(n+\frac{\sqrt{|c_0|}-k^{n/2}\sqrt{|c_n|+tm}}{\sqrt{|c_0|}}\right)tm\end{aligned}$$

where $m=\min_{|z|=1/k}|Q(z)|$. Equality holds in (1.9) for $P(z)=z^n+k^n$.

In the same paper, Authors [11] have also obtained following refinement of Theorem 1.3

Theorem 1.5. If $P(z)=\sum_{\nu=0}^{n}c_\nu z^\nu$ is a polynomial of degree n which has all its zeros in the disk $|z|\leq k, k\geq 1$, then for $0\leq t\leq 1$, we have

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(1.10) $\max_{|z|=1}|P^{'}(z)| \geq \frac{1}{1+k^n}\left(n+\frac{k^{n/2}\sqrt{|c_n|}-\sqrt{|c_0|+tm}}{k^{n/2}\sqrt{|c_n|}}\right)\left(\max_{|z|=1}|P(z)|+tm\right)$

where $m = \min_{|z|=k}|P(z)|$. Equality holds in (2.1) for $P(z) = z^n + k^n$.

Different versions of these Bernstein and Turán-type inequalities have appeared in the literature in more generalized forms in which the underlying polynomial is replaced by more general classes of functions. The one such generalization is moving from the domain of ordinary derivative of polynomials to their polar derivative which is defined as

Definition: Let $p(z)$ be a polynomial of degree n with complex coefficients and $\alpha \in \mathbb{C}$ be a complex number, then the polynomial

$$D_\alpha p(z) = np(z) + (\alpha - z)p^{'}(z)$$

is called polar derivative of $p(z)$ with pole α. Note that $D_\alpha p(z)$ is a polynomial of degree $n-1$ and it is a generalisation of the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha p(z)}{\alpha} = p^{'}(z)$$

uniformly with respect to z for $|z| \leq R, R > 0$.

Many of the generalizations of above mentioned inequalities involve the comparison of the polar derivative $D_\alpha P(z)$ with various choices of $p(z), \alpha$ and other parameters. For more information on the polar derivative of polynomials one can consult the comprehensive books of Marden [8], Milovanonic et al. [9] or Rahman and Schmeisser [13]. In 1998, Aziz and Rather [1] extended inequality () to polar derivative by proving that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k, k \geq 1$, then for every real or complex number α with $|\alpha| \geq k$,

(1.11) $\max_{|z|=1}|D_\alpha P(z)| \geq n\left(\frac{|\alpha|-k}{1+k^n}\right)\max_{|z|=1}|P(z)|$

Govil and Mctume [6] established the polar derivative extension of inequality (1.11) and proved

(1.12) $\max_{|z|=1}|D_\alpha P(z)| \geq n\left(\frac{|\alpha|-k}{1+k^n}\right)\max_{|z|=1}|P(z)| + n\left(\frac{|\alpha|-(1+k+k^n)}{1+k^n}\right)\min_{|z|=k}|P(z)|$

for any complex number α with $|\alpha| \geq 1 + k + k^n$.

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Also, using Dubinin lemma [3], Singh and Chanam [14] have proved the following improvement of inequality (1.11) due to Aziz and Rather [1]

Theorem 1.6. If $P(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n which has all its zeros in the disk $|z|\le k, k\ge 1$, then for any complex number α with $|\alpha|\ge k$, we have

$$(1.13)\max_{|z|=1}|D_\alpha P(z)|\ge \frac{(|\alpha|-k)}{(1+k^n)}\left(n+\frac{k^{n/2}\sqrt{|c_n|}-\sqrt{|c_0|}}{k^{n/2}\sqrt{|c_n|}}\right)\max_{|z|=1}|P(z)|$$

As a polar derivative generalization of Theorem 1.1, Mir and D. Breaz [12] obtained following result

Theorem 1.7. Let $P(z)=\sum_{\nu=0}^{n} c_\nu z^\nu$ be a polynomial of degree n having no zero in $|z|<k, k\le 1$and $Q(z)=z^n\overline{P(1/\bar{z})}$. If $|P^{'}(z)|$ and $|Q^{'}(z)|$ attain maximum at the same point on $|z|=1$, then

$$(1.14)\max_{|z|=1}|D_\alpha P(z)|\le n\left(\frac{|\alpha|+k^n}{1+k^n}\right)\max_{|z|=1}|P(z)|$$

The result is best possible and equality holds in (1.5) for $P(z)=z^n+k^n$.

2. Main Results

We begin, by presenting the following generalization and refinement of inequality (1.1), (1.2) and Theorem 1.6.

Theorem 2.1. If $P(z)=\sum_{\nu=0}^{n} c_\nu z^\nu$ is a polynomial of degree n which has all its zeros in the disk $|z|\le k, k\ge 1$, then for any complex number α with $|\alpha|\ge 1+k+k^n$, and $0\le t\le 1$, we have

$$(2.1)\quad \begin{aligned}\max_{|z|=1}|D_\alpha P(z)| \quad &\ge \frac{(|\alpha|-k)}{(1+k^n)}\left(n+\frac{k^{n/2}\sqrt{|c_n|}-\sqrt{|c_0|+tm}}{k^{n/2}\sqrt{|c_n|}}\right)\max_{|z|=1}|P(z)|\\ &+t\left(\frac{n\left(|\alpha|-(1+k+k^n)\right)}{(1+k^n)}+\frac{(|\alpha|-k)}{(1+k^n)}\frac{\left(k^{n/2}\sqrt{|c_n|}-\sqrt{|c_0|+tm}\right)}{k^{n/2}\sqrt{|c_n|}}\right)m\end{aligned}$$

where $m=\min_{|z|=k}|P(z)|$.

Remark 1. Since $P(z)=\sum_{\nu=0}^{n} c_\nu z^\nu$ has all its zeros in the disk $|z|\le k, k\ge 1$ and if $z_1, z_2, \dots, z_n$ are the zeros of $P(z)$, then

$$\left|\frac{c_0}{c_n}\right|=|z_1 z_2 \dots z_n|=|z_1||z_2|\dots|z_n|\le k^n$$

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As we see in the proof of Theorem 2.1 (given in next section), for every $\beta$with $|\beta| \leq 1$, the polynomial $P(z) + \beta m$ has all its zeros in the disk $|z| \leq k, k \geq 1$, hence

(2.2)$\left|\frac{c_0+\beta m}{c_n}\right| \leq k^n$

which is equivalent to

$$k^{n/2}\sqrt{|c_n|} \geq \sqrt{|c_0 + \beta m|}$$

If in (2.2), we choose the argument of β suitably, we get

(2.3)$\sqrt{|c_0| + |\beta| m} \leq k^{n/2}\sqrt{|c_n|}$.

If we take $|\beta| = t$ in (2.3), so that $0 \leq t \leq 1$, we get $\sqrt{|c_0| + tm} \leq k^{n/2}\sqrt{|c_n|}$.

Remark 2. For $t = 0$, Theorem 2.1 reduces to Theorem 1.6.

Remark 3. If we divide (2.1) by $|\alpha|$ and take $|\alpha| \to \infty$, we get Theorem 1.5 and thus Theorem 2.1 contains Theorem 1.5.

Remark 4. If we divide (2.1) by $|\alpha|$ and take $|\alpha| \to \infty$ and $t = 0$ we get Theorem 1.3 and thus Theorem 2.1 also contains Theorem 1.3. Theorem 2.1 in general provides much better information regarding $\max_{|z|=1}|D_\alpha P(z)|$, in case when $P(z)$ has all its zeros in $|z| < k, k \geq 1$. We illustrate this with the help of following example.

Example 2.1. Consider $P(z) = z^2 + 3z + 5/4$, which is polynomial of degree 2 having all its zeros in $|z| \leq 5/2$. We take $k = 3$and $\alpha = 15 + 8i$, so that $|\alpha| = 17$, then clearly $|\alpha| \geq 1 + k + k^n$. We find that

$$\min_{|z|=3}|P(z)| = 5/4 \quad \text{and} \quad \max_{|z|=1}|P(z)| = 21/4$$

For this polynomial, we obtain that

$$\max_{|z|=1}|D_\alpha P(z)| \geq 14.70 \quad \text{(by inequality (1.1))}$$
$$\max_{|z|=1}|D_\alpha P(z)| \geq 18.20 \quad \text{(by inequality (1.12))}$$
$$\max_{|z|=1}|D_\alpha P(z)| \geq 19.55 \quad \text{(by Theorem (1.6))}$$

While Theorem 2.1 (with = 1), gives

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$$\max_{|z|=1}|D_\alpha P(z)| \geq 20.25$$

which is much better than the bound given by above estimates.

Using Theorem 2.1, we prove the following generalisation and refinement of Theorem 1.4 and Theorem 1.7.

Theorem 2.2. Let $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ be a polynomial of degree n having no zero in $|z| < k, k \leq 1$and $Q(z) = z^n\overline{P(1/\bar{z})}$. If $|P^{'}(z)|$ and $|Q^{'}(z)|$ attain maximum at the same point on $|z| = 1$, then for $0 \leq t \leq 1$, we have

$$(2.4)\quad \max_{|z|=1}|D_\alpha P(z)| \leq \left[n|\alpha| - \frac{(|\alpha|-1)k^n}{(1+k^n)}\left(n + \frac{k^{n/2}\sqrt{|c_0|}-\sqrt{|c_n|+tm}}{\sqrt{|c_0|}}\right)\right]\max_{|z|=1}|P(z)| - \frac{(|\alpha|-1)k^n}{(1+k^n)}\left(n + \frac{k^{n/2}\sqrt{|c_n|}-\sqrt{|c_0|+tm}}{\sqrt{|c_0|}}\right)tm$$

where $m = \min_{|z|=1/k}|Q(z)|$. Equality holds in (2.4) for $P(z) = z^n + k^n$.

Remark 5. If we divide (2.4) by $|\alpha|$ and take $|\alpha| \to \infty$, we get Theorem 1.4 and thus Theorem 2.2 contains Theorem 1.4. Taking $t = 0$ in Theorem 2.2, we get the following polar derivative generalization of Theorem 1.7.

Corollary 1. Let $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ be a polynomial of degree n having no zero in $|z| < k, k \leq 1$and $Q(z) = z^n\overline{P(1/\bar{z})}$. If $|P^{'}(z)|$ and $|Q^{'}(z)|$ attain maximum at the same point on $|z| = 1$, then for $0 \leq t \leq 1$, we have

$$(2.5)\max_{|z|=1}|D_\alpha P(z)| \leq \left[n|\alpha| - \frac{(|\alpha|-1)k^n}{(1+k^n)}\left(n + \frac{k^{n/2}\sqrt{|c_0|}-\sqrt{|c_n|}}{\sqrt{|c_0|}}\right)\right]\max_{|z|=1}|P(z)|$$

where $m = \min_{|z|=1/k}|Q(z)|$. Equality holds in (2.4) for $P(z) = z^n + k^n$.

Remark 6. If we divide (2.4) by $|\alpha|$ and take $|\alpha| \to \infty$ and $t = 0$ we get Theorem 1.2.

Remark 7. If we divide (2.4) by $|\alpha|$ and take $|\alpha| \to \infty, t = 0$ and $k = 1$ we get the following improvement of inequality (1.2) due to Erdös and Lax for a subclass of polynomials having all its zeros in $|z| \geq 1$

Corollary 2. Let $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ be a polynomial of degree n having no zero in $|z| < 1$and $Q(z) = z^n\overline{P(1/\bar{z})}$. If $|P^{'}(z)|$ and $|Q^{'}(z)|$ attain maximum at the same point on $|z| = 1$, then

(2.6)$\max_{|z|=1}|P'(z)| \leq \frac{1}{2}\left(n - \frac{\sqrt{|c_0|}-\sqrt{|c_n|}}{\sqrt{|c_0|}}\right)\max_{|z|=1}|P(z)|$

In the same way, Theorem 2.2 in general provides much better information than Theorem regarding the maximum of $|D_\alpha P(z)|$on $|z| = 1$. We illustrate this with the help of following example

Example 2.2. Consider $P(z) = z^3 - z^2 + z - 1$, which is polynomial of degree 3. Clearly, $P(z)$ has all its zeros in $|z| \leq 1$. Further

$$Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)} = -P(z).$$

So that $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$. We take $= \frac{1}{2}$, so that $P(z) \neq 0$ in $|z| < k = \frac{1}{2}$ and we find numerically that $\max_{|z|=1}|P(z)| = 4$, $\min_{|z|=\frac{1}{1/2}}|Q(z)| = \min_{|z|=2}|Q(z)| = 5$. Taking $\alpha = \frac{3+i\sqrt{7}}{2}$, so that $|\alpha| = 2$, we obtain the following estimates

$\max_{|z|=1}|D_\alpha P(z)| \leq 22.66 \quad (by\ \ (1.14))$

While Theorem 2.2 gives

$$\max_{|z|=1}|D_\alpha P(z)| \leq 20.85$$

which is much better than the bound given by (1.14). For the proof our results, we need the following lemma due to Govil and Rahman [4].

Lemma 1. If $P(z)$ is a polynomial of degree n then on $|z| = 1$,

$$|P'(z)| + |Q'(z)| \leq n\max_{|z|=1}|P(z)|,$$

where $Q(z) = z^n\overline{P(1/\bar{z})}$.

3. Proofs of Theorems

Proof of Theorem 2.1. If $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ has a zero on $|z| = k$, then $m = \min_{|z|=k}|P(z)| = 0$ and the result follows from Theorem 1.3 in this case. Henceforth, we suppose that $P(z)$ has all its zeros in $|z| < k, k \geq 1$.

Let $H(z) = P(kz)$and $G(z) = z^n\overline{H(1/\bar{z})} = z^n\overline{P(k/\bar{z})}$, then all the zeros of $G(z)$ lie in $|z| > 1$ and $|H(z)| = |G(z)|$ for $|z| = 1$. This gives

$$\left|\overline{z^n P\left(\frac{k}{\bar{z}}\right)}\right| = |P(kz)| \geq m \quad \text{for} \quad |z| = 1$$

It follows by Minimum Modulus principle, that

$$\left|\overline{z^n P\left(\frac{k}{\bar{z}}\right)}\right| \geq m \quad \text{for} \quad |z| \leq 1$$

Replacing $z$by $1/\bar{z}$, it implies that

$$|P(kz)| \geq m|z|^n \quad \text{for} \quad |z| \geq 1$$

or

(3.1)$|P(z)| \geq m\left|\frac{z}{k}\right|^n \quad \text{for} \quad |z| \geq k$

Now, consider the polynomial

$$F(z) = P(z) + \beta m$$

whereβ is complex number with $|\beta| \leq 1$, then all the zeros of $F(z)$ lie in $|z| \leq k$. Because, if for some $z = z_1$, with $|z_1| > k$, we have

$$F(z_1) = P(z_1) + \beta m = 0$$

then

$$|P(z_1)| = |\beta m| \leq m < m\left|\frac{z_1}{k}\right|^n$$

which contradicts (3.1). Hence, for every complex number $\beta$with $|\beta| \leq 1$, the polynomial

$$F(z) = P(z) + \beta m = (c_0 + \beta m) + \sum_{\nu=1}^{n} c_\nu z^\nu$$

has all its zeros in $|z| \leq k$, where $k \geq 1$. Applying Theorem 1.6, to the polynomial $F(z)$, we get for every complex number β with $|\beta| \leq 1$ and $|z| = 1$

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(3.2) $\max_{|z|=1}|D_\alpha(P(z)+\beta m)| \geq \frac{|\alpha|-k}{1+k^n}\left(n+\frac{k^{n/2}\sqrt{|c_n|}-\sqrt{|c_0+\beta m|}}{k^{n/2}\sqrt{|c_n|}}\right)\left(\max_{|z|=1}|P(z)+\beta m|\right)$

For every $\beta \in \mathbb{C}$, we have

$$|c_0+\beta m| \leq |c_0|+|\beta|m,$$

Since the function $k(x) = n+\frac{k^{n/2}\sqrt{|c_n|}-\sqrt{x}}{k^{n/2}\sqrt{|c_n|}}$ is decreasing for $k \geq 1$, it follows from (3.2) that for every β with $|\beta| \leq 1$ and $|z|=1$

(3.3) $\max_{|z|=1}|D_\alpha(P(z)+\beta m)| \geq \frac{|\alpha|-k}{1+k^n}\left(n+\frac{k^{n/2}\sqrt{|c_n|}-\sqrt{|c_0|+|\beta|m}}{k^{n/2}\sqrt{|c_n|}}\right)\left(\max_{|z|=1}|P(z)+\beta m|\right)$

Choosing argument of β on R.H.S of (3.3) such that

$$\max_{|z|=1}|P(z)+\beta m| = \max_{|z|=1}|P(z)|+|\beta|m,$$

we obtain from (3.3) that

$$\max_{|z|=1}|D_\alpha P(z)|+|\beta|mn \geq \frac{|\alpha|-k}{1+k^n}\left(n+\frac{k^{\frac{n}{2}}\sqrt{|c_n|}-\sqrt{|c_0|+|\beta|m}}{k^{\frac{n}{2}}\sqrt{|c_n|}}\right)$$

(3.4) $\left(\max_{|z|=1}|P(z)|+|\beta|m\right)$

which on taking $|\beta| = t$, so that $0 \leq t \leq 1$ gives

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)| \quad &\geq \frac{(|\alpha|-k)}{(1+k^n)}\left(n+\frac{k^{n/2}\sqrt{|c_n|}-\sqrt{|c_0|+tm}}{k^{n/2}\sqrt{|c_n|}}\right)\max_{|z|=1}|P(z)| \\ &+t\left(\frac{n\left(|\alpha|-(1+k+k^n)\right)}{(1+k^n)}+\frac{(|\alpha|-k)}{(1+k^n)}\frac{\left(k^{n/2}\sqrt{|c_n|}-\sqrt{|c_0|+tm}\right)}{k^{n/2}\sqrt{|c_n|}}\right)m\end{aligned}$$

This completes the proof of the Theorem 2.1.

Proof of Theorem 2.2. Note that for any complex number $\alpha$with $|\alpha| \geq 1$, we have on $|z| = 1$

$$
\begin{aligned}
|D_\alpha P(z)| &= |nP(z) + (\alpha - z)P'(z)| \\
&= |nP(z) - zP'(z) + \alpha P'(z)| \\
&\leq |nP(z) - zP'(z)| + |\alpha||P'(z)| \\
&= |Q'(z)| + |\alpha||P'(z)| \\
&= n\max_{|z|=1}|P(z)| - |P'(z)| + |\alpha||P'(z)| \quad (by \quad \text{Lemma 1}) \\
&= n\max_{|z|=1}|P(z)| + (|\alpha| - 1)|P'(z)|
\end{aligned}
$$

Therefore, using Theorem 1.4, we have

$$
\begin{aligned}
\max_{|z|=1}|D_\alpha P(z)| \quad &\leq n\max_{|z|=1}|P(z)| + (|\alpha| - 1)\left[n - \frac{k^n}{(1+k^n)}\left(n + \frac{\sqrt{|c_0|} - k^{n/2}\sqrt{|c_n| + tm}}{\sqrt{|c_0|}}\right)\right]\max_{|z|=1}|P(z)| \\
&- \frac{(|\alpha| - 1)k^n}{(1+k^n)}\left(n + \frac{\sqrt{|c_0|} - k^{n/2}\sqrt{|c_n| + tm}}{\sqrt{|c_0|}}\right)tm
\end{aligned}
$$

That is

$$
\begin{aligned}
\max_{|z|=1}|D_\alpha P(z)| \quad &\leq \left[n|\alpha| - \frac{(|\alpha| - 1)k^n}{(1+k^n)}\left(n + \frac{\sqrt{|c_0|} - k^{n/2}\sqrt{|c_n| + tm}}{\sqrt{|c_0|}}\right)\right]\max_{|z|=1}|P(z)| \\
&- \frac{(|\alpha| - 1)k^n}{(1+k^n)}\left(n + \frac{\sqrt{|c_0|} - k^{n/2}\sqrt{|c_n| + tm}}{\sqrt{|c_0|}}\right)tm
\end{aligned}
$$

Data availablity : All data generated or analysed during this study are included in this article.

Conflict of Interest: On behalf of all authors, the corresponding author states that there is no conflict of interest.

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