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ON $|V,\lambda,\delta|_k$ SUMMABILITY FACTOR OF INFINITE SERIES:

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Abstract

Here…

1.1 DEFINITIONS AND NOTATIONS:

Definition 1:

Let $\sum a_n$ be an infinite series with Partial sum s_n. By u_n^α we denotes the n-th Cesár means of order $\alpha, (\alpha > -1)$ of the sequence $\{s_n\}$. Absolute summability of order k was defined for Cesár methods of order α by FLET [1], [2]. He also defined summability $|C,\alpha,\delta|_k \geqslant 1$, as follows,

A series is summable $|C,\alpha,\delta|_k$ if,

$$\sum_{n=1}^{\infty} n^{\delta k+k-1}|U_n^\alpha - U_{n-1}^\alpha|^k < \infty \qquad (1.1.1)$$

Definition 2: The series $\sum a_n$ will said to be Summable $|V,\lambda,\delta|_k, k \geqslant 1$, if

$$\sum_{n=1}^{\infty} \lambda_n^{\delta k+k-1}|V_{n+1}(\lambda) - V_n(\lambda)|^k < \infty \qquad (1.1.2)$$

for $\lambda_n = n$ it is reduces to $|C,I,\delta|_k$ and for $k = 1, \delta = 0$ it is same as $|V,\lambda\}$ summability.

Definition 3: Let $\sum a_n$ is an infinite series then it is said to be stongly bounded by logarithmic means with index 1 or simply bounded $\{R, \log n, 1\}$, if,

$$\sum_{v=1}^{n} \frac{|S_v|}{v} = o(\log n),\ as\ n \to \infty \qquad (1.1.3)$$

for any sequence $\{\epsilon_n\}$ we write

$$\Delta\epsilon_n = \epsilon_n - \epsilon_{n+1}\Delta^2\epsilon_n = \Delta(\Delta\epsilon_n)$$

1.2 INTRODUCTION:

Generalizing the result of PRASAD [3] for $|V, \lambda|$ Summability which is given by theorem SINHA, CHANDRA and KUMAR [4] have proved the following theorem:

Theorem: If,

$$\epsilon_m \mu_m = O(1)\ as\ m \to \infty \qquad (1.1.1)$$

$$\sum_{1}^{m} \lambda_n \mu_n |\Delta^2 \epsilon_n| = O(1) \qquad (1.1.2)$$

and

$$\sum_{1}^{m} \frac{|t_v'|}{\lambda_v} = O(\rho_m \mu_m)\ as\ m \to \infty \qquad (1.1.3)$$

where

$\mu_n = \sum_{i=1}^{n} \lambda_i^{-1}$ and $\{\rho_n\}$ are positive non-decreasing sequences such that:

$$\Delta_n \mu_n p_n \Delta\left(\frac{1}{\rho_n}\right) = O(1)\ as\ n \to \infty \qquad (1.2.4)$$

then $\sum \frac{a_n \epsilon_n}{\rho_n}$ is summable $|V, \lambda|$.

The object of this paper is to prove a more general theorem for $|V, \lambda, \delta|_k$ summability which generalized all the above theorems. However, our theorem is as follows.

1.3. Theorem: If,

$$\epsilon_m \mu_m = O(1),\ as\ m \to \infty \tag{1.3.1}$$

$$\sum_{n=1}^{m} \lambda_n \mu_n |\Delta^2 \epsilon_n| = O(1) \tag{1.3.2}$$

$$\sum_{v=1}^{m} \frac{|t_v'|^{\delta k + k}}{\lambda_v} = \rho_m \mu_m \tag{1.3.3}$$

$$k \geqslant 1, m \to \infty, 0 \leqslant \delta < \frac{1}{k}$$

where

$\mu_n = \sum_{i=1}^{n} \lambda_i^{-1}$ and $\{\rho_n\}$ are positive non-decreasing sequences such that

$$\lambda_n \mu_n \rho_n \Delta \left(\frac{1}{P_n}\right) = O(1)\ as\ n \to \infty \tag{1.3.4}$$

then

$$\sum \frac{a_n \epsilon_n}{\rho_n}\ is\ summable\ |V, \lambda, \delta|_k, k \geqslant 1, 0 \leqslant \delta \leqslant \frac{1}{k}$$

1.4 Proof of the theorem:

Let $T_n = v_{n+1}(\lambda; \epsilon_n) - v_n(\lambda; \epsilon_n)$ where $v_n(\lambda; \epsilon_n)$ is the n-th de - la vallée Poussin means of the series $\sum a_n \frac{\epsilon_n}{\rho_n}$

Then to prove the theorem it is sufficient to prove that:

$$\sum_{n=1}^{\infty} \lambda_n^{\delta k+k-1}|T_n|^k < \infty$$

let $\sum'$ be the summation over all n Satisfying $\lambda_{n+1} = \lambda_n$ and

$\sum''$ be the summation over all n satisfying $\lambda_{n+1} > \lambda_n$

we have

$$T_n = \frac{1}{\lambda_n \lambda_{n+1}} \sum_{v=n-\lambda_{n+2}}^{n+1} [(\lambda_{n+1} - \lambda_n)^{v-n+1} + \lambda_n] \frac{a_v \epsilon_v}{\rho_v}$$

when $\lambda_{n+1} = \lambda_n$, then we have,

$$T_n = \frac{1}{\lambda_{n+1}} \sum_{v=n-\lambda_n+2}^{n+1} \frac{a_v \epsilon_v}{\rho_v}$$

Applying Abel's transformation, we have

$$T_n = [\Sigma_1 + \Sigma_2 - \Sigma_3]$$

where,

$$\Sigma_1 = \frac{1}{\lambda_{n+1}} \sum_{v=n-\lambda_n+2}^{n} \Delta\left(\frac{\epsilon_n}{V\rho_v}\right) \sum_{r=0}^{v} ra_r \qquad (1.4.1)$$

$$= \frac{1}{\lambda_{n+1}} \sum_{v=n-\lambda_n+2}^{n} \Delta\left(\frac{\epsilon_n}{V\rho_v}\right) (V+1)t_v'$$

$$= \frac{1}{\lambda_{n+1}} \sum_{v=n-\lambda_{n+2}}^{n} \Delta\left(\frac{\epsilon_v}{V\rho_v}\right) (V+1)t_v' \qquad (1.4.2)$$

$$\Sigma_2 = \frac{1}{\lambda_{n+1}} \frac{\epsilon_{n+1}}{(n+1)\rho_{n+1}} \sum_{r=0}^{n+1} ra_r = \frac{1}{\lambda_{n+1}} \frac{\epsilon_{n+1}}{\rho_{n+1}} t_{n+1}'$$

$$\sum_3 = \frac{\epsilon_{n-\lambda_n+2}}{\lambda_{n+1}(n-\lambda_n+2)\rho_{n-\lambda_n+2}}\sum_{r=0}^{n-\lambda_n+1} ra_r$$

$$= \frac{\epsilon_{n-\lambda_n+2}t'_{n-\lambda_n+1}}{\lambda_{n+1}\rho_{n-\lambda_n+2}}$$

Now

Considering (1.4.1) we have

$$\sum_1 = \sum_{11} + \sum_{12} + \sum_{13} + \sum_{14}$$

Where

$$\sum_{11} = \frac{1}{\lambda_n}\sum_{v=n-\lambda_n+2}^{n} \frac{\Delta\epsilon_n t'_v}{\rho_v}$$

$$\sum_{12} = \frac{1}{\lambda_n}\sum_{v=n-\lambda_n+2}^{n} \frac{\Delta\epsilon_v t'_v}{v\rho_v} \sum_{13} = \frac{1}{\lambda_n}\sum_{v=n-\lambda_n+2}^{n} \frac{\epsilon_{v+1} t'_v}{v\rho_v}$$

and

$$\sum_{14} = \frac{1}{\lambda_n}\sum_{v=n-\lambda_{n+2}}^{n} \epsilon_{v+1}t'_v\Delta\left(\frac{1}{\rho_v}\right)$$

By Minkouski.s inequility, it is therefore, sufficient to prove that

$$\sum' \lambda_n^{\delta k+k-1}\left|\sum_{1r}\right|^k < \infty,\ for\ r = 1,2,3,4$$

$$\sum' \lambda_n^{\delta k+k-1}\left|\sum_2\right|^k < \infty$$

$$\sum' \lambda_n^{\delta k+k-1}\left|\sum_3\right|^k < \infty$$

Now

$$\sum' \lambda_n^{\delta k+k-1}\left|\sum_{11}\right|^k = \sum_n' \lambda_n^{\delta k+k-1}\left|\frac{1}{\lambda_n}\sum_{v=n-\lambda_n+2}^{n} \frac{\Delta\epsilon_v t'_v}{\rho_v}\right|^k$$

$$= O(1)\left[\sum{}' \lambda_n^{\delta k-1}\left\{\sum_{v=n-\lambda_n+2}^{n} \frac{|\Delta\epsilon_v||t_v'|}{\rho_v}\right\}^k\right]$$

$$= O(1)\left[\sum{}' \lambda_n^{\delta k-1}\left\{\sum_{v=n-\lambda_n+2}^{n} \frac{|\Delta\epsilon_v||t_v'|^k}{\rho_v}\right\}\left\{\sum_{v=n-\lambda_n+2}^{n} \frac{|\Delta\epsilon_v|}{\rho_v}\right\}^{k-1}\right]$$

$$= O(1)\left[\sum{}' \lambda_n^{\delta k-1}\sum_{v=n-\lambda_n+2}^{n} \frac{|\Delta\epsilon_v||t_v'|^k}{\rho_v}\right]$$

$$= O(1)\left[\sum_{v=1}^{\infty} \frac{|t_v'|^k|\Delta\epsilon_v|}{\rho_v}\sum_{n=v}^{v+\lambda_v-1} \lambda_n^{\delta k-1}\right]$$

$$= O(1)\left[\sum_{v=1}^{\infty} \frac{|t_v'|^k|\Delta\epsilon_v|}{\rho_v}\right]$$

Now

$$\sum_{v=1}^{m} \frac{|t_v'|}{\lambda_v}\cdot\frac{|\Delta\epsilon_v|\lambda v}{\rho_v}\sum_{v=1}^{m-1} \Delta\left\{\frac{|\Delta\epsilon_v|\lambda v}{\rho_v}\right\}\sum_{r=1}^{v} \frac{|t_v'|^k}{\lambda_r}+\frac{\lambda_m|\Delta\epsilon_{m+1}|}{\rho_{m+1}}\sum_{v=1}^{m} \frac{|t_v'|^k}{\lambda v}$$

$$\sum_{11}^{(1)} + \sum_{11}^{(2)} + \sum_{11}^{(3)} + \sum_{11}^{(4)}$$

where

$$\sum_{11\ (2)}^{(1)} = \sum_{v=1}^{m-1} |\Delta^2\epsilon_v|\lambda_v\mu_v = O(1)$$

$$\sum_{11}^{(2)} = \sum_{v=1}^{m-1} |\Delta\epsilon_{v+1}|\Delta\lambda_v\mu_v = O(1)$$

$$\sum_{11}^{(3)} = \sum_{v=1}^{m-1} |\Delta\epsilon_{v+1}|\lambda_{v+1}\Delta\left(\frac{1}{\rho_v}\right)\rho_v\mu_v = 0$$

$$\sum_{11}^{(4)} = \lambda_m|\Delta\epsilon_{m+1}|\mu_m = O(1)$$

as $m \to \infty$, by virtue of conditions $(1\cdot3\cdot1), (1\cdot3\cdot2)$ and by the hypothesis of theorem.

Hence

$$\sum{}' \lambda_n^{\lambda k+k-1} \left| \sum_{11} \right|^k < \infty$$

Now consider

$$\sum{}' \lambda_n^{\delta k+k-1} \left| \sum_{12} \right|^k = \sum{}' \lambda_n^{\delta k+k-1} \left| \frac{1}{\lambda_n} \sum_{v=n-\lambda_n+2}^{n} \frac{\Delta \epsilon_n t_v^{'}}{V \rho_v} \right|^k$$

$$= O(1) \left[\sum{}' \lambda_n^{\delta k-1} \left\{ \sum_{v=n-\lambda_n+2}^{n} \frac{|\Delta \epsilon_n| t_v^{'}}{v \rho_v} \right\}^k \right]$$

$$= O(1) \left[\sum{}' \lambda_n^{\delta k-1} \left\{ \sum_{v=n-\lambda_n+2}^{n} \frac{|\Delta \epsilon_v| |t_v^{'}|^k}{v \rho_v} \right\} \left\{ \sum_{v=n-\lambda_n+2}^{n} \frac{|\Delta \epsilon_v|}{v \rho_v} \right\}^{k-1} \right]$$

$$= O(1) \left[\sum{}' \lambda_n^{\delta k-1} \sum_{v=n-\lambda_n+2}^{n} \frac{|\Delta \epsilon_v| |t_v^{'}|^k}{v \rho_v} \right]$$

$$= O(1) \left[\sum_{v=1}^{\infty} \frac{|\Delta \epsilon_v| |t_v^{'}|^k}{v \rho_v} \sum_{n=v}^{v+\lambda_v-1} \lambda_n \delta k - 1 \right]$$

$$= O(1) \left[\sum_{v=1}^{\infty} \frac{|\Delta \epsilon_v| |t_v^{'}|^k}{v \rho_v} \right]$$

Now

$$\sum_{v=1}^{m} \frac{|\Delta \epsilon_v| \lambda_v |t'_v|^k}{\lambda_v v \rho_v} = \sum_{v=1}^{m-1} \Delta \left\{ \frac{|\Delta \epsilon_v| \lambda v}{v \rho_v} \right\} \sum_{r=1}^{v} \frac{|t_r^{'}|^k}{\lambda_r} + \frac{|\Delta \epsilon_m| \lambda_m}{m \rho_m} \sum_{n=1}^{m} \frac{|t_r^{'}|^k}{\lambda_r}$$

$$= O(1) \sum_{12}^{(1)} + \sum_{12}^{(2)} + \sum_{12}^{(3)} + \sum_{12}^{(4)} + \sum_{12}^{(5)}$$

where

$$\sum_{12}^{(1)} = \sum_{v=1}^{m-1} \frac{|\Delta^2 \epsilon_v| \lambda_v \mu_v}{V} = O(1)$$

$$\sum_{12}^{(2)} = \sum_{v=1}^{m-1} \frac{|\Delta \epsilon_{v+1}| \Delta \lambda_v \mu_v}{v} = O(1)$$

$$\sum_{12}^{(3)} = \sum_{v=1}^{m-1} \frac{|\Delta \epsilon_{v+1}| \lambda_{v+1} \mu_v}{v(v+1)} = O(1)$$

$$\sum_{12}^{(4)} = \sum_{v=1}^{m-1} \frac{|\Delta \epsilon_{v+1}| \lambda_{v+1}}{(v+1)} \Delta \left(\frac{1}{\rho_v} \right) \rho_v \mu_v = O(1)$$

and

$$\sum_{12}^{(5)} = \frac{|\Delta\epsilon_m|\lambda_m\mu_m}{m} = O(1)$$

as $m \to \infty$ by the hypothesis of the theorem

hence $\sum'\lambda_n^{\delta k+k-1}\left|\sum_{12}\right|^k < \infty.$

Again

$$\sum' \lambda_n^{\delta k+k-1}\left|\sum_{13}\right|^k = \sum{}' \lambda_n^{k+\delta k-1} \left|\frac{1}{\lambda_n}\sum_{v=n-\lambda_n+2}^{n} \frac{\epsilon_{v+1}t_v'}{v\rho_n}\right|^k =$$

$$O(1)\left[\sum{}' \lambda_n^{\delta k-1}\left\{\sum_{v=n}^{n} \frac{|\epsilon_v||t_v'|}{v\rho_v}\right\}^k\right] =$$

$$O(1)\left[\sum'\lambda_n^{\delta k-1}\left\{\sum_{v=n-\lambda_n+2}^{n} \frac{|\epsilon_v||t_v'|^k}{v\rho_n}\right\}\left\{\sum_{v=n-\lambda_n+2}^{n} \frac{|\epsilon_v|}{v\rho_n}\right\}^{k-1}\right]$$

$$= O(1)\left[\sum{}'_n \lambda_n^{\delta k-1}\sum_{v=n-\lambda_n+2}^{n} \frac{|\epsilon_v||t_v'|^k}{v\rho_{v+1}}\right]$$

$$= O(1)\left[\sum_{v=1}^{\infty} \frac{|\epsilon_v||t_v'|^k}{v\rho_v}\sum_{n=v}^{v+\lambda_v-1} \lambda_n^{\delta k-1}\right]$$

$$= O(1)\left[\sum_{v=1}^{\infty} \frac{|\epsilon_v||t_v'|^k}{v\rho_v}\right]$$

Now

$$\sum_{v=1}^{m} \frac{|\epsilon_v||t_v'|^k}{v\rho_v} = \sum_{v=1}^{m} \frac{|\epsilon_v|\lambda_v|t_v'|^k}{v\rho_v\lambda_v}$$

$$= O(1)\left[\sum_{v=1}^{m-1} \Delta\frac{|\epsilon_v|\lambda_v}{v\rho_v}\sum_{r=1}^{v}\frac{|t_r'|^k}{\lambda_r} + \frac{|\epsilon_m|\lambda_m}{m\rho_m}\sum_{r=1}^{m}\frac{|t_r'|^k}{\lambda_r}\right]$$

$$= O(1)\left[\sum_{13}^{(1)} + \sum_{13}^{(2)} + \sum_{13}^{(3)} + \sum_{13}^{(4)} + \sum_{13}^{(5)}\right]$$

$$\sum_{13}^{(1)} = \sum_{v=1}^{m-1} \frac{|\Delta\epsilon_v|\lambda v\mu_v}{v} = O(1)$$

$$\sum_{13}^{(2)} = \sum_{v=1}^{m-1} \frac{|\epsilon_{v+1}|\Delta\lambda_v\mu_k}{v} = O(1)$$

$$\sum_{13}^{(3)} = \sum_{v=1}^{m-1} \frac{|\epsilon_{v+1}|\lambda_{v+1}\mu_v\rho_v}{v}\Delta\left(\frac{1}{\rho_v}\right) = O(1)$$

$\sum_{13}^{(4)} \quad = \sum_{v=1}^{m-1} \quad \frac{|\epsilon_{v+1}|\lambda_{v+1}\mu_v}{v(v+1)} = O(1)$

and

$\sum_{13}^{(5)} \quad = \frac{|\epsilon_m|\lambda_m \mu_m}{m} = O(1)$

on $m \to \infty$, by virtue of conditions (1.3.1), (1.3.2) and by the hypothesis of the theorem.

Hence

$\sum_n' \quad \lambda_n^{\delta k+k-1} \left|\sum_{13} \quad \right|^k < \infty$

Further

$\sum' \lambda_n^{\delta k+k-1} \left|\sum_{14} \quad \right|^k = \sum' \lambda_n^{\delta k+k-1} \left|\frac{1}{\lambda_n} \sum_{v=n-\lambda_{n+2}}^{n} \quad \epsilon_{v+1} t_v' \Delta\left(\frac{1}{\rho_n}\right)\right|^k \quad =$
$O(1)\left[\sum' \lambda_n^{\delta k-1} \{\sum_{v=n-\lambda_n+2}^{n} \quad |\epsilon_v||t_v'|\Delta\left(\frac{1}{P_v}\right)]^k\right] \quad =$
$O(1)\left[\sum' \lambda_n^{\delta_{k}-1} \left\{\sum_{v=n-\lambda_n+2}^{n} \quad |\epsilon_v||t_v'|^k \Delta\left(\frac{1}{\rho_v}\right)\right\} \left\{\sum_{v=n-\lambda_n+2}^{n} \quad |\epsilon_v|\Delta\left(\frac{1}{\rho_v}\right)\right\}^{k-1}\right] \quad =$
$O(1)\left[\sum' \lambda_n^{\delta k-1} \sum_{v=n-\lambda_n+2}^{n} \quad |\epsilon_v \parallel t_v'|^k \Delta\left(\frac{1}{\rho_v}\right)\right] \quad =$
$O(1)\left[\sum_{v=1}^{\infty} \quad |\epsilon_v||t_v'|^k \Delta\left(\frac{1}{\rho_v}\right) \sum_{n=v}^{v+\lambda_n-1} \quad \delta_n^{\delta k-1}\right] \quad = O(1)\left[\sum_{v=1}^{\infty} \quad |\epsilon_v \parallel t_v'|^k \Delta\left(\frac{1}{\rho_v}\right)\right]$

Now

$$\sum_{v=1}^{m} |\epsilon_v||t_v'|^k \Delta\left(\frac{1}{\rho_v}\right) = \sum_{v=1}^{m} |\epsilon_v|\lambda_v \Delta\left(\frac{1}{\rho_n}\right) \frac{|t_v'|^k}{\lambda_v}$$

$$= O(1)\left[\sum_{v=1}^{m-1} \Delta\left\{|\epsilon_v|\lambda_v \Delta\left(\frac{1}{\rho_v}\right)\right\} \sum_{r=1}^{v} \frac{|t_r'|^k}{\lambda_r} + |\epsilon_m|\lambda_m \Delta\left(\frac{1}{\rho_m}\right) \sum_{r=1}^{m} \frac{|t_r'|^k}{\lambda_r}\right]$$

$$= O(1)\left[\sum_{14}^{(1)} + \sum_{14}^{(2)} + \sum_{14}^{(3)} + \sum_{14}^{(4)}\right]$$

where

$$\sum_{14}^{(1)} = \sum_{v=1}^{m-1} |\Delta\epsilon_v|\lambda_v \Delta\left(\frac{1}{\rho_v}\right)\rho_v\mu_v = O(1) \quad \sum_{14}^{(2)}$$
$$= \sum_{v=1}^{m-1} |\epsilon_{v+1}|\Delta\lambda_v \Delta\left(\frac{1}{\rho_v}\right)\rho_v\mu_v = O(1) \quad \sum_{14}^{(3)}$$
$$= \sum_{v=1}^{m-1} |\epsilon_{v+1}|\lambda_{v+1}\Delta^2\left(\frac{\rho_v}{\rho_v}\right)\rho_v\mu_v = O(1) \quad and \quad \sum_{14}^{(4)}$$
$$= |\epsilon_m|\lambda_m \Delta\left(\frac{1}{\rho_m}\right)\rho_m\mu_m = O(1)$$

as $m \to \infty$ by virtue of the hypothesis

Hence,

$$\Sigma'\lambda_n^{k-1}\left|\sum_{14}\right|^k < \infty$$

moreover, we have

$$\Sigma'\lambda_n^{\delta k-1}|\Sigma_2|^k + \Sigma'\lambda_n^{\delta k+k-1}|\Sigma_3|^k$$
$$= O(1)\left[\Sigma'\lambda_n^{\delta k-1}\frac{|\epsilon_n|^k|t_n^{'}|^k}{|\rho_n|^k}\right]$$
$$= O(1)\left[\Sigma'\lambda_n^{\delta k-1}\frac{|\epsilon_n||t_n^{'}|^k}{\rho_n}\right]$$

Now

$$\Sigma_1^m \quad \lambda_n^{\delta k-1}\frac{|\epsilon_n||t_n^{'}|^k}{\rho_n} =$$

$$\Sigma_1^{m-1} \quad \Delta\left(\frac{|\epsilon_n|}{\rho_n}\right)\Sigma_{r=1}^n \quad \lambda_r^{\delta k-1}|t_r^{'}| + + \frac{|\epsilon_n|}{\rho_n}\Sigma_{r=1}^n \quad \lambda_r^{\delta k-1}|t_r^{'}|^k = O(1)[\Sigma^{(1)} +$$

$$\Sigma^{(2)} + \Sigma^{(3)}]$$

where

$$\sum^{(1)} = \sum_{n=1}^{m-1} |\Delta\epsilon_n|\mu_n = O(1)\sum^{(2)} = \sum_{n=1}^{m-1} |\epsilon_{n+1}|\Delta\left(\frac{1}{\rho_n}\right)\rho_n\mu_n = O(1)$$

and

$$\sum^{(3)} = |\epsilon_m|\mu_m = O(1)$$

as $m \to \infty$, by virtue of conditions (1.3.1), (1.3.2) and by hypothesis of theorem.

hence

$$\Sigma'\lambda_n^{\delta k+k-1}\left|\sum_2\right|^k + \Sigma'\lambda_n^{\delta k+k-1}\left|\sum_3\right|^k < \infty$$

Therefore,

$$\Sigma'\lambda_n^{\delta k+k-1}|T_n|^k < \infty$$

when $\lambda_{n+1} > \lambda_n$, then we have

$$T_n = \frac{1}{\lambda_n\lambda_{n+1}}\left\{\sum_{v=n-\lambda_{n+2}}^{n+1} (\lambda_n + v - n - 1)\frac{a_v\epsilon_v}{\rho_v}\right\}T_n$$
$$= \frac{1}{\lambda_n\lambda_{n+1}}\left\{\sum_{v=n-\lambda_n+2}^{n+1} (\lambda_n + v - n - 1)va_v\frac{\epsilon_v}{v\rho_v}\right\}$$

On applying Abel's transformation we have.

$T_n = [\Sigma_1^1 + \Sigma_2^1 + \Sigma_3^1]$

where

$$\Sigma_1^1 = \frac{1}{\lambda_n^2}\Sigma_{v=n-\lambda_{n+2}}^{n} \Delta\left\{(\lambda_n + v - n - 1)\frac{\epsilon_v}{v\rho_v}\right\}vt_v', \Sigma_2^1 = \frac{\epsilon_{n+1}t_{n+1}'}{\lambda_{n+1}\rho_{n+1}}$$

and

$$\sum_{3}^{1} = \frac{\epsilon_{n-\lambda_{n+2}} t'_{n-\lambda_n+1}}{\lambda_n \lambda_{n-\lambda_{n+1}} \rho_{n-\lambda_n+2}}$$

It is therefore sufficient to show that

$$\sum \lambda_n^{\delta k+k-1} |\Sigma_r^1 \quad |^k < \infty \; for \; r = 1,2,3$$

we have

$$\Sigma_n'' \quad \lambda_n^{\delta k+k-1} |\Sigma_1' \quad |^k = \Sigma'' \quad \frac{1}{\lambda_n^{k+1-\delta k}} \left| \Sigma_{k=n-\lambda_n+2}^n \quad \left\{ \Delta(\lambda_n + v - n - h) \frac{\epsilon_n}{v\rho_v} \right\} vt_v' \right|^k \leqslant \Sigma'' \quad \frac{1}{\lambda_n^{k+1-\delta k}} \left[\Sigma_{v=n-\lambda_n+2}^n \quad \left| \Delta \left\{ (\lambda_n + v - n - 1) \frac{\epsilon_u}{v\rho_v} \right\} v\rho t_v' \right|^k \right]$$

Since $\left| \Delta \left\{ (\lambda_n + v - n - 1) \frac{\epsilon_v}{v\rho_v} \right\} \right| \le \lambda_v \Delta \left(\frac{|\epsilon_v}{v\rho_v} \right) + \frac{|\epsilon_v|}{v\rho_v}$

therefore,

$$\Sigma'' \lambda_n^{k-1} |\Sigma_1'|^k = O(1)[\Sigma_{11}^1 + \Sigma_{12}^1]$$

where

$$\Sigma_{11}^1 \quad = \Sigma'' \quad \frac{1}{\lambda^{k+1-\delta k}} \left\{ \Sigma_{v=n-\lambda_n+2}^n \quad \lambda_{v\Delta} \left(\frac{|\epsilon_v|}{v\rho_v} \right) v |t_v'|^k \right\} \quad (1.4.2) \quad \Sigma_{12}^1 \quad = \Sigma'' \quad \frac{1}{\lambda_n^{k+1-\delta k}} \left\{ \Sigma_{v=n-\lambda_n+2}^n \quad \frac{|\epsilon_v||t_v'|^k}{\rho_v} \right\}$$

Now considering (1.4.2) we have

$$\Sigma_{11}^1 \quad = \Sigma_{11}^{1\;(1)} \quad + \Sigma_{11}^{1\;(2)} \quad + \Sigma_{11}^{1\;(3)}$$

Where,

$$\Sigma_{11}^{1\;(1)} \quad \Sigma'' \quad \frac{1}{\lambda^{k+1-\delta k}} \left\{ \Sigma_{v=n-\lambda_n+2}^n \quad \frac{\lambda_v |\Delta \epsilon_v| |t_v'|^k}{\rho_v} \right\}$$

$$= \Sigma_{v=1}^\infty \quad \frac{|t_v'| \lambda_v |\Delta \epsilon_v|}{\rho_v} \Sigma_{n \geqslant v}'' \quad \frac{1}{\lambda_n^{k+1-\delta k}}$$

$= O(1)\sum_{v=1}^{\infty} \frac{|t_v'|^k |\Delta\epsilon_v|}{\rho_v} = O(1)$ *on proved earlier* $\sum_{11}^{1\ (2)} =$

$\sum_n'' \frac{1}{\lambda_n^{k+1-\delta k}} \left\{\sum_{v=n-\lambda_n+2}^{n} \frac{|\epsilon_{v+1}||t_v'|^k \lambda_v}{(v+1)\rho_v}\right\}$

$= O(1)\sum'' \frac{1}{\lambda_n^{k+1-\delta k}} \left\{\sum_{v=n-\lambda_n+2}^{n} \frac{|\epsilon_v||t_v'|^k \lambda_v}{v\rho_v}\right\}$

$= O(1)\sum_{v=1}^{\infty} \frac{|\epsilon_v||t_v'|^k \lambda v}{v\rho_v} \sum_{n\geqslant v}'' \frac{1}{\lambda_n^{k+1-\delta k}} = O(1)\sum_{v=1}^{\infty} \frac{|\epsilon_v||t_v'|^k}{v\rho_v}$

$= O(1)$ as proved earlier.

Hence

$\sum_{11}^{1} \quad \sum_{11}^{1(1)} + \sum_{11}^{1(2)} + \sum_{11}^{1(3)} = O(1)$

Now

$\sum_{12}^{1} = \sum'' \frac{1}{\lambda_n^{k+1-\delta k}} \left\{\sum_{v=n-\lambda_n+2}^{n} \frac{|\epsilon_v||t_v'|^k}{\rho_v}\right\}$

$= O(1)\sum_{v=1}^{\infty} \frac{|\epsilon_u||t_v'|^k}{\rho_v} \sum_{n\geqslant v} \frac{1}{\lambda_n^{k+1-\delta k}} = O(1)\sum_{v=1}^{\infty} \frac{|\epsilon_v||t_v'|^k}{\rho_v \lambda_v} =$

$O(1)$ *as proved earlier*

Hence

$\sum'' \lambda_n^{\delta k+k-1} |\sum_1^1 \quad |^k = O(1)[O(1) + O(1)]$

therefore

$\sum'' \lambda_n^{\delta k+k-1} |\sum_1^1 \quad |^k < \infty$

Also

$\sum'' \lambda_n^{\delta k+k-1} |\sum_2^1 \quad |^k = O(1), as$

as proved in the previous case for Σ_2. Lastly

$$\sum'' \quad \lambda_n^{\delta k+k-1} |\Sigma_3^1 \quad |^k = O(1)$$

as proved in the previous case for Σ_3.

This complete the proof of Theorem.

References

[1] FLET, T.M. : On an Extension of absolute summability and some theorems for little wood and paley; Proc. London Math Soc. 7 (113-141) 1957

[2] FLET, T.M. : Some more theorems concerning absolute summability of Fourier series and Power series; Proc. Lond, Math. Soc. (3)8 (357-387) (1958).

[3] PRASAD, J. : Some Contribution to absolute summability; Ph.D. Thesis. Rohilkhand Univ. (1981).

[4] SINHA, R., : : On $|V, \lambda|$ Summability factors of CHANDRA, S. infinite series; Abstract: Proceeding of AND KUMAR, P. Indian Sci Congress Association section of Math. P. 27 (1983)

[5] CHANDRA, S. : On absolute summability fields of infinite series; Ph.D. Thesis, Rohilkhand University (1984).