STUDY OF NONEXPANSIVE MAPPINGS THEORY

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Abstract

Complications in the general study of FPP were noted early. A major obstacle is the obvious fact that fixed point properties for nonexpansive mappings are not invariantunder reenormings. There are other hindrances as wel l. It has been known virtuallyfrom the outset that FPP for a Banach space depends strongly on 'nice'geometrical properties of the space.On the other hand, two closed convex subset K_1, K_2 X may have f.p.p. yet $K_1 \cap K_2$ may fail to have f.p.p. Indeed, even much more can be said.Goebel and kuczomow have shown to constructed a descending sequence $\{K_n\}$ of non empty bounded closed convex subsets of ℓ^1 which has the property that n is odd , K_n has f.p.p., if n is even K_n fails to have f.p.p. and in fact the sequence $\{K_n\}$ may be constructed so that $\cap K_n$ falls into either category. The space ℓ^1 provides the setting for another interesting example. It is possible to construct a family $\{K\epsilon\}$ ($\epsilon > 0$) of bounded closed convex sets in ℓ^1 each of which has f.p.p, but which converges as $\epsilon \to 0$ in the hausdroff meric to a non empty bounded closed convex K_0 which fails to have f.p.p.

Keywords: *nonexpansive, renormings, descending sequence, nonempty bounded*

Introduction:

The central questions of metric fixed point theory, especially as related to nonexpansive Mappings, usually involve the study of the following topics.

(I) Conditions which imply existence of fixed points.

(II) The structure of the fixed point sets.

(III) Asymptotic regularity.

(IV) The approximation of fixed points.

(V) Applications.

Here we take up, in order, some of the central results in each of the above categories.

Existence of fixed points: We begin with the study of nonexpansive mappings in a Banach space setting. If X is a Banach space and D $\subseteq$ X, then a mapping T:D → X is said to be *nonexpansive* if for each x,y ϵD,

$$\| T(x) - T(y) \| \leq \| x - y \|$$

The study of the existence of fixed points for nonexpansive mappings has generally fallen into three categories. We shall say that a Banach space has FPP if each of its nonempty bounded closed convex subsets has the fixed point property for nonexpansive self-mappings (which we denote f.p.p.); wc-FPP if each of its weakly compact convex subsets has f.p.p.; and B- FPP if its unit ball (hence any ball) has f.p.p. This latter category is primarily relevant to dual spaces where the unit ball is always compact in its weak* topology relative to any predual. The classical nonreflexive space ℓ^1 provides an example of a space which has **B-FPP** but not **FPP** (Karlovitz, Lim). Also $c_0$provides an example of a space which has wc-FPP but neither FPP nor B-FPP (Maurey).

Clearly one of the central goals of the theory should be to characterize those Banach spaces which have FPP. It is known that essentially all classical reflexive spaces, and in particular all uniformly convex spaces, have FPP, hence wc-FPP, via a geometric property they share called normal structure. As our point of departure, we shall state and prove the original 1965 fixed point theorem of Krik. It is an examination of the *proof* of this theorem which provides the basic of much that follows in the next section of this report.

Defination 1. A Banach space X is said to have *normalstructure* if any bounded convex subset K of X which contains more than one point contains a point x_0 such that

$$\text{Sup}\, \{\| x_0 - x \| : x \epsilon K\} < \text{diam}\, (K) := \sup\, \{\| x - y \| : x, y \epsilon K\}.$$

Such a point x_0 is called a *nondiametral point* of K.

In what follows we shall use the symbol B($x; r$) to denote the *closed ball* centered at x ϵK with radius r > 0. Thus:

$B\{x; r) = \{y \in K: \| x - y \| \leq r\}$.

Also we need some additional notation.

$$\mathrm{Diam}(K) = \sup\{\| u - \upsilon \in K\};$$

$$r_x(K) = \sup\{\| x - \upsilon \| : \upsilon \in K\}, \quad (x \in K);$$

$$r(K) = \inf\{r_x(K) : x \in K\}.$$

If *X* is reflexive and if *K* is a bounded closed and convex subset of *X* then it readily follows from the weak compactness of *K* that the set

$C(K) := \{z \in K : r_z(K) = r(K))$

called the *chebyshev center* of K, is a *nonempty* closed and convex subset of *K*.

Review of Literature:

For a chronological and methodological perspective, we list below (by name) a few of the more well-known fixed point theorems of functional analysis.

(a) The Zermelo-Bourbaki-Kneder Theorem (1908-1955)

(b) The Brouwer Theorem (1912)

(c) Banach's Contraction Mapping Principle (1922)

(d) The Schauder Theorem (1930)

(e) The Leray-Schauder Theorem (1934)

(f) The Schauder-Tychonff Theorem (1935)

(g) The Markov-Kakutani Theorem (1936)

(h) Tarski's Theorem (1955)

(i) The Browder-Gohde-Kirk Theorem (1956)

(j) The Ryll-Nardzewski Theorem (1966)

(k) Sadovskii's Theorem (1967)

(l) Caristi's Theorem (1976)

(m) Maurey's Theorem (1981)

Most of these theorems are well-known to specialists in fixed point theory. One might roughly characterize them as follows: (a) and (h) are set –theoretic; (b), (d), (e), and (f) are more topological in nature; the linear structure of the space plays a large role in (g) and (j); (c), (i), and (m) are primarily metric in nature; and (k) provides an example of a result which bridges the matric and topological theories. In our consideration of the metric theory, since (c) is well understood, we shall concentrate on the theory as it pertains to (i), (1), and (m), the rather surprising connections (i) and (1) have with (a), and on a number of more recent developments. We refer to Zeidler for a through discussion of the remaining theorems listed as well as numerous other fixed point theorems.

Theorem 1. Let *X*be a reflexive Banach space which has normal structure.

Then X has FPP.

Proof. Let *K* be a nonempty bounded closed and convex subset of *X*, and suppose $T : K \rightarrow K$is nonexpansive. Suppose $\mathfrak{I}$ denotes the collection of all nonempty closed convex T-invariant subsets of *K*. Then if $\mathfrak{I}$ is ordered by set inclusion, it follows from the weak comptances of the members of *K* (X is reflexive) that every descending chain in $\mathfrak{I}$ has a lower bound— namely the intersection of its members. Thus by Zorn's Lemma, $\mathfrak{I}$ has a minimal element, say K_0.

Obviously$_{conv}$T (K_0) is nonempty, closed, convex, and *T*-invariant; thus by minimality it cannot be a proper subset of $K_{0,}$ so

$$K_0 = conv\, T(K_0).$$

Let $\upsilon \in C(K_0)$; thus $r_u(K_0) = r(K_0)$. Since $\|T(u) - T(\upsilon)\| \leq \| u - \upsilon \| \leq r(K_0)$

for all $\upsilon \in K_0$, it follows that $T(K_0) \subset B(T(u); r(K_0))$. Consequently,

$$K_0 = \overline{conv}\, T(K_0) \subset B(T(u); r(K_0))$$

Showing that $r_{T(u)}(K_0) = r(K_0)$; thus $T(u)$ € $C(K_0)$. We conclude that $C(K_0)$ is T-invariant. The minimality of K_0 implies that $K_0 = C(K_0)$ and in view of normal structure this in trun implies that K_0 consists of a single point which is fixed under T.

Complications in the general study of FPP were noted early. A major obstacle is the obvious fact that fixed point properties for nonexpansive mappings are not invariant under renormings. There are other hindrances as well. It has been known virtually from the outset that FPP for a Banach space depends strongly on ‗nice' geometrical properties of the space. On the other hand, two closed convex subsets K_1, K_2□X *may have* f.p.p. yet $K_1 \cap K_2$ *may fail to have* f.p.p! Indeed, even much more can be said. Goebel and Kuczomow have shown to constructed a descending sequence $\{K_n\}$of nonempty bounded closed convex subsets of ℓ^1 which has the property that n is odd, K_n has f.p.p., if n is even K_n fails to have f.p.p., and in fact the sequence $\{K_n\}$may be constructed so that $\cap K_n$ falls into either category. The space ℓ^1 provides the setting for another interesting example. It is possible to construct a family {K€} (€ >0) of bounded closed convex sets in ℓ^1 each of which has f.p.p., but which converges as € →0 in the Hausdorff metric to a nonempty bounded closed convex K_0 which fails to have f.p.p.

Karlovitz first noted that even in reflexive spaces normal structure is not essential for FPP. An example is provided by the James's spaces X_β, $\beta \geq 0$, defined by:

$$X_\beta = \{x€\ell^2 : \|x\|_\beta = \max\{\|x\|_{\ell^2}, \beta\|x\|_\infty\}\}.$$

RC James observed that while X_β is reflexive (since it is isomorphic to ℓ^2), it fails to have

normal structure if $\beta = \sqrt{2}$. In fact ,X_β structure which implies $\beta < \sqrt{2}$.. Even more is known. The concept of asymptotic normal structure was introduced by Baillon and Schoneberg in 1981. A Banach space X has *asymptoticnormal structure* if each nonempty bounded closed and convex subset K of X which contains more than one point has the property: If $\{x_n\}$□ Ksatisfies $\|x_n - x_n+1\| \to 0$ then there exists x€ K such that

Lim lim $\|x_n - x\| <$diam(K).

$j \to \infty$

But Baillon and Schoneberg observe that X_β has asymptotic normal structure ⇔ $\beta < 2$, and they prove the following:

Theorem 2. In a reflexive Banach space, asymptotic normal structure⇒FPP.

In the same research Baillon and Schoneberg went on to show that even X_2 has FPP, thus showing that asymptotic normal structure is not a necessary condition for FPP. (Surprisingly, P.K. Lin proved in 1985 that X_β has FPP for all $\beta > 0$.)

It is actually shown in that in an arbitrary Banach space asymptotic normal structure implies wc-FPP. There has been an interesting further development regarding wc-FPP. In A. Jimenez-Melado and E. Llorens Fuster introduced a generalization of uniform convexity called orthogonal convexity and proved that weekly compact convex subsets of orthogonally convex spaces have the fixed point property for nonexpansive mappings.

Orthogonal convexity is defined as follows: For points $x,y$of a Banach space X and $\lambda > 0$, let

$$M_\lambda(x,y)=\{z \in X : \max\{\|z-x\|,\|z-y.\|\}\le\tfrac{1}{2}(1+\lambda)\,\|x-y\|\}$$

If A *is a bounded subset of* X,*let*$|A|=\sup\{\|x\|: x\in A\}$, and for a bounded sequence $\{x_n\}$in X and $\lambda > 0$, let

$$D(\{x_n\}) = \limsup_{i\to\infty}\,(\limsup_{j\to\infty}\|x_i - x_j\|);$$

$$A\lambda(\{x_n\}) = \limsup_{i\to\infty}\,(\limsup_{j\to\infty}|M\lambda(x_i, x_j)|).$$

A Banach space is said to be *orthogonallyconvex* if for each sequence $\{x_n\}$ in X which converges weakly to 0 and for which $D(\{x_n\}) > 0$, there exists $\lambda > 0$ such that

A $\lambda(\{x_n\})< D(\{x_n\})$. It is shown in every uniformly convex space is orthogonally convex. Other examples enclude Banach spaces with the Schur property (hence ℓ_1), c_0, c, and James's space J.

In 1971 it was observed by Day-James-Swaminathan that every separable space has an equivalent norm which has normal structure. *Thus every separable reflexive space has an equivalent norm which has* FPP. (It appears to be an open question whether every reflexive Banach space has an equivalent norm which has normal structure.)

The question of whether reflexive is essential for FPP remains open, but there is some recent evidence that it might be. First, it is known that the classical nonreflexivity space c_0 and in ℓ^1 fail to have FPP. Also, Bessaga and Pelczynski have shown that if X is any Banach space with an unconditional basis, then X is non-reflexive $\Leftrightarrow X$ contains a subspace isomorphic to c_0 or ℓ^1.

Thus all classical nonreflexive can be renormed so that they fail to have FPP.

This raises an obvious question: Can c_0 or or ℓ^1 be renormed so that they have FPP? Recall that any renorming of ℓ^1 contains almost isometric copies of ℓ^1 suggesting, at least for ℓ^1, that the answer should be no. If indeed the answer is no, then by the Bessaga-Pelczynski result, in any space with an unconditional basis, FPP$\Rightarrow$reflexivity.

The space L^1: As we have noted ℓ^1 (hence L^1) fails to have FPP.

However, in 181, Alspach proved much more, namely that L^1 fails to have wcFPP. At the same time, Maurey proved that all reflexive subspaces of L^1 do have FPP (hence wc-FPP. There has been another recent development. Dowling and Lennard have shown that nonreflexive subspaces of L^1fail to have FPP. Thus: ***A subspace of L^1has FPP $\Leftrightarrow$ is reflexive.***

Conclusion:

The above examples illustrate why the problem of classifying Banach spaces which have FPP or sets which have f.p.p. migh be extremely difficult. However,Theorem 1 raises the obvious question of precisely how are reflexivity,normal structure and FPP related. But the equation remains:

Does FPP⇒ reflexivity?

Of course the reverse implication remains unknown as well. In fact the following question also remain open:

Does superreflexivity⇒ FPP?

Recall that a superreflexive space is one which has the property that every space whichis finitely representable in it must itself be reflexive. In theorem 2 ,Superreflexive spaces are also characterized by that fact that they all have equivalent uniformly convex norms. Maurey proved that superreflexive spaces have FPP for *isometric suggests* that the answer to the above might be yes.

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