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Mathematical Modelling of the Solutions of Partial Differential Equation

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Abstract: Examining the mathematical modelling of partial differential equation solutions with physical implications was the goal of this effort. We specifically examined the homogeneous one-dimensional wave solution's well-posedness and consistency. The method of change of variable was used to obtain the general solution of the wave equation, and this general solution ultimately led to the d'Alembert's formula, which is the only solution to the problem. Next, we demonstrate the existence, uniqueness, and stability of the d'Alembert's formula. After that, we analyzed the results using the answer we had acquired, displayed the behavior of our results in a table, and came to the conclusion that the idea of a well-posed issue is crucial in applied mathematics.

Keyword: Partial Derivatives; Partial Differential Equation

1. Introduction

In general, it may be impossible or at the very least difficult to find the exact solution to partial differential equation problems. Partial differential equations were initially studied as a means of examining physical science models. Physical rules like momentum, conservation laws, balancing forces (Newton's law), and others are the usual source of PDEs (Strauss, 2008). This paper derives the string's equation of motion, which takes the form of a second-order partial differential equation, under specific assumptions. The one-dimensional wave equation, or governing partial differential equation, depicts the transverse vibration of an elastic string (King and Billingham, 2000).The analytical solution has been obtained using method of change of variable. The solution of wave equation was one of the major mathematical problems of the mid eighteenth century. The wave equation was first derived and studied by D'Alembert in 1746. He introduced the one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

The wave equation stood then generalized to 2 and 3 dimensions, i.e.

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \Delta u(x,t)$$

Where

$$\Delta = \sum_{i-1}^{3} \frac{\partial^2}{\partial x_i^2}$$

The wave equation, which describes the propagation of many waves, including sound and water waves, is a second-order linear hyperbolic PDE. It appears in a variety of disciplines, including electromagnetics, fluid dynamics, and acoustics (Sajjadi 2008). A partial derivative of the independent variable, which is an unknown function in multiple variables, is found in a partial differential equation (PDE).

$$\frac{\partial u}{\partial x} = u_x, \frac{\partial u}{\partial y} = u_y \& \frac{\partial u}{\partial t} = u_t$$

We can inscribe the overall first order PDE for $u(x,t)$ as

$$F[x,t,u(x,t),u(x,t),u(x,t)] = F(x,t,u,u_x,u_t) = 0 \dots \dots \dots . . (1)$$

Although one can study PDEs with many independent variables as one wish, but in this research work we will be primarily concerned with PDEs in two independent variables.

2. Statement of the problem

The main focus of this research project is on the good qualities and consequences of a given partial differential equation solution. The homogeneous one-dimensional wave equation in particular piques our interest in the mathematical modelling of the consistency and well-posedness of the solution or solutions to certain PDEs. Some function u=u(x,y,z,t) will measure different physical quantities. This may rely on all temporal variables, a subset of them, or none (Guo, 2009). The shortened notation that follows will be used to represent the partial derivatives of u:

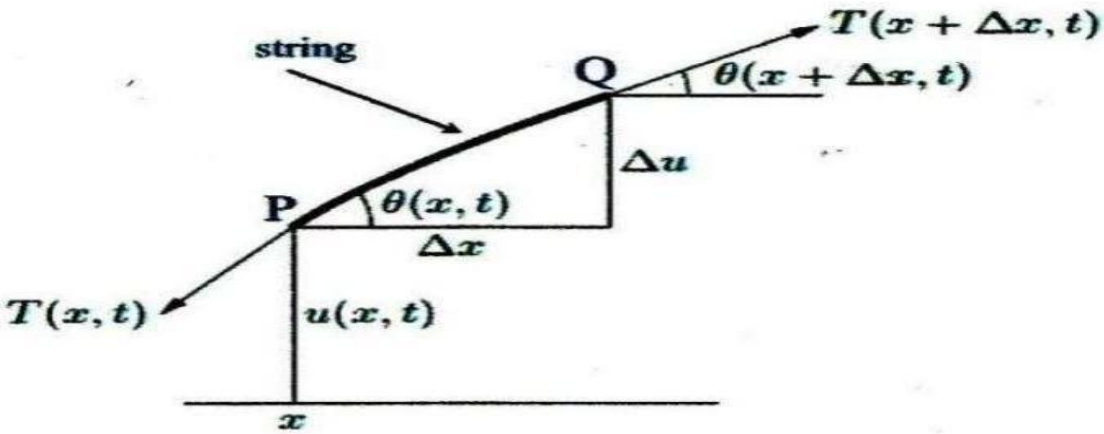
$$u_x = \frac{\partial u}{\partial y}, u_{xx} = \frac{\partial^2 u}{\partial x^2}, u_{xy} = \frac{\partial^2 u}{\partial x\, \partial y}, u_{xt} = \frac{\partial^2 u}{\partial x\, \partial t}, u_t = \frac{\partial u}{\partial t} \text{ etc.}$$

3. METHODS

We will look at a specific kind of problem related to partial differential equations that are hyperbolically linear. This issue will be discussed in relation to the homogeneous one-dimensional wave equation of the type

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Where C is a constant and the independent variables are x and t. undoubtedly, one of the most significant classical equations in mathematical physics is the wave equation. There are numerous physical applications for the wave equation, ranging from sound waves in air to magnetic waves in the Sun's atmosphere. On the other hand, waves on a stretched elastic thread are the easiest systems to picture and explain. The string is initially horizontal and has two fixed ends, let's say a left end (L) and a right end (R): When we shake the string from end L onward, we see a wave propagate across the string. The goal is to attempt to calculate the vertical displacement of the string from the X-axis, u(x,t), as a function of timet and location X: In other words, the displacement from equilibrium at Position X and time t is represented as u(x,t): A little portion of the string's displacement between Points P and Q is displayed in;Where



- $\theta(x,t)$ is the angle amid the string and a straight line at location$x$ and time t
- $T(x,t)$ is the tautness in the string at location$x$ and time t;
- $\rho(x)$ is the mass density of the thread at position x :

To originate the wave equation we necessity to make particular simplifying expectations: (1) The density of the string, ρ is continuous so that the mass of the string amid$P$ and Q is simply ρperiods the length of the string among$P$ and Q where the distance of the string is Δsagreed by

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta u)^2} = \Delta x \sqrt{1 + \left(\frac{\Delta u}{\Delta x}\right)^2} \approx \Delta x \sqrt{\left(\frac{\partial u}{\partial x}\right)^2}$$

(2) The displacement, $u(x,t)$ and its derivatives are assumed small so that

$$\Delta s \approx \Delta x$$

and the mass of the helping of the thread is

$\rho \Delta x$

(3) The solitary forces acting on this helping of the cordstand the tensions $T(x,t)$ at P and $T(x+\Delta x,t)$at Q. The gravitational force stands neglected.

(4) We have a little string element that only moves vertically. Thus, there must be no net horizontal force acting on it. Subsequently, we examine the forces operating on the standard string segment depicted previously. These forces consist of:

(i) Tension heaving to the right, which has scale$T(x+\Delta x,t)$ and entertainments at an angle $\theta(x+\Delta x,t)$upstairs the horizontal.

(ii) Tension drawing to the left, which consumesmagnitude $T(x,t)$, and entertainments at an angle $\theta(x,t)$, upstairs the horizontal.

We may now separate the forces into their vertical and horizontal components.- Horizontal: The small string's net horizontal force is

$$T(x+\Delta x,t)\cos\theta(x+\Delta x,t)-T(x,t)\cos\theta(x,t).$$

we must need

$$T(x,t)\cos\theta(x,t)=T(x+\Delta x,t)\cos\theta(x+\Delta x,t)=T.$$

Vertical: At Ptension force stands$-T(x,t)\sin\theta(x,t)$somewhere as at Q the Force is $T(x+\Delta x,t)\sin\theta(x+\Delta x,t)$.

Bounces

$$\rho\Delta x\frac{\partial^2 u}{\partial t^2}=T(x+\Delta x,t)\sin\theta(x+\Delta x,t)-T(x,t)\sin\theta(x,t).$$

$$\frac{\rho}{\rho}\Delta x\frac{\partial^2 u}{\partial t^2}=\frac{T(x+\Delta x,t)\sin\theta(x+\Delta x,t)}{T(x+\Delta x,t)\sin\theta(x+\Delta x,t)}=-\frac{T(x,t)\sin\theta(x,t)}{T(x,t)\cos\theta(x,t)}$$

$$\tan\theta(x+\Delta x,t)-\tan\theta(x,t)$$

But

$$\tan\theta(x,t)=\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}=u_x(x,t).$$

Also,

$$\tan\theta(x+\Delta x,t)=u_x(x+\Delta x,t)$$

$$\frac{\rho}{T}\Delta x u_{tt}(x,t)=u_x(x+\Delta x,t)-u_x(x,t)$$

$$\Delta x\to 0$$

$$\frac{\rho}{T}\Delta x u_{tt}(x,t)=u_{xx}(x,t)$$

Or

$$u_{tt}(x,t)=c^2u_{xx}(x,t)$$

Where $c^2=\frac{T}{\rho}$,

The transverse vibration of the string is represented by this PDE. It's also known as the one-dimensional wave equation. Over the complete real line, $-\infty<x<+\infty$, we solve the wave equation. Real-world physical conditions typically occur at fixed intervals. For two reasons, we can justify taking x on the entire real line. Without the complexities of boundary conditions, the most basic properties of the PDEs can be discovered with the greatest ease.

$$u_{tt} = c^2 u_{xx} \text{for} -\infty < x < \infty, t > 0$$
$$u(x,0) = f(x) \text{for} -\infty < x < \infty \text{ ....(2)}$$
$$u_t(x,0) = g(x) \text{for} -\infty < x < \infty \text{............(3)}$$

3.1 Solution via change of variable

Since the equation is hyperbolic we present the novel variable ε, ηdistinct by

$$\varepsilon = x + ct$$
$$\eta = x - ct$$

we have

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \varepsilon} \cdot \frac{\partial \varepsilon}{\partial \eta} + \frac{\partial}{\partial \eta} \cdot \frac{\partial \eta}{\partial t}$$
$$= C \frac{\partial}{\partial \varepsilon} + \frac{\partial}{\partial \eta}(-C)$$
$$C\left(\frac{\partial}{\partial \varepsilon} - \frac{\partial}{\partial \eta}\right)$$

So,

$$\frac{\partial^2}{\partial t^2} = C\left(\frac{\partial}{\partial \varepsilon} - \frac{\partial}{\partial \eta}\right) C\left(\frac{\partial}{\partial \varepsilon} - \frac{\partial}{\partial \eta}\right)$$
$$\frac{\partial^2}{\partial t^2} = C^2\left(\frac{\partial}{\partial \varepsilon} - \frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \varepsilon} - \frac{\partial}{\partial \eta}\right)$$
$$\frac{\partial^2}{\partial t^2} = C^2\left(\frac{\partial^2}{\partial \varepsilon^2} - \frac{2\,\partial^2}{\partial \varepsilon\,\partial \eta} + \frac{\partial^2}{\partial \eta^2}\right)$$

Likewise

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \varepsilon} \cdot \frac{\partial \varepsilon}{\partial x} + \frac{\partial}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}$$

weneed

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \varepsilon} \cdot 1 + \frac{\partial}{\partial \eta} \cdot 1$$
$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \varepsilon} + \frac{\partial}{\partial \eta}$$

So,

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial \varepsilon} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \varepsilon} + \frac{\partial}{\partial \eta}\right)$$
$$= \frac{\partial^2}{\partial \varepsilon^2} + \frac{2\,\partial^2}{\partial \varepsilon\, \partial \eta} + \frac{\partial^2}{\partial \eta^2}$$

Let transform $x, t \to \varepsilon, \eta$s.t.

$$U(x,t) = W(\varepsilon, \eta)$$

$$U_{tt} - C^2 U_{xx} = 0$$

Becomes

$$C^2\left(W_{\varepsilon\varepsilon} - 2W_{\varepsilon\eta} + W_{\eta\eta}\right) - C^2\left(W \quad + 2W_{\varepsilon\eta} + W_{\eta\eta}\right) = 0$$

$$\Rightarrow -4C^2 W_{\varepsilon\eta} = 0$$

Thus

$$\Rightarrow W_{\varepsilon\eta} = 0$$

$$\frac{\partial^2 w}{\partial \varepsilon\, \partial \eta} = 0 \dots\dots\dots\dots\dots\dots (4)$$

By integrating equation (4) twice, the general solution can be found with ease. Let's first assume that you integrate with regard to E and observe that, in order to obtain, the integration constant needs to rely on η.

$$\frac{\partial w}{\partial \eta} = G(\eta)$$

Currently integrate with respect to η and sign that the constant of mixing depends on ε.

$$w = \int_0^{\eta} G(\eta) d\eta + F(\varepsilon)$$

So

$$\int_0^{\eta} G(\eta) d\eta = G(\eta)$$

So that

$$w = G(\eta) + F(\varepsilon)$$

Remember that we transformed

$$U(x,t) = w(\varepsilon, \eta)$$

So

$$U(x,t) = F(x + ct) + G(x - ct) \dots\dots\dots (5)$$

This is the wave equation's general solution by d'Alembert, where f and g are arbitrary functions. In order to distinguish a specific physical solution from the general solution (5), we next take into consideration a few beginning conditions in addition to the wave equation (1). Since the equation is of second order in time t, we can consider two initial data points of the equation (1): f(x) and g(x), which are specified for the initial displacement u(x,0) and the initial velocity u_t (x,0), respectively. f and g are arbitrary functions of a single variable. In order to meet equation (1) above, we now need to calculate the functions F and G in the general equation (5).

Given;

$$u(x,0) = f(x) \text{for} -\infty < x < \infty$$

$$u_t(x,0) = g(x) \text{for} -\infty < x < \infty$$

put$t = 0$ in (5),

$$U(x,0) = f(x) = F(x) + G(x)$$

$$U(x,0) = F(x) + G(x) = f(x) \ldots\ldots(6)$$

put$t = 0$ to get

$$U_t(x,0) = CF^{'}(x) + CG^{'}(x)$$

$$CF^{'}(x) - CG^{'}(x) = g(x)$$

$$C(F^{'}(x) - G^{'}(x)) = \frac{g(x)}{C} \ldots..(7)$$

Integrating (ii)

$$F(x) - G(x) = \int_0^x \frac{g(s)}{C} ds + k \ldots\ldots\ldots(8)$$

Adding (i) and (ii)

$$2F(x) = \int_0^x \frac{g(s)}{C} ds + k + f(x) \ldots\ldots\ldots..(9)$$

$$F(x) = \frac{1}{2} \int_C^x g(s) ds + \frac{k}{2} + \frac{f(x)}{2}$$

Toowithdrawing (i) since (ii) yield

$$2G(x) = -\frac{1}{c} \int_0^x g(s) ds - k + f(x)$$

$$G(x) = -\frac{1}{2c} \int_0^x g(s) ds - \frac{k}{2} + \frac{f(x)}{2} \ldots\ldots\ldots..(10)$$

Memory that

$$U(x,t) = F(x+ct) + G(x-ct) \ldots\ldots\ldots\ldots(11)$$

Therefore since (IV)

$$F(x+ct) = \frac{f(x+ct)}{2} + \frac{1}{2c} \int_0^{x+ct} g(s) ds + \frac{k}{2}$$

Besides

$$G(x-ct) = \frac{f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^0 g(s) ds \frac{k}{2} \ldots\ldots\ldots(12)$$

Later, calculation (vi) and (vii) yield.

$$U(x,t) = f(x+ct) + f(x-ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds \ldots\ldots\ldots\ldots..(13)$$

D'Alembert's formula for solving the aforementioned starting value problem is this. The solution is uniquely determined by the initial data f and g, as our derivation of d'Alembert's formula demonstrates that any solution of (1), (2), and (3) that is twice continuously differentiable must have the representation (13). Therefore, the special solution to (1), (2), and (3) is represented by d'Alembert's formula.

3.2 Uniqueness of the Solution

The energy function E(t) for the wave equation is defined in this part. It is demonstrated that energy is conserved for the Cauchy problems (1), (2), and (3), and we utilise this fact to prove that the solutions to the aforementioned Cauchy problems are unique. Assume for now that

$$u = u(x,t)\text{..........(14)}$$

To be a fluid solution to the derivatives and the Cauchy issue

$$u_t(x,t)\text{and}u_x(x,t)$$

Are square integrable for each $t \geq 0$.

$$E(t) = \int_{-\infty}^{\infty}\left(\frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2\right)dx\text{........(15)}$$

is finite; kinetic and potential energy combined make up E(t). The energy that could be

$$PE(t) = \int_{-\infty}^{\infty}\frac{C^2}{2}u_x^2 dx\text{.........(16)}$$

Is the tension and kinetic energy combined to store energy in the string?

- $KE(t) = \int_{-\infty}^{\infty}\frac{1}{2}u_t^2 dx\text{.....(17)}$

Is the custom of $\frac{1}{2}mv^2$ in classical procedure of a rigid body through mass m and velocity v.Let us reproduce the PDE

$$u_{tt} = c^2 u_{xx}\text{.........(18)}$$

byu_t and integrate by parts.

Increasing

$$u_t u_{tt} = c^2 u_{xx} u_t$$

$$\int_{-\infty}^{\infty} u_t u_{tt}\, dx = \int_{-\infty}^{\infty} c^2 u_{xx} u_t\, dx\text{.......(19)}$$

Integrating by parts

$$\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\frac{\boldsymbol{U}_t^2}{2}dx = c^2 u_x u_t|_{-\infty}^{+\infty} - c^2\int_{-\infty}^{\infty} u_x u_{xt}\, dx$$
$$= -c^2\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\frac{\boldsymbol{U}_x^2}{2}dx$$

$$E'(t) = \frac{d}{dt}\int_{-\infty}^{\infty}\left(\frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2\right)dx = 0 \ldots\ldots\ldots(20)$$

Consequently, we have overall energy conservation: E(t)= constant, which allows us to infer

$$E(t) = E(0) = \frac{1}{2}\int_{-\infty}^{\infty}(\varphi(x)^2 + c^2\phi'(x)^2)dx, t > 0 \ldots\ldots(21)$$

As we will describe later, this identity is a crucial instrument for the presence, regularity, and uniqueness of solutions.

3.3 Uniqueness of Solution

To prove uniqueness, we demonstration

$$u_1 = u_2 \ldots\ldots..(22)$$

Let outline

$$u(x,t) = u_1(x.t) - u_2(x,t) \ldots\ldots\ldots(23)$$

Then u satisfies the homogeneous wave equation, with zero initial data:

$$u_{tt} = c^2 u_{xx} - \infty < x < \infty \ldots\ldots\ldots.(24)$$

$$u(x,0) = 0, u_t(x,0) = 0 \ldots\ldots\ldots\ldots\ldots(25)$$

Later$E(t) = E(0) = 0$ for this problematic, we obligate

$$E(t) = \int_{-\infty}^{\infty}\left(\frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2\right)dx = 0 \ldots\ldots(26)$$

Then

$$u_t = 0, \text{ And } u_x = 0 \ldots\ldots\ldots\ldots(27)$$

$u(x,0) = 0$, so the constant is zero.

Later

$$u = u_1 - u_2 = 0$$

4. Result

$$U_{tt} - 25U_{xx} = 0 \quad -\infty < x < \infty, \quad t > 0$$

$$U(x,0) = f(x) \quad -\infty < x < \infty,$$

$$U(x,0) = \sin x \quad -\infty < x < \infty,$$

$$U_t(x,0) = 0 \quad -\infty < x < \infty$$

Explanation

Comparing through the standard wave equation

$$U_{tt} - C^2 U_{xx} = 0 \qquad -\infty < x < \infty, \quad t > 0 \ldots\ldots\ldots(28)$$

$$U(x,0) = f(x) \quad -\infty < x < \infty,$$

$$U_t(x,0) = g(x) \quad -\infty < x < \infty$$

So from the above question

$$C^2 = 25 \quad \Rightarrow C = 5$$

$$f(x) = \sin x$$

$$g(x) = 0$$

Therefore, using

$U(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x)ds$.........(29)

We obligate

$$U(x,t) = \frac{1}{2}[\sin(x+ct) + \sin(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} 0ds$$

$$U(x,t) = \frac{1}{2}[\sin(x+ct) + \sin(x-ct)] + 0$$

$$U(x,t) = \frac{1}{2}[\sin(x+5t) + \sin(x-5t)] \text{.........(30)}$$

Recall that

$$\sin(A+B) + \sin(A-B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \text{.........(31)}$$

Consequently, laying (2) keen on (1) yield

$$U(x,t) = \frac{1}{2}\left[2\sin\left(\frac{x-5t+x-5t}{2}\right)\cos\left(\frac{x+5t-x+5t}{2}\right)\right] \text{............(32)}$$

Therefore

$$U(x,t) = \sin x \cos 5t \text{..............(33)}$$

Table 1: viewing the values of $u(x,t)$ at changing x and t

S/N	$u(x,t)$	x	t
1	0.0771	5^0	0
2	0.1710	10^0	2
3	0.2432	15^0	4
4	0.2962	20^0	6
5	0.3237	25^0	8
6	0.3214	30^0	10
7	0.2868	35^0	12
8	0.2198	40^0	14
9	0.1228	45^0	16
10	0.000	50^0	18

Table 2: Viewing result of $u(x,t)$ at immovable$x$ and varying t

S/N	$u(x,t)$	Fixed x	Varying t
1	0.0872	5	0
2	0.0859		2
3	0.0819		4
4	0.0755		6
5	0.0668		8

Table 3: Viewing result of $u(x,t)$ at fixed t and varying x

S/N	$u(x,t)$	Varying x	Fixed t
1	0.0872	5	0
2	0.1736	10	
3	0.2588	15	
4	0.3420	20	
5	0.4226	25	
6	0.5000	30	
7	0.5736	35	

5. Conclusion

In conclusion, the concept of a well-posed issue holds significance in the field of applied mathematics. Even if the well-posed issues are nearly entirely covered by the classical theory of partial differential equations, ill-posed problems can nevertheless be fascinating from a mathematical and scientific standpoint.

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