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A FIVE COLOR MAP PROBLEM

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ABSTRACT

This paper presents a solution to the problem of putting five different colors on a map, where there is a minimum distance between each of the colors.We propose a solution using 4 trapezoids and calculate the colored area of the map. We then introduce a refinement into the solution by rounding the lines of the trapezoid to increase the colored area

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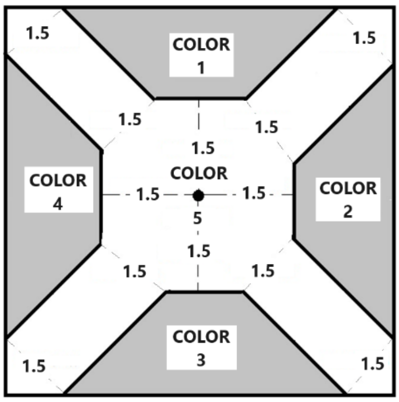
The Problem

Consider a square map where we arbitrarily set the length of each side as 6. The map's area is thus 36. There are five different colors on the map. Assume that the distance between one color and another color must be at least 1.5.

A Solution

One solution is to put one color in a trapezoidal shape on each of the four sides of the map, and put a dot[i] of Color 5 in the center (see Figure 1)

Figure 1

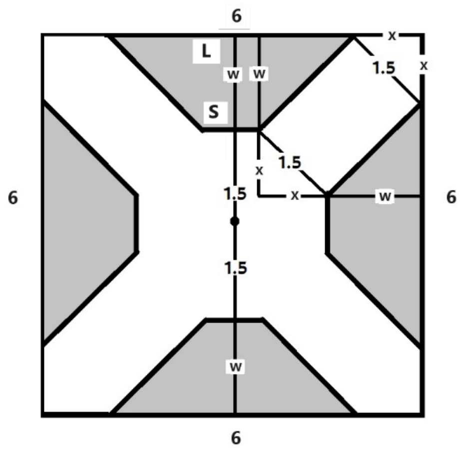


In this solution, Color 5 is exactly 1.5 units distant from each one of the other colors; Color 1 is exactly 1.5 units distant from Colors 2 and 4; Color 2 is exactly 1.5 units distant from Colors 1 and 3; Color 3 is exactly 1.5 units distant from Colors 2 and 4;andColor 4 is exactly 1.5 units distant from Colors1 and 3.

Calculating the colored area

Consider Figure 2.

Figure 2



The width of each trapezoid, W = 1.5.

Proof: Consider the vertical line that goes through Color 5 in the center.The total length of the line is the map length, 6, so:

$$W + 1.5 + 1.5 + W = 6$$
$$W = 1.5$$ **Q.E.D.**

The length of the longer side of each trapezoidL = 6 – 3 / √2≈3.88

Proof: For the triangle at each corner, we get:

$$X^2 + X^2 = 1.5^2$$
$$X = 1.5/\sqrt{2}$$

Since the length of the side is six and there are two corners, we get:

$$L + 2X = 6$$
$$L + 3/\sqrt{2} = 6$$
$$L = 6 - 3/\sqrt{2}$$ **Q.E.D.**

The length of the shorter side of each trapezoidS =3 – 3/√2 ≈0.88.

Proof: The length from the end of the shorter side to the edge of the map is X + W, so,

$$S = 6 - 2(W + X) = 6 - 3 - 3/\sqrt{2} = 3 - 3/\sqrt{2}$$ **Q.E.D.**

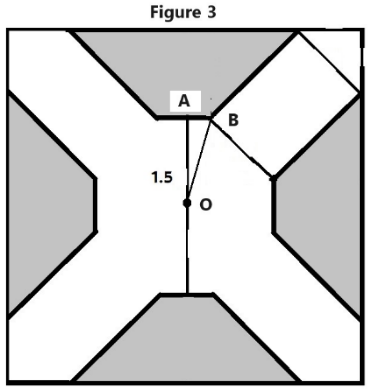
The area of a trapezoid = ½ xW x (L + S).Since there are four trapezoids, the total colored area is:

$$4 \times \frac{1}{2} \times 1.5 \times (6 - 3/\sqrt{2} + 3 - 3/\sqrt{2}) = 27 - 18\sqrt{2} \approx 14.27$$

A Refinement

However, the above solution can be improved. Consider Figure 3.

Figure 3



Since OB is the hypotenuse of the right triangle OAB, it must be longer than OA. Thus, the shorter side of each trapezoid can be rounded (see Figure 4), so that the distance from Color 5 to every point on the rounded part is 1.5, which increases the colored area.

Figure 4

Additions

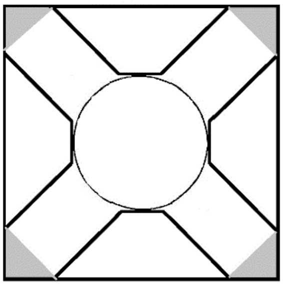
Calculating the colored area of the map is not a trivial problem. The rest of the paper presents the solution for calculating this area.

Calculating the colored area of the map

The colored area can be calculated by calculating the non-colored area and subtracting that from the total area of the map (36). This is broken down into four independent steps.

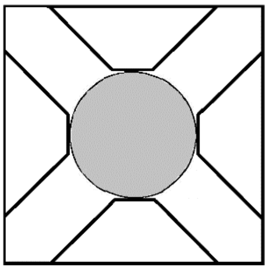
A. Calculate the area of the four corner triangles that are not colored (see Figure 5).

Figure 5



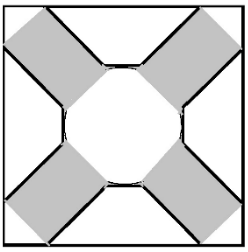
B. Add to that the area of the circle that is not colored (see Figure 6).

Figure 6



C. Add to that the four rectangles that are not colored (see Figure 7).

Figure 7

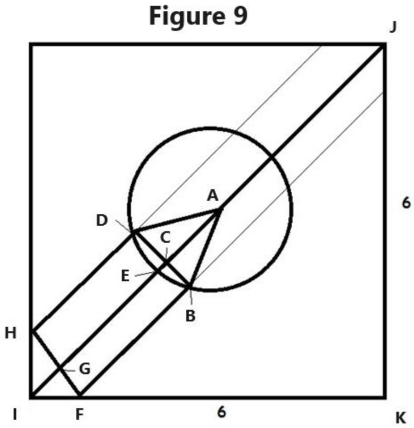


D. The final step is to recognize thatthe area of the rectangles that is also in the circle is being double counted (see Figure 8). The areas overlapping must be calculated and then subtracted to get the total non-colored area.

Figure 8

To solve this problem, we present Figure 9:

Figure 9



A. Calculating the area of the four corner triangles:

HF, the hypotenuse of the triangleHIF = 1.5
Since HIF is a right isosceles triangle, we get:

$$HI = IF = 1.5 / \sqrt{2}$$

The area of each triangle is:

$$(1.5 / \sqrt{2} \times 1.5 / \sqrt{2}) / 2 = 1.5^2 / 4$$

Since there are four triangles, the total area of the triangles is1.5^2= 2.25.

B. Calculating the area of the circle:

$$AE = \text{circle radius} = 1.5$$

So, the area of the circle is:

$$\Pi\ 1.5^2 = 2.25\pi$$

C. Calculating the area of the four rectangles:

The area of one rectangle is DH x HF.

$$HF = 1.5$$

$$DH = CG = AI - AC - GI = 3\sqrt{2} - .75 - .75\sqrt{3} \text{ (see Proof below).}$$

So, the area of all four rectangles is:

$$4 \times DH \times HF = 4 \times (3\sqrt{2} - .75 - .75\sqrt{3}) \times 1.5 = 18\sqrt{2} - 4.5 - 4.5\sqrt{3}$$

Proof that CG = 3 √2 - .75 - .75 √3:

$$JK = IK = 6$$

Since IJK is a right isosceles triangle, we get:

$$IJ = 6\sqrt{2}$$

Since AI is half of IJ, we get:

$$AI = 3\sqrt{2}$$

$$AE = \text{radius of the circle} = 1.5$$

$$IE = AI - AE = 3\sqrt{2} - 1.5$$

$$EC = 1.5 - .75\sqrt{3} \text{ (see Proof below)}$$

$$IG = .75 \text{ (see Proof below), so}$$
$$CG = AI - IG - AC = AI - IG - (AE - EC) =$$
$$3\sqrt{2} - .75 - (1.5 - (1.5 - .75\sqrt{3}) = 3\sqrt{2} - .75 - .75\sqrt{3}$$ **Q.E.D.**

Proof the EC = 1.5 - .75 √3:

$$BD = 2\sqrt{(AE^2 - AC^2)}$$ [ii]
$$AC = \sqrt{(AE^2 - BD^2 / 4)}$$

Since BD = HF = 1.5 and AE = 1.5

$$AC = \sqrt{(1.5^2 - 1.5^2 /4)} = .75\sqrt{3}$$

Since

$$AE = AC + EC$$
$$EC = AE - AC = 1.5 - .75\sqrt{3}$$ **Q.E.D.**

Proof that IG = .75:

$$HG^2 + IG^2 = HI^2$$
$$HG = HF / 2 = 1.5 / 2 = .75$$

We have already shown that HI = 1.5 / √2, so

$$.75^2 + IG^2 = (1.5 / \sqrt{2})^2$$
$$IG = .75$$ **Q.E.D.**

D. Calculating the total of the overlapping areas:

$$BD = HF = 1.5$$
$$AD = AB = 1.5$$

Thus, the triangle ABCD is equilateral.
Which means that the angle DAB = 60^0
Since a circle is 360^0, the part of the circle ADEB is 1/6 of the circle.
Since the area of the total circle is $\pi 1.5^2$, the area of ADEB is $\pi 1.5^2 / 6$
The area of an equilateral triangle whose side = S is: $S^2\sqrt{3} / 4$.
Since the triangle ABCD is equilateral, and each side = 1.5, the area of the triangle is: $1.5^2\sqrt{3} / 4$
We calculate an overlapping area as:

$$BCDE = ABED - ABCD = \pi 1.5^2 / 6 - 1.5^2\sqrt{3} / 4$$

Since there are four overlapping areas, their total area = $1.5\pi - 1.5^2\sqrt{3}$

Thus, the total of the non-colored area is:
The area of the four triangles, plus the area of the circle, plus the area of the four rectangles, minus the four overlapping areas, or:

$$(2.25) + (2.25\pi) + (18\sqrt{2} - 4.5 - 4.5\sqrt{3}) - (1.5\pi - 1.5^2\sqrt{3}) =$$
$$.75\pi - 2.25 + 18\sqrt{2} - 2.25\sqrt{3}$$

So, the area that is colored is:

$$36 - (.75\pi - 2.25 + 18\sqrt{2} - 2.25\sqrt{3}) \approx 14.34$$

[i] The dot is assumed to be dimensionless.
[ii] The proof of this can be found in any geometry textbook.