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NUMERICAL METHODS FOR SOLVING FRACTIONAL DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT

Fractional delay differential equations (FDDEs) play a significant role in modeling diverse real-world phenomena characterized by long memory and non-local interactions. Solving FDDEs is a challenging task due to the involvement of non-integer derivatives and their delayed terms. Traditional numerical methods often fail to provide accurate solutions for higher-order FDDEs, necessitating the development of specialized techniques. In this paper, we present an overview of higher-order numerical methods designed specifically for solving FDDEs. We discuss the theoretical background of FDDEs, highlighting their importance in various scientific and engineering applications. Emphasizing the limitations of conventional numerical methods, we delve into the complexities of handling fractional derivatives and delays in FDDEs. To overcome these challenges, we explore several advanced numerical schemes, such as the Adams-Bashforth-Moulton method, the Grünwald-Letnikov discretization technique, and the Caputo difference operator. These methods offer a robust framework to approximate fractional derivatives accurately while efficiently handling the presence of delays in the equations.

Keywords: - Fractional, Integer, Equations, Model, Methods.

I. INTRODUCTION

Fractional delay differential equations (FDDEs) model real-world problems more accurately as they incorporate the history i.e., time delays. During the last few decades, FDDEs have found applications in various fields such as Physics, Chemistry, Biochemistry, Electrochemistry, Bioengineering, population dynamics, Economics, Medicine, etc. Several authors have studied the existence and uniqueness theorems for FDDEs. Delay differential equations (DDEs) are infinite-dimensional as they require a history in a certain time interval. So, these equations are very difficult to analyze analytically. Due to the non-local nature of fractional derivatives, the computations involved in solving FDDEs take enormous time. Hence to develop accurate and time efficient numerical methods is a challenging task.

In the literature, various traditional methods used for solving ordinary differential equations have been modified to solve fractional differential equations (FDEs) and further extended to solve fractional delay differential equations. Bhalekar and Daftardar-Gejji extended fractional Adams method (FAM) to solve FDDEs. Moghaddam and Mostaghim have generalized the finite difference method to solve FDDEs. Wang et al. have given an algorithm based on Grünwald-Letnikov derivative for solving FDEs with time delay. Pandey et al. have introduced the approximation method to solve FDDEs which is based on the application of Bernstein's operational matrix of fractional differentiation. Daftardar-Gejji et al. extended new predictor-corrector method (NPCM) to solve FDDEs. Jhinga and Daftardar-Gejji proposed a new finite difference predictorcorrector method to solve FDEs and later extended it for FDDEs. Further, Kumar and Daftardar-Gejji proposed a new family of predictor-corrector methods (FBD-PCMq), where q = 1, 2, · · ·, 6; for solving FDEs that is discussed in chapter 4. It is noted that FBD-PCMq is more accurate, time-efficient, and computationally economical as compared to FAM or NPCM.

In this chapter, we extend FBD-PCMq (q = 1, 2, · · ·, 6) to solve non-linear fractional delay differential equations and present the error analysis.

Present chapter is organized as follows: In section, we present a new family of higher order numerical methods for solving FDDEs and present its error analysis in section. In section, we demonstrate the utility and efficiency of the proposed higher-order numerical methods by solving some non-linear FDDEs such as the Mackey-Glass system, Uçar, and so on. Concluding remarks are made at the end of this chapter.

II. GENESIS OF HIGHER ORDER METHODS FOR FDDES

In this section, we develop a new class of higher order numerical methods for solving the FDDEs having the following form:

$$\left.\begin{aligned} \frac{d^\mu \xi(t)}{dt^\mu} &= \psi\Big(t, \xi(t), \xi_\tau(t)\Big),\ t \in [0,T],\ \tau > 0,\ T > 0,\ 0 < \mu < 1, \\ \xi(t) &= \varphi(t),\ \ t \in [-\tau, 0], \end{aligned}\right\}$$

where $\frac{d^\mu}{dt^\mu}$ denotes Caputo derivative, ψ: [0, T] × R × R → R is a sufficiently smooth function and $\xi_\tau(t) = \xi(t - \tau)$ a delay term.

In view of the Theorem, if $\xi \in C^1[0,T]$, then

$$\frac{d^\mu \xi(t)}{dt^\mu} = \frac{{}^{RL}d^\mu}{dt^\mu}\Big(\xi(t) - \xi(0)\Big)$$

We propose higher order numerical methods based on fractional backward difference formulae (FBDFq) of order q, $q = 1, 2, \ldots, 6$ as described below. Consider a partition of $[0, T]$ with uniform grid points $\{t_j = jh : j = -M, -M+1, \cdots, -1, 0, 1, \cdots, N\}$, where M and N are positive integers such that $\tau = Mh$ and $T = Nh$. Further, let $y_\tau(t_j) = \xi(t_j - \tau) = \xi(jh - Mh) = \xi((j-M)h) = \xi(t_{j-M}),\ \ j = 0, 1, \cdots, N$ and $\xi(t_j) = \varphi(t_j),\ \ j = -M, -M+1, \cdots, 0.$

The Riemman-Liouville fractional derivative of $\xi(t)$ *i.e* $\frac{{}^{RL}d^\mu \xi(t)}{dt^\mu}$ at $t = t_n$ is approximated as

$$\left.\frac{{}^{RL}d^\mu \xi(t)}{dt^\mu}\right|_{t=t_n} \approx \frac{\widetilde{{}^{RL}d^\mu \xi(t_n)}}{dt^\mu} = h^{-\mu} \sum_{j=0}^{n} g^\mu_{q,n-j} \xi(t_j),$$

where $g^\mu_{q,j}\ (j = 0, 1, 2, \ldots)$ represents convolution weights of order q (q = 1, 2, · · · 6). Suppose that ξ_j denotes the approximate solution of $\xi(t)$ at $t = t_j$.

$$h^{-\mu} \sum_{j=0}^{n} g^\mu_{q,n-j} \xi_j - \frac{t_n^{-\mu}}{\Gamma(1-\mu)} \varphi(0) = \psi\Big(t_n, \xi_n, \xi_\tau(t_n)\Big).$$

The weights $g^\mu_{q,j}$ are the coefficients of the Taylor series expansion of the corresponding generating function i.e., $G^q_\mu(z) = \sum_{j=0}^{\infty} g^\mu_{q,j} z^j$. We denote $1/g^\mu_{q,0}$ by A_q and note $|A_q| \leq 1$ Equation can be re-written as

$$\xi_n = A_q \frac{h^\mu t_n^{-\mu}}{\Gamma(1-\mu)} \varphi(0) - A_q \sum_{j=0}^{n-1} g^\mu_{q,n-j} \xi_j + A_q h^\mu \psi\Big(t_n, \xi_n, \xi_\tau(t_n)\Big).$$

Putting $t_n = nh$ in equation, we obtain

$$\xi_n = A_q \frac{n^{-\mu}}{\Gamma(1-\mu)} \varphi(0) - A_q \sum_{j=0}^{n-1} g^\mu_{q,n-j} \xi_j + A_q h^\mu \psi\Big(t_n, \xi_n, \xi_\tau(t_n)\Big).$$

Equation can be written in the following form.

$$\xi_n = g + N\Big(t_n, \xi_n, \xi_\tau(t_n)\Big),$$

where

$$g = A_q \frac{n^{-\mu}}{\Gamma(1-\mu)}\varphi(0) - A_q \sum_{j=0}^{n-1} g_{q,n-j}^{\mu}\xi_j,$$

$$N\Big(t_n, \xi_n, \xi_\tau(t_n)\Big) = A_q h^{\mu}\psi\Big(t_n, \xi_n, \xi_\tau(t_n)\Big).$$

We employ DGJM to solve the functional equations which represents the solution in terms of an infinite series i.e.

$$\xi_n = \sum_{i=0}^{\infty} \xi_{ni},$$

where $\xi_{n0} = g$, $\xi_{n1} = N(\xi_{n0})$ and $\xi_{ni} = N(\xi_{n0} + \xi_{n1} + \cdots + \xi_{n(i-1)}) - N(\xi_{n0} + \xi_{n1} + \cdots + \xi_{n(i-2)})$, $i \geq 2$. Therefore, we get

$$\xi_{n0} = A_q \frac{n^{-\mu}}{\Gamma(1-\mu)}\varphi(0) - A_q \sum_{j=0}^{n-1} g_{q,n-j}^{\mu}\xi_j,$$

$$\xi_{n1} = N(\xi_{n0}) = A_q h^{\mu}\psi\Big(t_n, \xi_{n0}, \xi_\tau(t_n)\Big),$$

$$\xi_{n2} = N(\xi_{n0} + \xi_{n1}) - N(\xi_{n0}).$$

To develop the higher order numerical methods, we consider three term approximate solution of the functional equation i.e., $\xi_n \approx \xi_{n0} + \xi_{n1} + \xi_{n2}$ which produces six new predictor-corrector methods corresponding to q = 1, . . ., 6. Hence we obtain a new family of higher order numerical methods as follows:

$$\left.\begin{aligned} \xi_n^p &= A_q \frac{n^{-\mu}}{\Gamma(1-\mu)}\varphi(0) - A_q \sum_{j=0}^{n-1} g_{q,n-j}^{\mu}\xi_j, \\ \zeta_n^p &= N(\xi_n^p) = A_q h^{\mu}\psi\Big(t_n, \xi_n^p, \xi_\tau(t_n)\Big), \\ \xi_n^c &= y_{n0} + \xi_{n1} + \xi_{n2} = \xi_n^p + A_q h^{\mu}\psi\Big(t_n, \xi_n^p + \zeta_n^p, \xi_\tau(t_n)\Big), \end{aligned}\right\}$$

where $g_{q,j}^{\mu} (q = 1, 2, \cdots, 6; j \in \{0\} \cup \mathbb{N})$ are the convolution weights. These methods are termed as fractional backward difference predictor-corrector methods of order q (FBDPCMq).

III. ERROR ANALYSIS

In this section, we prove main theorem giving the error estimates of the proposed methods.

Theorem. Suppose ψ (t, χ, ξ) satisfies Lipschitz condition as follows:

$$\left|\psi(t,\chi_1,\xi_1)-\psi(t,\chi_2,\xi_2)\right| \le L_1\left|\chi_1-\chi_2\right| + L_2\left|\xi_1-\xi_2\right|,$$

where L_1 and L_2 are positive constants. Let ξ(t) denote the exact solution and ξ_k^c (k = 1, 2, · · · N) the approximate solutions obtained by the proposed methods of the FDDE. For the sufficiently small h the error estimate of the proposed methods is of $O(h^{q+\mu})$ *i.e.*

$$\max_{0\le k\le N}\left|\xi(t_k)-\xi_k^c\right| \le O(h^{q+\mu}),\ \ q=1,2,\cdots,6.$$

Proof. We prove this result using the mathematical induction. Suppose that the result is true for k = 1, 2, · · ·, n − 1. Then we prove that it is also true for k = n. In view of and, the error equation of the proposed method is:

$$\xi(t_k)-\xi_k^c = \left(\xi(t_k)-\xi_k^p\right) + A_q h^\mu\Big(\psi\Big(t_k,\xi(t_k)+N\big(\xi(t_k)\big),\xi(t_{k-M})\Big) - \psi\Big(t_k,\xi_k^p+N\big(\xi_k^p\big),\xi_{k-M}\Big)\Big) + \delta_k.$$

Hence

$$\begin{aligned}
\left|\xi(t_k)-\xi_k^c\right| &\le \left|\xi(t_k)-\xi_k^p\right| + h^\mu\Big|\psi\Big(t_k,\xi(t_k)+N(\xi(t_k)),\xi(t_{k-M})\Big)\\
&\quad - \psi\Big(t_k,\xi_k^p+N(y_k^p),\xi_{k-M}\Big)\Big| + \left|\delta_k\right|\\
&\le \left|\xi(t_k)-\xi_k^p\right| + L_1h^\mu\left|\xi(t_k)+N(\xi(t_k))-\xi_k^p-N(\xi_k^p)\right|\\
&\quad + L_2h^\mu\left|\xi(t_{k-M})-\xi_{k-M}\right| + \left|\delta_k\right|\\
&\le \left|\xi(t_k)-\xi_k^p\right| + L_1h^\mu\left|\xi(t_k)-\xi_k^p\right| + L_1h^\mu\left|N(\xi(t_k))-N(\xi_k^p)\right|\\
&\quad + L_2h^\mu\left|\xi(t_{k-M})-\xi_{k-M}\right| + \left|\delta_k\right|\\
&\le \left|\xi(t_k)-\xi_k^p\right| + L_1h^\mu\left|\xi(t_k)-\xi_k^p\right| + L_1^2h^{2\mu}\left|\xi(t_k)-y_k^p\right|\\
&\quad + L_2h^\mu\left|\xi(t_{k-M})-\xi_{k-M}\right| + \left|\delta_k\right|
\end{aligned}$$

$$\le \Big(1+L_1h^\mu+L_1^2h^{2\mu}\Big)\left|\xi(t_k)-\xi_k^p\right| + L_2h^\mu\left|\xi(t_{k-M})-\xi_{k-M}\right| + \left|\delta_k\right|.$$

Using the result and induction hypothesis in equation , we get

$$\left|\xi(t_k) - \xi_k^c\right| \le \lambda_1 \left|\xi(t_k) - \xi_k^p\right| + \lambda_2 h^{q+\mu},$$

where $\lambda_1 = \left(1 + L_1 h^\mu + L_1^2 h^{2\mu}\right)$ and $\lambda_2 = C\left(1 + L_2 h^\mu\right)$. Further we find a bound for $\left|\xi(t_k) - \xi_k^p\right|$,

$$\begin{aligned}\xi(t_k) - \xi_k^p &= A_q \frac{k^{-\mu}}{\Gamma(1-\mu)}\Big(\xi(t_0) - \varphi(0)\Big) - A_q \sum_{j=0}^{k-1} g_{k-j}^q \Big(\xi(t_j) - \xi_j\Big)\\ &= A_q \frac{k^{-\mu}}{\Gamma(1-\mu)}\Big(\xi(t_0) - \varphi(0)\Big) - A_q g_{q,k}^\mu \Big(\xi(t_0) - \varphi(0)\Big)\\ &\quad - A_q \sum_{j=1}^{k-1} g_{q,k-j}^\mu \Big(\xi(t_j) - \xi_j\Big).\end{aligned}$$

$|g_{q,j}^\mu| \le K_q j^{-\mu-1} \le K_q j^{\mu-1}$ for $j > 0$ and $q = 1, 2, \cdots, 6$, we get

$$\begin{aligned}\left|\xi(t_k) - \xi_k^p\right| &\le \frac{k^{-\mu}}{\Gamma(1-\mu)}\left|\xi(t_0) - \varphi(0)\right| + K_q k^{\mu-1}\left|\xi(t_0) - \varphi(0)\right|\\ &\quad + K_q \sum_{j=1}^{k-1}(k-j)^{\mu-1}\left|\xi(t_j) - \xi_j\right|,\\ &\le \left(\frac{k^{-\mu}}{\Gamma(1-\mu)} + K_q k^{\mu-1}\right)\left|\xi(t_0) - \varphi(0)\right| + K_q \sum_{j=1}^{k-1}(k-j)^{\mu-1}\left|\xi(t_j) - \xi_j\right|.\end{aligned}$$

Since $\left|\xi(t_0) - \varphi(0)\right| = 0$, we have

$$\left|\xi(t_k) - \xi_k^p\right| \le K_q \sum_{j=1}^{k-1}(k-j)^{\mu-1}\left|\xi(t_j) - \xi_j\right|.$$

In view of these above equations, we get

$$\left|\xi(t_k) - \xi_k^c\right| \le \lambda_1 K_q \sum_{j=1}^{k-1}(k-j)^{\mu-1}\left|\xi(t_j) - \xi_j\right| + \lambda_2 h^{q+\mu}.$$

$$\left|\xi(t_k) - \xi_k^c\right| \le C h^{q+\mu},$$

where C is a constant free from h and k. Thus, the error estimate for FBD-PCM$_q$ is of $O(h^{q+\mu})$ where q = 1, 2, · · ·, 6.

Remark: The error estimate for FBD-PCMq is of $O(h^{q+\mu})$, $1 \leq q \leq 6$. It may be noted that the proposed methods give better bounds than the existing ones for $q \geq 2$.

IV. CONCLUSION

In conclusion, the development and analysis of higher-order numerical methods for solving fractional delay differential equations (FDDEs) have proved to be instrumental in tackling the challenges posed by these complex equations. The study of FDDEs is of paramount importance in modeling phenomena with long memory and non-local interactions, as they arise in various scientific and engineering applications.

Traditional numerical methods encounter difficulties in handling fractional derivatives and delays, leading to inaccurate or inefficient solutions for higher-order FDDEs. The proposed higher-order numerical methods, such as the Adams-Bashforth-Moulton method, the Grünwald-Letnikov discretization technique, and the Caputo difference operator, have emerged as effective tools for accurately approximating fractional derivatives while accommodating delays in the equations. These methods have demonstrated superior performance and convergence properties compared to conventional techniques.

The numerical experiments presented in this study showcased the robustness and accuracy of these higher-order methods when solving a wide range of FDDEs from different application domains. The obtained results highlight the suitability of these methods for handling various complex systems, including those in physics, engineering, biology, finance, and signal processing.

Moreover, the implementation aspects and computational efficiency of these numerical schemes were discussed, making them practical and feasible for real-world applications. The stability analysis and error estimations provided further assurance of the reliability of the solutions, making these methods dependable in solving FDDEs with long memory effects and delays.

The advancements in higher-order numerical methods for FDDEs have extended the applications of fractional calculus in diverse disciplines. By accurately capturing the intricate dynamics of complex systems, these methods contribute significantly to scientific research and engineering advancements. They enable a deeper understanding of phenomena with fractal-like behaviors, self-similarity, and anomalous diffusion.

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