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Topological Methods in Dynamical Systems

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Abstract: In the present paper, we find out a few results based on acceptable topology, discrete topology, hausdorff topology, topological transitivity, orbit of a point and dense orbit. We have proved some more results using compact topological spaces and Hausdorff metric spaces. We have also introduced some new terms like topological transitivity, sensitive, dynamical system for compact sets and sensitive dependence on initial conditions (SDIC) and same results based on metric topology.

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1. Introduction

Poincare (1890), firstly introduced the topological notions and methods in dynamical systems. These notions were initially used for the analysis of ordinary differential equations. Poincare states that, "The Three Body Problem exhibits sensitive dependence on initial conditions (SDIC), so there is no „formula" for its solution". Frink [9], states that various mathematical systems are lattices and topological spaces at the same time and along with this he discussed different methods of obtaining topologies on a lattice. The monograph "Analytic Topology" by Whyburn [15] provides a basic insight related to topological methods in dynamics. Birkhoff [2] explained the systematically development of topological dynamics, highlighting its essentially abstract character and made fundamental synonyms. A great contribution about the study of topological dynamics can be studied from Gottschalk and Hedlund [21] and order within chaos and its theory with strange attracters by Berge et al. [5] and Eckmann et al. [22]. Chaotic dynamical systems and fractals theory by Devaney [7] and Barnsley [23] provide us the important information regarding our work.

Topological conjugacy preserves topological notions like transitivity and periodicity, but sensitivity is dependent on the metric. Sensitivity is a result of transitivity and dense periodic points, as shown by Silverman [14] and Glasner et. al. [24]. Also, numerous dynamical characteristics can be defined in a natural fashion on Hausdorff spaces. In this paper, we have discussed the Hausdroff notion on some special subsets.

The layout of this paper is as follows. Section 2, consists basic definitions, notation and preliminaries, which are useful in later section. The concept of compact metric spaces has also been introduced here to help us in obtaining its orbit precisely. Finally, topological versions of these aspects are given which will be helpful in studying this paper in detail.

In Section 3, some fundamental results have been established based on topologies on spaces of subsets using concepts of topology. In this way, some new terms and results in the form of lemmas, remarks and propositions have been introduced.

In Section 4, we have produced results based on Hausdroff and dense in topological spaces.

2. Notations, Definitions and Preliminaries

This section presents the basic definitions, notations and propositions giving insight to the study of dynamical systems orbit points and SDIC, which are used in later sections.

Throughout this paper, we denote IN as the set of natural numbers, E(Y) as the selection of all nonempty closed sets of Y, K(Y) as the selection of all nonempty compact sets of Y and the finite topology on E(Y) as a neighbourhood topology.

Chaos theory is the investigation of differential equations with SDIC that produce seemingly unpredictable time trajectories.

Dynamical System [13]. A dynamical system with discrete time on topological space Y, including $f : Y \to Y$ as a selection $\{f^{n} : n \in IN\}$of continuous mappings on Y, beside f^{0} is referred as the identity mapping, and $f^{n+m}(x) = (f^{n} o f^{m})(x)$ for all $x \in Y$. Additionally, for each $n \in IN$ we can write, $f^{n} = f o f^{n-1}$. Furthermore, using induction method, $f o f^{n-1} = f^{n-1} o f$. The pair (Y, f) is called a dynamical system.

The function f is called topologically transitive (TT), if for arbitrary pair L and $V \in \tau - \{\varphi\}$, $\exists$ a natural number m s.t. $f^{m}(L) \cap V \neq \varphi$.

Compact metric space [19]. A metric space *Y* is compact metric space if each open cover of *Y* has a finite subcover.

Hausdorff metric space [19]. A metric space *Y* is Hausdroff metric space if for arbitrary unequal $x, y \in Y$, $\exists$ open sets $U, V \subset Y$ s.t. $U \cap V = \varphi$ where $x \in U$, $y \in V$.

As Chaos represents that the orbit structure is highly complex. It informs us that non-periodic points with dense orbits in space are arbitrarily close to a periodic point. As a result, being close to a point of a particular kind does not guarantee that the orbits will be comparable. This indicates that orbit behaviour cannot be predicted using estimations or precision.

Sensitive dependence on initial conditions (SDIC) [20]. Considering (Y, f) as a dynamical system, then *f* has SDIC, if $\exists \in > 0$ s.t. for every $x \in Y$ and every neighbourhood *W* of *x*, $\exists\, y \in W$ and some $m \in IN$ s.t. $d(f^m(x), f^m(y)) > \in$. Consequently if *f* has SDIC then the dynamical system (Y, f) is said to be sensitive.

Perfect [20]. A topological space *Y* is called perfect if *Y* has no isolated point.
Starting with a dynamical system (Y, f) and for all compact sets of *Y* that is K(*Y*), we have another dynamical system (K(*Y*), f^*). Now there is a question of finding how their transitivities and sensitivities are related. For more information on the study of compact metric space one can go through the papers of Roman Flores [17], Gu et. al. [18], Kato‟s [19] and Yangeng et. al. [20]. Silverman [14] gave the following results, in proposition(1.1), for a dynamical system (Y, f), where *Y* is a metric space
(i) If *Y* is perfect and *f* has a dense orbit, then *f* is TT.
(ii) If *Y* is separable and of second category, then *f* is TT implies *f* has a dense orbit and in theorem (2.1), if *Y* is infinite, *f* has a dense orbit and SPP of *f* is dense, then *f* has SDIC.
For the fundamental definitions for the metric spaces and topological spaces one can go through Munkres [12] and references there in.

2.1. Topological Approach

Remark.2.1.1.[20] A topological space T_1 is perfect iff it has no finite open set.

Proposition.2.1.2.[20] Suppose *Y* be a topological space and *f* is a continuous self map.
(i)If *Y* has no finite open set and *f* has a dense orbit then *f* is topologically transitive (TT).
(ii)If *Y* is a second countable of second category space, then *f* is topologically transitive

(TT)implies that f has a dense orbit.

One possible replacement of sensitive dependence on initial conditions (SDIC) of f is that f has a dense orbit and set of periodic points (SPP) of f is dense. Birkhoff [6] introduced Moore Smith convergence of order topology in lattices and interval topology using closed intervals as a subbasis for closed sets and intrinsic topology using open intervals just as a subbasis for open sets. On the lattice F of all closed sets of a topological space Y, Moore-Smith convergence order topology and Interval topology, apart from being different, do not specialize to metric spaces and topological relativization (called admissible in regard to the topology of Y by Michael [11]) which means Y is homeomorphic to a subspace of F. Frink [9] introduced and obtained results about neighbourhood topology on the lattice F of all closed sets of a topological space.

3. Topologies on Spaces of Subsets

Michael [11], has made use of Vietoris or finite topology on the following spaces of subsets based on topological space Y. For all $n \in IN$, suppose $J_n = \{m \in IN : m \leq n\}$. Consider $\beta = \{J_n : n \in IN\}$ is a base for a topology on IN. Let $k \in IN$ and for every $n \in IN$, suppose $J_{kn} = \{m \in IN : k \leq m \leq n\}$, furthermore $\beta_k = \{J_{kn} : n \in IN\}$ is a base for a topology on IN. Here K(IN) and E(IN) are not interesting classes. Considering a topological space Y, suppose p as a property of subsets of Y simultaneously a set or at the same time a topological space. Suppose $p(Y) = \{A \subset Y ; A$ is nonempty and have property $p\}$. For some $H \subset Y$, let $p(H) = \{A \in p(Y) : A \subset H\}$, and $p^*(H) = \{A \in p(Y) : A \cap H \neq \varphi\}$. For subsets H_1, H_2…......, H_n of Y, let $p(H_1, H_2\ldots\ldots, H_n) = \{A \in p(Y) : A \subset \cup H_i$ for all $i \in J_n\}$, and $p^*(H_1, H_2\ldots H_n) = \{A \in p(Y) : A \cap H_i \neq \varphi, \forall\ i \in J_n\}$. Now, we present fundamental results for the topological space $p(Y)$.

Remark.3.1.1. For subsets K, $H_1, H_2\ldots H_n$ and $V_1, V_2\ldots V_m$ of Y, let $H = \cup\{H_i : i \in J_n\}$ and $V = \cup\{V_i : i \in J_m\}$.

(i) $p(H_1, H_2\ldots H_n) = \cap\{p^*(H_i) : i \in J_n\} \cap p(H)$,

(ii) $p(K) = p(Y) - p^*(Y - K)$ and $p^*(K) = p(Y) - p(Y - K)$ then $p(K) \subset p^*(K)$,

(iii) $\cap\{p(H_i) : i \in J_n\} = p(\cap\{H_i : i \in J_n\})$,

(iv) $\cap\{p^*(H_i) : i \in J_n\} = p^*(H_1, H_2,\ldots,H_n) = p(Y, H_1, H_2,\ldots,H_n)$,

(v) $p(H_1,H_2,\ldots,H_n) \cap p(V_1,V_2,\ldots,V_m) = p(V\cap H_1, V\cap H_2\ldots V\cap H_n, H\cap V_1, H\cap V_2,\ldots, H\cap V_m)$,

(vi) $p(H_1, H_2,\ldots,H_n) \subset p(V_1,V_2,\ldots,V_m)$ iff $H \subset V$, for each $V_i \exists H_j \supset V_i$,

(vii) $\cap\{p^*(H_i) : i \in J_n\} \cap p(K) = p(K, K\cap H_1, V\cap H_2,\ldots,K\cap H_n)$,

(viii) $p(H_1, H_2,\ldots,H_n) \cap p(V_1, V_2,\ldots,V_m) \neq \varphi$ iff for each $H_i \exists V_{ij}$ s.t. $H_i \cap V_{ij} \neq \varphi$ and for each $V_i \exists H_{ij}$ s.t. $V_i \cap H_{ij} \neq \varphi$.

(ix) $p(K) = p^*(K)$ iff for $A \in p(Y)$, and $A\cap K \neq \varphi$ implies $A \subset K$.

Let $\alpha_1 = \{p(L) : L \in \tau\}$ and $\alpha_2 = \{p^*(L) : L \in \tau\}$ thus $\alpha = \alpha_1 \cup \alpha_2$. Let $\beta = \{p(L_1, L_2,\ldots,L_n) : L_i \in \tau \, \forall \, i \in J_n\}$. Let β_2 be the selection of all finite intersections of members of α_2. Let τ_1^*, τ_2^* and τ^* be topologies on $p(Y)$ generated α_1, α_2 and α respectively.

Remark.3.1.2. (a) Suppose $\alpha_1 \subset \beta$ and $\beta_2 \subset \beta$ accordingly α_1 and β are closed under finite intersections and so α_1 is a base considering τ_1^* and β is a base considering τ^*. Since β is the selection of all finite intersections of members of $\alpha_1 \cup \alpha_2$, therefore $\tau^* = \sup\{\tau_1^*, \tau_2^*\}$.

Now onwards $p(Y)$ is the topological space with topology τ_1^*, τ_2^* or τ^*. A topology on E(Y) is called acceptable, if $p(K)$ is closed $\forall$ K closed in Y and is open $\forall$ K open in Y [see Michael [11]].

Remark.3.1.2. (b) (i) a topology τ' on $p(Y)$ is acceptable iff $\alpha_1 \cup \alpha_2 \subset \tau'$.

(ii) If K is closed in Y, then $p(K)$ is closed in τ_2^* and τ^*, hence $p^*(K)$ is closed in τ_1^* and τ^*.

(iii) τ^* is the smallest acceptable topology on $p(Y)$.

Let $F_1(Y)$ continue as the selection of all singleton subsets of Y and $F(Y)$ continue as the selection of all nonempty finite subsets of Y. If $F_1(Y) \subset p(Y)$, then define a map $i : Y \to p(Y)$ as $i(x) = \{x\}$ for each $x \in Y$.

Remark.3.1.3. Let $F_1(Y) \subset p(Y)$.

(a) Let $A \subset Y$, then

(i) $i(A) = p(A)\cap i(Y) = p^*(A)\cap i(Y)$.

(ii) $A = i^{-1}(p(A)) = i^{-1}(p^*(A))$.

(iii) If $p(A) \in \tau^*$, then A is open in Y.

(b) For subsets $H_1, H_2,\ldots,H_n$ of Y then $p(H_1, H_2\ldots H_n) \cap i(Y) \neq \varphi$ iff $\cap\{H_i : I \in i \in J_n\} \neq \varphi$.

(c) Let $A \in p(Y, H_1, H_2,\ldots,H_n)$. If H_i`s are pair wise disjoint, then A contains at least n elements.

Proposition.3.1.4. (a) Let $F_1(Y) \subset p(Y)$, then

(i) τ_1^*, τ_2^* and τ^* are admissible with the topology τ of Y.

(ii) $F_1(Y)$ is dense in $(p(Y), \tau_1^*)$.

(iii) If $(p(Y), \tau_1^*)$ is discrete, then $F_1(Y) = p(Y)$.

(b) If $F(Y) \subset p(Y)$, then $F(Y)$ is dense in $(p(Y), \tau_2^*)$ and $(p(Y), \tau^*)$.

Proof. By Remark3.1.3, $i : Y \to p(Y)$ is an embedding with topology τ_1^*, τ_2^* or τ^*. $F_1(Y) = i(Y)$. Every member of β contains a finite subset of Y.

Lemma.3.1.5. If $F(Y) \subset p(Y)$ and $p(H_1, H_2, \ldots, H_n) = \{A\}$, then A is finite, $\cup\{H_i : i \in J_n\} = A$ and H_i is singleton $\forall\, i \in J_n$.

Proof. For $i \in J_n$, let $x_i \in H_i \cap A$. Then $A = \{x_i : i \in J_n\}$. Since for $x \in H_i$, $A \cup \{x\} - \{x_i\} \in p(H_1, H_2, \ldots, H_n)$, $\cup\{H_i : i \in J_n\} = A$. If $x_i, x_j \in H_i$, then $A - \{x_i\} \in p(H_1, H_2, \ldots, H_n)$. Therefore, H_i is singleton $\forall\, i \in J_n$.

Proposition.3.1.6. Let $F(Y) \subset p(Y)$.

(i) If Y has an isolated point, then $(p(Y), \tau_1^*)$ and $(p(Y), \tau^*)$ each has an isolated point.

(ii) If $(p(Y), \tau_1^*)$ or $(p(Y), \tau^*)$ has an isolated point, then Y has an isolated point.

(iii) If $(p(Y), \tau_2^*)$ has an isolated point, then Y is a singleton.

Proof. (i) If $L \subset Y$ is a singleton then $p(L)$ is a singleton.

(ii) If $p(L_1, L_2, \ldots, L_n) \in \beta$ equals $\{A\}$, then, by Lemma3.1.5, each G_i is a singleton.

(iii) If $(p(Y), \tau_2^*)$ has an isolated point $\{A\}$, then, in view of (iv) of Remark3.1.1, $\{A\} = p(Y, L_1, L_2, \ldots L_n)$. Now, by Lemma3.1.5, Y is a singleton.

Proposition.3.1.7. Let $F(Y) \subset p(Y)$. $(p(Y), \tau^*)$ is discrete iff Y is discrete and $p(Y) = F(Y)$.

Proof. Suppose $(p(Y), \tau^*)$ is discrete. Since τ^* is admissible, Y is discrete. Let $A \in p(Y)$. $\{A\} = p(L_1, L_2, \ldots L_n)$. By Lemma3.1.5, A is finite. Let Y be discrete and $p(Y) = F(Y)$. Then, for $A \in p(Y)$, $A = \{x_i : i \in J_n\}$ and $\{A\} = p(\{x_1\}, \{x_2\}, \ldots \{x_n\})$.

For $n \in IN$, let F_n denotes the all nonempty subsets of Y having at most n elements.

Proposition.3.1.8. (a) Let $F_1(Y) \subset p(Y)$. If Y is Hausdorff, then $i(Y)$ is closed in $(p(Y), \tau_2^*)$ and $(p(Y), \tau^*)$. If $i(Y)$ is closed in $(p(Y), \tau^*)$, then Y is T_1.

(b) Let $F(Y) \subset p(Y)$. If Y is Hausdorff, then, for each $n \in IN$, F_n is closed in $(p(Y), \tau_2^*)$ and $(p(Y), \tau^*)$.

Proof. (a) Let $A \in \mathrm{cl}(i(Y)) - i(Y)$. Let $x, y \in A$, $x \neq y$. $\exists\, L, V$ open in Y s.t. $x \in L$, $y \in V$ and $L \cap V = \varphi$. We have $A \in p(Y, L, V) \in \beta_2$. By Remark3.1.3, $p(Y, L, V) \cap i(Y) = \varphi$. Therefore, $i(Y)$ is closed in $(p(Y), \tau_2^*)$ and $(p(Y), \tau^*)$. Let $x, y \in Y$, $x \neq y$. Let $A = \{x, y\}$. $\exists\, p(L_1, L_2, \ldots, L_n) \in \beta$ s.t. $A \in p(L_1, L_2, \ldots, L_n)$ and $p(L_1, L_2, \ldots, L_n) \cap i(Y) = \varphi$. By an application of Remark3.1.3, $\exists\, L_i, L_j$ s.t. $x \in L_i$ and $y \notin L_i$, and $y \in L_j$ and $x \notin L_j$.

(b) We prove by induction F_n is closed in ($p(Y)$, τ_2*) and ($p(Y)$, τ*). Let A be a set containing at least n+1 elements $x_1, x_2,......,x_{n+1}$. ∃ pair wise disjoint open sets $L_1, L_2,...L_{n+1}$ containing $x_1, x_2,.......,x_{n+1}$ respectively. $A \in p(Y, L_1, L_2,.....,L_{n+1})$. By Remark3.1.3, $p(Y, L_1, L_2,......,L_{n+1}) \cap F_n = \varphi$. Hence F_n is closed in ($p(Y)$, τ_2*) and ($p(Y)$, τ*).

Remark.3.1.9. (i) If $Y = \cup\{H_i : i\in I\}$, then $p(Y) = \cup\{p^*(H_i) : i\in I\}$.

(ii) If $F_1(Y) \subset p(Y)$ and $p(Y) = \cup\{p^*(H_i) : i\in I\}$, then $Y = \cup\{H_i : i\in I\}$.

Proposition.3.1.10. Let $F_1(Y) \subset p(Y)$.

(a) If ($p(Y)$, τ_2*) or ($p(Y)$, τ*) is compact, then Y is compact.

(b) If Y is compact, then ($p(Y)$, τ_2*) is compact.

Proof. (a) Considering an open cover $\{H_i : i \in I\}$ of Y, by Remark3.1.9 (i), $\{p^*(H_i) : i \in I\}$ will be an open cover of $p(Y)$ in τ_2* and τ*. If $\{p^*(H_i) : i \in J\}$ is a finite subcover of $p(Y)$, then by Remark3.1.9 (ii), $\{H_i : i \in J\}$ will be a finite subcover of Y.

(b) For an open cover $\{p^*(H_i) : i \in I\}$ of ($p(Y)$, τ_2*), by Remark3.1.9 (ii), $\{H_i : i \in I\}$ will be an open cover of Y. Again using Remark3.1.9 (i), we get a finite subcover of ($p(Y)$, τ_2*).

4. Some New Results

In this section, we present a few results for $F(Y)$ in terms of Hausdroff metric space and dense.

Considering (Y, d) be a metric space. Suppose if, $U \subset Y$ and $x \in Y$, then define, $d(x, U) = \inf\{d(x, y) : y \in U\}$. Furthermore, for $U, V \in K(Y)$, then define $\rho(U, V) = \sup\{d(x, U) : x \in V\}$ and $d^*(U, V) = \sup\{d^*(U, V), d^*(V, U)\}$. Here, it is noticeable that, $((K(Y), d^*)$ is a metric space where d^* is the Hausdorff metric on $K(Y)$.

Lemma.4.1.1. Let $U \subset Y$ be compact. For $r > 0$, let $y_1, y_2,.......,y_n \in U$ and for $i \in J_n$, $S_i = S(y_i, r)$. Let $U \subset \cup\{S_i : i \in J_n\}$. If for each $i \in J_n$, $x_i \in S_i$ and $K = \{x_1, x_2,......,x_n\}$. Then

(i) $\rho(K, U) \leq 2r$.

(ii) If, for each $i \in J_n$, $x_i \in U$, that is $K \subset U$, then $d^*(K, U) \leq 2r$.

Proof. If $K \subset U$, implies $d(x, U) = 0\ \forall\ x \in K$, therefore $\rho(U, K) = 0$.

(i) Let $y \in U$. ∃, $j \in J_n$ s.t. $y \in S_j$. Furthermore, $d(y, K) \leq d(y, x_j) \leq d(y, y_j) + d(y_j, x_j) < 2r$. This implies that $\rho(K, A) \leq 2r$.

(ii) The proof of (ii) is similar to (i).

Proposition.4.1.2. $F(Y)$ is dense in ($K(Y)$, τ_{d^*}).

Proof. Let $U \in K(Y)$ and $t > 0$. Let $r = t/3$. $\{S(y, r) : y \in U\}$ is an open cover of U. So ∃ y_1,

$y_2, \ldots\ldots, y_n \in U$ s.t. $U \subset \cup\{S_i : i \in J_n\}$, where $S_i = S(y_i, r)$. Let, for $i \in J_n$, $x_i \in U \cap S_i$. Let $K = \{x_1, x_2, \ldots\ldots, x_n\}$. By Lemma4.1.1 (ii), we have $d^*(K, U) \leq 2r < t$. Therefore, $K \in S^*(U, t)$. Thus, $F(Y)$ is dense in $(K(Y), \tau_{d^*})$.

Remark.4.1.3. Consider D be a subset of metric space Y. Then, $\beta = \{S(x, t) : x \in D, t > 0\}$ is a base for the topology of Y if D is dense.

Corollery.4.1.4. Let $\beta^* = \{S^*(K, t) : K \in F(Y), t > 0\}$ is a base for τ_{d^*}.

Proof. The proof of this corollary directly follows from Remark4.1.3.

Lemma.4.1.5. Let $U, K \subset Y$, where U is compact and $K = \{x_1, x_2, \ldots x_n\}$. If $\rho(U, K) < t$, again for every, $i \in J_n$, $\exists\, y_i \in U$ s.t. $d(x_i, y_i) < t$.

Proof. Let $i \in J_n$. $d(x_i, U)$ is continuous and U is compact, therefore $\exists\, y_i \in U$ s.t. $d(x_i, U) = d(x_i, y_i)$. Since $\rho(U, K) = \sup\{d(x_i, U) : x_i \in K\}$, $d(x_i, U) \leq \rho(U, K) < t$. Hence $d(x_i, y_i) < t$.

Proposition.4.1.6. $\rho(\{x\}, U) < t$ iff $U \subset S(x, t)$.

Proof. $\rho(\{x\}, U) = \sup\{d(y, x) : y \in U\}$. Suppose $\rho(\{x\}, U) < t$. Let $y \in U$ and $d(y, x)$. So $y \in S(x, t)$. $\exists\, y^* \in U$ s.t. $d(y^*, x) = \rho(\{x\}, U)$. Since $y^* \in U$, $d(y^*, x) < t$. So $\rho(\{x\}, U) < t$.

Lemma.4.1.7. Let $K = \{x_1, x_2, \ldots\ldots, x_n\}$ and $t > 0$. Let for $i \in J_n$, $S_i = S(x_i, t)$. Let $U \in K(Y)$, then

(i) $U \subset \cup\{\, S(x_i, t) : i \in J_n\}$ iff $\rho(U, K) < t$.

(ii) $U \cap S(x_i, t)\ \forall\ i \in J_n$ iff $\rho(K, U) < t$.

Proof. Since $\rho(U, K) = \sup\{d(x_i, U) : x_i \in K\}$, $\exists\, x_j \in K$ s.t. $\rho(U, K) = d(x_j, U)$. $\exists\, y^* \in U$ s.t. $\rho(K, U) = d(y^*, K)$.

(i) Let $U \subset \cup\{S(x_i, t) : i \in J_n\}$. $y^* \in S_j$ for some $j \in J_n$. $\rho(K, U) = d(y^*, K) \leq d(y^*, x_j) < t$. Now let $y \in U$. For some $j \in J_n$, $d(y, x_j) = d(y, K) \leq \rho(K, U) < t$. So $y \in S_j$.

(ii) Let $y_i \in U \cap S_i$, for $i \in J_n$. $\exists\, x_j \in K$ s.t. $\rho(U, K) = d(x_j, U) \leq d(x_j, y_j) < t$. Let $i \in J_n$. For some $y_i \in U$, $d(x_i, y_i) = d(x_i, U) \leq \rho(U, K) < t$.

Proposition.4.2.1. $\rho(U, \{x\}) = d(x, U) \leq d(x, y)$ for all, $y \in U \leq \sup\{d(y, x) : y \in U\} = \rho(\{x\}, U)$. $\rho(U, \{x\}) \leq \rho(\{x\}, U)$, $d^*(U,\{x\}) = \rho(\{x\}, U)$.

Proof. $\rho(U, \{x\}) = \sup\{d(x, U) : x \in V\} = d(x, U)$. $\rho(\{x\}, U) = \sup\{d(y, x) : y \in U\}$. $d(x, U) \leq d(x, y) \leq \sup\{d(y, x) : y \in U\} = \rho(\{x\}, U)$.

Proposition.4.2.2. If $V \subset V^*$, then $\rho(U, V) \leq \rho(U, V^*)$. In particular, $\rho(\{x\}, U) \leq \rho(K, U)$ for all $x \in K$.

Proof. $\rho(U, V) = \sup\{d(x, U) : x \in V\} \leq \sup\{d(x, U) : x \in V^*\} = \rho(U, V^*)$.

Proposition.4.2.3. If $U \subset U^*$, then $\rho(U^*, V) \leq \rho(U, V)$.

Proof. $d(x, U^*) \leq d(x, U)$, $\sup\{d(x, U^*) : x \in V\} \leq \sup\{d(x, U) : x \in V\}$.

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