International Journal of Engineering, Science and Mathematics
Vol. 8 Issue 1, January 2019,
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijesm.co.in, Email: ijesmj@gmail.com
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A Characterization of Weyl-Heisenberg frame for Hilbert Spaces

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Abstract:

In this paper, we discuss some results related to Weyl-Heisenberg frames. The sufficient condition for Weyl-Heisenberg frames to be Rotor frames is given.

Keywords: Separable Hilbert Space, Weyl-Heisenberg frame, Frames.

1. Introduction:

Frames for Hilbert spaces were formally defined by R.J. Duffin and A.C. Schaffer [1] in 1952 to deal with non harmonic Fourier series. After a couple of years, frames were brought to life in 1986 by Daubechies, Grossmann, Meyer in the context of Painless non orthogonal expansions [2]. D. Gabor following his fundamental work in Gabor transform, this transform being called the Weyl- Heisenberg wavelet transform. Christopher E. Heil and David F. Walnut investigated some properties of Gabor frame [3] in 1989. Peter G. Casazza and Ole Christensen discussed some results in [4] and [5] and C. Easwaran Nambudiri and K. Parthasarathy are discussed characterization and generalized Weyl-Heisenberg frame in [7] and [8].

In this paper, we have to discuss some result on generalized Weyl-Heisenberg frame (Gabor frame) and we will give a new identity of Weyl-Heisenberg frame, we use rotation operator instead of modulation operator in Gabor frame.

2. Preliminaries and Notations:

Let H be a separable Hilbert space and $L(H)$ be a set of all bounded linear operator on H,

We can define the following operators

$$T : l^2 \to H , \quad Ta = \sum_{n=1}^{\infty} a_n f_n \quad , for\ all\ a = \{a_n\} \in l^2$$

is called synthesis operator or pre frame operator,

The adjoint operator $T^{*}: \mathcal{H} \to l^2$, $T^{*}f = \{\langle f, f_n\rangle\}_{n=1}^{\infty}$
is called the analysis operator. The composition operator T with its adjoint T^{*} it denoted by $S = TT^{*}$,

That is, $S: H \to H$, $Sf = \sum_{n=1}^{\infty} \langle f, f_n\rangle f_n \quad for\ all\ f \in H$
is called the frame operator.

We begin with frame definitions. Let $\mathcal{H}$ be separable Hilbert space with the inner product $\langle \cdot , \cdot \rangle$ linear in the first entry and all index sets are assumed to be countable.

Definition 2.1 (see [1]) Let $\mathcal{H}$ be separable Hilbert space and a sequece $\{f_n\}_{n=1}^{\infty} \subset H$ is called an ordinary frames, If there exists Constants A , B > 0, $such\ that$

$$A\,\|f\|^2 \le \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \le B\|f\|^2 \quad , for\ all \quad f \in H.$$

Definition 3.2 A sequence $\{f_n\}_{n=1}^{\infty} \subset H$ is called a Bessel sequence. If there exists Constant B > 0, $such\ that\ \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \le B\|f\|^2 , for\ f \in H.$

5. ROTOR FRAMES

Rotor frames are a special class of frames in $L^2(R) \subset H$ of the form $\{V_{n\phi} T_{m\tau}\, g: n, m\epsilon Z\ and\ \alpha, \beta\epsilon R\}$ which is generator by a single function through translation and rotation where $V_{n\phi}\ and\ T_{m\tau}$ are rotation operator and translation operator respectively .

It is known that the frame operator of the Rotor frame commutes with involved $V_{n\phi}\ and\ T_{m\tau}$ and those two operators satisfies positive, invertible and bounded linear operator. In [3] and [16] have been discussed modulation and translation operator. Here Rotor frame consists of Rotation and Translation operators. These operators are given $V_{n\phi}\, g(n) = g(n)e^{i\phi_n}$ and $T_{m\tau}\, g(m) = g(m - \tau)$ where $\phi_n = [\mathcal{H}_1, \mathcal{H}_2] = C^{-1}(Sup|\langle x, y\rangle|:$

$x \in \mathcal{H}_1 \ominus \mathcal{H}_2\ , y \in \mathcal{H}_2 \ominus \mathcal{H}_1\ and\ \|x\| = \|y\| = 1\}$. $\mathcal{H}_1 and\ \mathcal{H}_2$ are subspace of Hilbert space. The cosine of the angle of two subspaces is denoted by $C[\mathcal{H}_1, \mathcal{H}_2]$ as [18].If two subspace are orthogonal , the cosine of the angle is zero .Here we have to find the angle between a frame in subspace $\mathcal{H}_1$ and reference frame in subspace $\mathcal{H}_2$ in H. If two subspaces are closed, then its cosine of the angle of them is less than 1 and its converse is also true.

Definition: 5.1 Let H be separable Hilbert space and $\{V_{n\phi}T_{m\tau}g: n, m\epsilon Z\}$ $in\ H$ is said to be Rotor frames if there exist constant $A_i > 0\ and\ B_i > 0\ for\ all\ i \in N$ such that

$A_i \|f\|^2 \le \sum_{i=1}^{\infty} |\langle f, V_{n\phi}T_{m\tau}g\rangle|^2 \le B_i \|f\|^2, for\ f \in H$, for all $n, m\epsilon Z$.

Example 5.2. Let $H_1\ and\ H_2$ be two subspaces of Hilbert space H and here

$\phi_n = [H_1, H_2] = C^{-1}(Sup\{|\langle f, g\rangle| : f \in H_1 \ominus H_2\ , g \in H_2 \ominus H_1\ and\ \|f\| = \|g\| = 1)\}$, τ is real number and $(n) \in H$, the cosine of the angle of two subspaces is denoted by $C[H_1, H_2]$. For $f_1 = (1\ 0\ 0)\ in\ H_1$, $g_1 = \left(\frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}\ 0\right)\ in\ H_2$ such that $\|f_1\| = 1$, $\|g_1\| = 1$

$$\text{Now } \langle f_1\ ,\ g_1\ \rangle = \langle (1,0,0), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)\rangle$$

$$= \frac{1}{\sqrt{2}}$$

$$|\langle f_1\ ,\ g_1\ \rangle| = \frac{1}{\sqrt{2}}$$

$$\phi_1 = [H_1, H_2] \quad = C^{-1}(Sup\{|\langle f_1, g_1\rangle| : f_1 \in H_1 \ominus H_2\ , g_1 \in H_2 \ominus H_1\ )\}$$

$$= C^{-1}\left(Sup\left\{\frac{1}{\sqrt{2}} : f_1 \in H_1 \ominus H_2\ , g_1 \in H_2 \ominus H_1\right\}\right)$$

$$= Cosine^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\text{i.e } \phi_1 = \frac{\pi}{4}$$

Suppose $f_2 = (1\ 0\ 0)$, $g_2 = \left(\frac{1}{2}\ \frac{1}{2}\ 0\right)$, then we have $\phi_2 = \frac{\pi}{3}$.

here $V_{n\phi}\,g(n) = g(n)e^{i\phi_n}$ is rotation operator and $T_{m\tau}\,g(m) = g(m-\tau)$ is translation operator.

Theorem: 5.1 Let H be separable Hilbert space and if there is bounded linear invertible

Operators on $L^2(R)$ which commutes with $V_{n\phi}\ and\ T_{m\tau}$, then $\{V_{n\phi}T_{m\tau}g: n, m\epsilon Z\}$ is Rotor

Frames.

Proof: Let $S > 0$ bounded linar invertible operator on $L^2(R)$ which is commutes with

Rotation operator $V_{n\phi}$ and translation operator $T_{m\tau}$, then its positive root $s^{\frac{1}{p}-1}$
Where $0 < p \le 1$ is also bounded linear invertible operator on $L^2(R)$. The frame operator $s^{\frac{1}{p}-1} where\ \ 0 < p \le 1$. We have to show that $\{V_{n\phi}T_{m\tau}g: n, m\epsilon Z\}$ is Rotor

frames $on\ L^2(R)$

with projection operator $s^{\frac{1}{p}-1} where\ 0 < p \leq 1$ and inevitability of $s^{\frac{1}{p}-1}$ ensures that $\left\{s^{\frac{1}{p}-1} V_{n\phi} T_{m\tau}\, g = V_{n\phi} T_{m\tau}\, g s^{\frac{1}{p}-1}\right\}$, for all integer $n, m.$,

Since $Q = s^{\frac{1}{p}-1}$ is projection operator, that is, $Q^2 = Q$

$$s^{\frac{1}{p}-1} f = \left(s^{\frac{1}{p}-1}\right)^2 f \qquad \text{(Idempotent)}$$

$$= s^{\frac{1}{p}-1}\left(s^{\frac{1}{p}-1} f\right)$$

$$= s^{\frac{1}{p}-1}\left(\sum_{n=1}^{\infty} \langle s^{\frac{1}{p}-1} f, V_{n\phi} T_{m\tau}\, g\rangle V_{n\alpha} T_{m\tau}\, g\right)$$

$for\ f\ \in H$

$$= \sum_{n=1}^{\infty} \langle s^{\frac{1}{p}-1} f, V_{n\phi} T_{m\tau}\, g\rangle V_{n\phi} T_{m\tau}\, s^{\frac{1}{p}-1} g$$

$$= \sum_{n=1}^{\infty} \langle f, V_{n\phi} T_{m\tau}\, s^{\frac{1}{p}-1} g\rangle V_{n\phi} T_{m\tau}\, s^{\frac{1}{p}-1} g$$

Therefore, $s^{\frac{1}{p}-1} f = \sum_{n=1}^{\infty} \langle f, V_{n\phi} T_{m\tau}\, s^{\frac{1}{p}-1} g\rangle V_{n\phi} T_{m\tau}\, s^{\frac{1}{p}-1} g$

where $s^{\frac{1}{p}-1}$ is frame operator of the frames $\left\{s^{\frac{1}{p}-1} V_{n\phi} T_{m\tau}\, g = V_{n\phi} T_{m\tau}\, g s^{\frac{1}{p}-1}\right\}$

now $\quad ||s^{\frac{1}{p}-1} f||^2 = \langle s^{\frac{1}{p}-1} f, s^{\frac{1}{p}-1} f\rangle$

$$= \langle \sum_{n=1}^{\infty} \langle f, V_{n\phi} T_{m\tau}\, s^{\frac{1}{p}-1} g\rangle V_{n\phi} T_{m\tau}\, s^{\frac{1}{p}-1} g\ , \sum_{n=1}^{\infty} \langle f, V_{n\phi} T_{m\tau}\, s^{\frac{1}{p}-1} g\rangle V_{n\phi} T_{m\tau}\, s^{\frac{1}{p}-1} g\ \rangle$$

$$= \sum_{n=1}^{\infty} \langle f, V_{n\phi} T_{m\tau}\, s^{\frac{1}{p}-1} g\rangle \underline{\langle f, V_{n\phi} T_{m\tau}\, s^{\frac{1}{p}-1} g\rangle}$$

$$= \sum_{n=1}^{\infty} \left|\langle f, V_{n\phi} T_{m\tau}\, s^{\frac{1}{p}-1} g\rangle\right|^2$$

for $g \in H$ and positive operator $s^{\frac{1}{p}-1}$ and let $B_i = \sum_{n,m=1}^{\infty} ||V_{n\phi} T_{m\tau}\, s^{\frac{1}{p}-1} g||^2$ and

$A_i = inf\ inf \left\{\left|S^{\frac{1}{p}-1} g\right|\ ,\ for\ g\ \in H\right\}$ such that

$\sum_{n=1}^{\infty} \left|\langle f, V_{n\phi} T_{m\tau} s^{\frac{1}{p}-1} g\rangle\right|^2 \leq B_n ||f||^2$ is Bessel's sequence

This completes the proof.

Conclusion:

The frame theory concepts can be adopted for many applications such as Signal processing, image processing, communication systems, information processing, and so on.

In Rotor frames, the main advantage of a vector rotation is elimination of position dependency from the machine electrical variables and also applicable in communication system. The formulation presented in this paper is well suited for information retrieval systems where the system retrieves collection of relevant documents for the given query. In order to compare the documents with the query against similarity to estimate the closeness, the frames theory would help to represent documents and the input query where, documents and query will be expressed in terms of vectors.

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