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SOLUTION OF NON-LINEAR RICCATI EQUATION USING LAPLACE TRANSFORM AND COMPARISON OF SOLUTIONS GRAPHICALLY

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Abstract: In this paper, we have used Laplace transform-homotopy pertur bation method (LT-HPM) to solve non-linear Riccati equation. Figures of exact solution and approximate solution by this method is drawn and also compared with solution calculated by another method.

Key words: Laplace transform, homotopy perturbation method, Riccati equation.

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1. Introduction:

The differential equation of the following form:

$$A(x)\,\frac{dy(x)}{dx} + B(x)y(x) + C(x)y^2(x) \;=\; G(x), a \;\leq\; x \;\leq\; b, \tag{1.1}$$

is called Riccati differential equation, where A(x), B(x), C(x) and G(x) are real functions on R. Which plays an important role in the theory of control problems and other fields like dynamic games, linear system with Markovian jumps, and stochastic control. Solution of this equation can be calculated using classical numeri cal methods like e.g. the forward Euler method or Runge-Kutta method. Through Adomian decomposition method solution of this equation is previously found [1]. The homotopy perturbation method (HPM) [2] is a useful method to solve many non-linear equations. Some Riccati differential equations have been solved by HPM [3]. It is a semi analytical method and this method is attracted and simultaneously adopted by the researcher rapidly [4, 5, 6] since it is easy to use and gives a bet ter approximate solution in compare to another traditional perturbation method. Here Laplace transform is also added to the HPM to give a new technique Laplace transform-homotopy perturbation method (LT-HPM) [7]. Through LT-HPM many linear and non-linear differential equation have been also solved which shows the efficiency of the method [8, 9]. In [10] fractional order Riccati differential equation is solved using new homotopy perturbation method.

2. Homotopy perturbation method

To describe the HPM considering following nonlinear differential equation:

$$A(u) = f(r),\ r \in \Omega$$

(2.1)

subject to the condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0,\ r \in \Gamma,$$

(2.2)

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω and $\frac{\partial}{\partial n}$ denotes differentiation along the normal vector drawn outwards from Ω. The operator A can generally be divided into two parts M and N. Therefore, (2.1) can be rewritten as follows:

$$M(u) + N(u) = f(r), \quad r \in \Omega.$$

(2.3)

A homotopy $v(r,p) : \Omega \times [0,1] \rightarrow R$ is constructed, which satisfies

$$H(v,p) = (1-p)[M(v) - M(u_0)] + p[A(v) - f(r)] = 0,$$

(2.4)

which is equivalent to

$$H(v,p) = M(v) - M(u_0) + pM(u_0) + p[N(v) - f(r)] = 0,$$

(2.5)

where $p \in [0,1]$ is an embedding parameter, and u0 is an initial approximation of (2.3). We have

$$H(v,0) = M(v) - M(u_0) = 0,\ H(v,1) = A(v) - f(r) = 0.$$

(2.6)

As p changes from 0 to 1, $H(v,p)$ changes from $M(v) - M(u_0)$ to $A(v) - f(r)$. This change in topology, called deformation and $M(v) - M(u_0)$ and $A(v) - f(r)$ are called homotopic. According to the HPM, p is a small parameter, and the solution of (2.4) can be expressed as a series in p in the form

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots$$

(2.7)

when p →1, (2.4) becomes given problem, and (2.1) and (2.7) becomes the approximate solution of (2.1), i.e.,

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \ldots$$

(2.8)

If (2.1) admits a unique solution, then this method produces the unique solution. If (2.1) does not possess unique solution, the HPM will give a solution among many other possible solutions. The convergence of the series in (2.8) is discussed in [3].

3. Laplace transform-homotopy perturbation method (LT-HPM)

In this section we employ LT-HPM to find analytical approximate solutions of PDE's. For this purpose, LT-HPM follows the same steps of standard HPM until (2.5); next we apply Laplace transform on both sides of homotopy equation (2.5) to obtain

$$\mathcal{L}(M(v) - M(u_0) + p[M(u_0) + N(v) - f(r)]\,) = 0,$$

(3.1)

using the differential property of LT, we have

$$s^n \mathcal{L}(v) - s^{n-1}\, v(0) - s^{n-2} v'(0) - \ldots - v^{(n-1)}(0)$$
$$= \mathcal{L}(M(u_0) - p[M(u_0) + N(v) - f(r)])\,,$$

(3.2)

or

$$\mathcal{L}(v) = \frac{1}{s^n} \left[s^{n-1} v(0) + s^{n-2} v'(0) + \cdots + v^{(n-1)}(0) + \right.$$
$$\left. \mathcal{L}(M(u_0) - p[M(u_0) + N(v) - f(r))\right]$$

(3.3)

Thus, by application of inverse Laplace transform in (3.3), we have

$$v = \mathcal{L}^{-1} \left[\frac{1}{s^n} \left(s^{n-1} v(0) + s^{n-2} v'(0) + \ldots + v^{(n-1)}(0) \right.\right.$$
$$\left.\left. + \mathcal{L}(M(u_0) - p[M(u_0) + N(v) - f(r)\,)\,\right)\right]$$

(3.4)

Assuming that the solutions of (2.3) can be expressed as a power series of p

$$v = \sum_{n=0}^{\infty} p^n v_n$$

(3.5)

Then substituting (3.5) into (3.4), we get

$$\sum_{n=0}^{\infty} p^n v_n = \mathcal{L}^{-1} \left[\frac{1}{s^n} \left(s^{n-1} v(0) + s^{n-2} v'(0) + \ldots + v^{(n-1)}(0) \right.\right.$$
$$\left.\left. + \mathcal{L}(M(u_0) - p[M(u_0) + N(\textstyle\sum_{n=0}^{\infty} p^n v_n) - f(r)\,)\,\right)\right]$$

(3.6)

Comparing coefficient of p with the like powers leads to

$$p^0 \;:\; v_0 = \mathcal{L}^{-1}\left[\frac{1}{s^n}\left(s^{n-1}v(0) + s^{n-2}v'(0) + \ldots + v^{(n-1)}(0) + \mathcal{L}(\mathrm{M}(\mathrm{u}_0)\right)\right]$$

$$p^1 \;:\; v_1 = \mathcal{L}^{-1}\left[\frac{1}{s^n}\left(\mathcal{L}\{-\mathrm{N}(\mathrm{v}_0) - \mathrm{M}(\mathrm{u}_0) + \mathrm{f}(\mathrm{r})\}\right)\right]$$

$$p^2 : v_2 = \mathcal{L}^{-1}\left[\frac{1}{s^n}\left(\mathcal{L}\{-\mathrm{N}(\mathrm{v}_0, \mathrm{v}_1)\}\right)\right]$$

$$\ldots$$

(3.7)

Assuming that the initial approximation has the form $v(0) = v_0 = \alpha_0$, $v'(0) = \alpha_1, \ldots, v^{n-1}(0) = \alpha_{n-1}$, therefore the exact solution may be obtained as follows:

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \ldots$$

(3.8)

4. Solution of Riccati differential equation

Problem 1: Consider Riccati differential equation as taken in [1]

$$\frac{dy}{dx} = 1 + 2y - y^2, \; \forall\, x \in [0,4],$$

(4.1)

with initial condition $y(0) = 0$. Exact solution of this is equation is given as

$$y(x) = 1 + \sqrt{2}\,tanh\left[\sqrt{2}x + \frac{1}{2}\log\frac{\sqrt{2}-1}{\sqrt{2}+1}\right].$$

(4.2)

Its Taylor series expansion about $x = 0$ is given by

$$y(x) = x + x^2 + \frac{x^3}{3} - \frac{x^4}{3} - 7\frac{x^5}{15} + O(x^6)\,.$$

(4.3)

Now proceeding as the steps given in LT-HPM, we construct the homotopy as

$$(1 - p)(y' - y_0') + p(y' + y^2 - 2y - 1) = 0,$$

which can be written as

$$y' = y_0' + p(-y_0' - y^2 + 2y + 1).$$

(4.4)

Applying Laplace transform to both sides of (4.4), we get

$$\mathcal{L}(y') = \mathcal{L}[y_0' + p(-y_0' - y^2 + 2y + 1)]\,,$$

now using the differential property of Laplace transform, where we are taking $\mathcal{L}(y(x)) = Y(s)$, we get

$$sY\,(s)\; - y(0)\; = \mathcal{L}[y_0'\; + p(-y_0'\; - y^2\; + 2y\; + 1)]\,,$$

$$Y(s)\; = \frac{1}{s}\mathcal{L}[y_0'\; + p(-y_0'\; - y^2\; + 2y\; + 1)\,.$$

Applying the inverse Laplace transform both sides, we get

$$y(x)\; = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[y_0'\; + p(-y_0'\; - y^2\; + 2y\; + 1)\right].$$

(4.5)

Let the solution of (4.5) is of the form

$$\mathrm{y(x)}\; = \sum_{n=0}^{\infty} \mathrm{p^n} v_n.$$

(4.6)

Taking initial approximation as $v_0(x)\; =\; x$, which satisfies the initial condition $y(0)\; =\; 0$, and also from (4.1), we have $y_0'\; =\; 1$.

Substituting (4.6) in (4.5) and using values of v_0 and y_0', we get

$\sum_{n=0}^{\infty} \mathrm{p^n} v_n = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\big(1 + p(-(x + pv_1 + p^2v_2 + \cdots)^2 + 2(x + pv_1 + p^2v_2 + \right.$ *...)* (4.7)

Comparing the coefficient of like power of p, we get

$p^0\; :\; v_0\; = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}(1)\right]$,

$p^1\; :\; v_1\; = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}(-x^2 + 2x)\right]$,

$p^2\; :\; v_2\; = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}(v_1(2 - x))\right]$,

$p^3\; :\; v_3\; = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}(-v_1^2 + v_2(2 - x))\right]$,

$$\cdots\,.$$

Therefore

$$v_0 = x,$$

$$v_1 = x^2 - \frac{x^3}{3},$$

$$v_2 = \frac{x^5}{15} - \frac{5x^4}{12} + \frac{2x^3}{3},$$

$$v_3 = -\frac{168x^7}{6615} + \frac{73x^6}{360} - \frac{x^5}{2} + \frac{x^4}{3}$$

$$\ldots\,.$$

Hence the approximate solution of (4.1) is obtained as:

$$y(x) = x + x^2 + \frac{x^3}{3} - \frac{x^4}{12} - \frac{13x^5}{30} + O(x^6)$$

(4.8)

Moreover Riccati differential equation is also solved by using Adomian decomposi tion method [1], and its solution is given as

$$y(x) = x + x^2 + \frac{x^3}{3} - \frac{x^4}{12} - \frac{13x^5}{30} + O(x^6), \qquad \forall\, x \in [0,4]$$

(4.9)

From the figure-2 we can see the approximate solution obtained from the LT-HPM and by ADM is coincide with the exact solution in the neighbourhood of $x = 0$. The solution obtained in power series by our method is also similar to the first few terms of Taylor series expansion of the exact solution about $x = 0$.

LAPLACE TRANSFORM-HOMOTOPY PERTURBATION METHOD

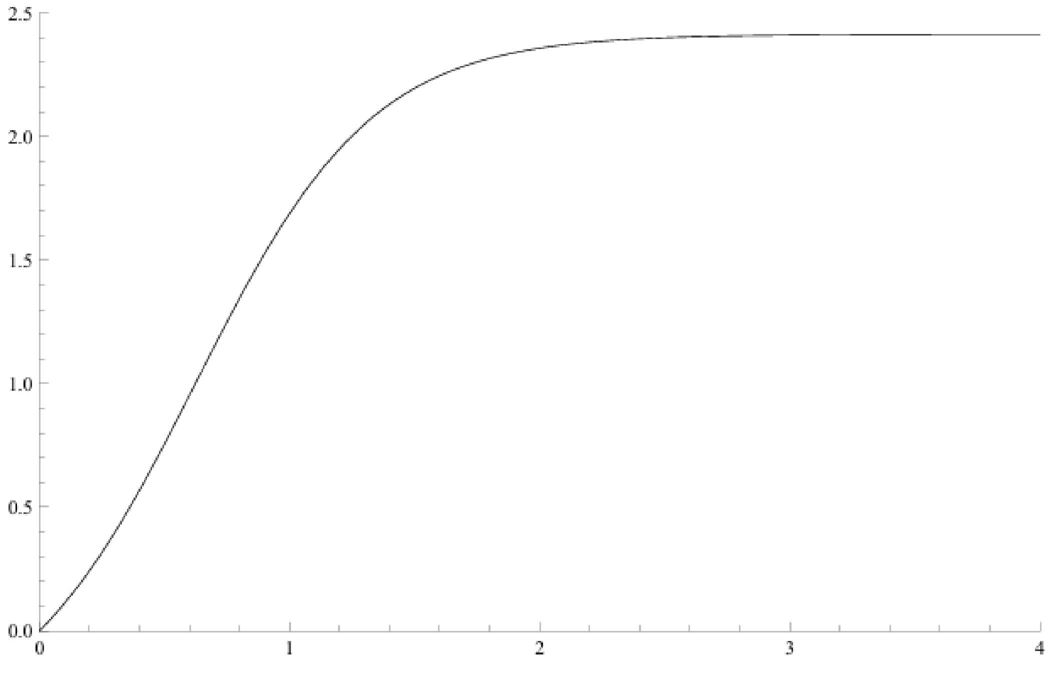

Figure 1. Exact solution

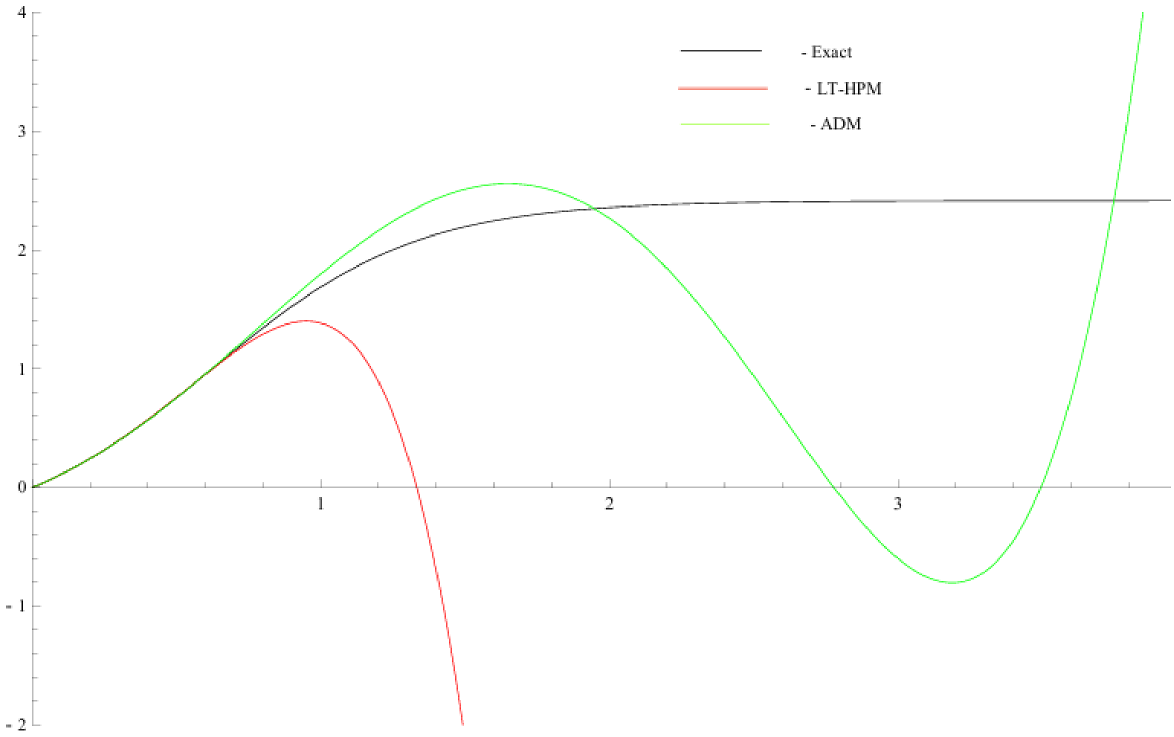

Figure 2. Exact solution, LT-HPM, and ADM method solution.

Problem 2 Considering Riccati differential equation in another form as

$$\frac{dy}{dx} = 1 - y^2, \quad y(0) = 0. \tag{4.10}$$

Its exact solution is given as

$$y(x) = \frac{e^{2x} - 1}{e^{2x} + 1}. \tag{4.11}$$

Figure-3 represents the exact solution. The Taylor series expansion of the exact solution is given as

$$y(x) = x - \frac{x^3}{3} + \frac{x^5}{15} - \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots . \tag{4.12}$$

By proceeding as done in previous problem, using LT-HPM solution of the equation (4.10) is given as

$$y(x) = x - \frac{x^3}{3} + \frac{x^5}{15} + O(x^6). \tag{4.13}$$

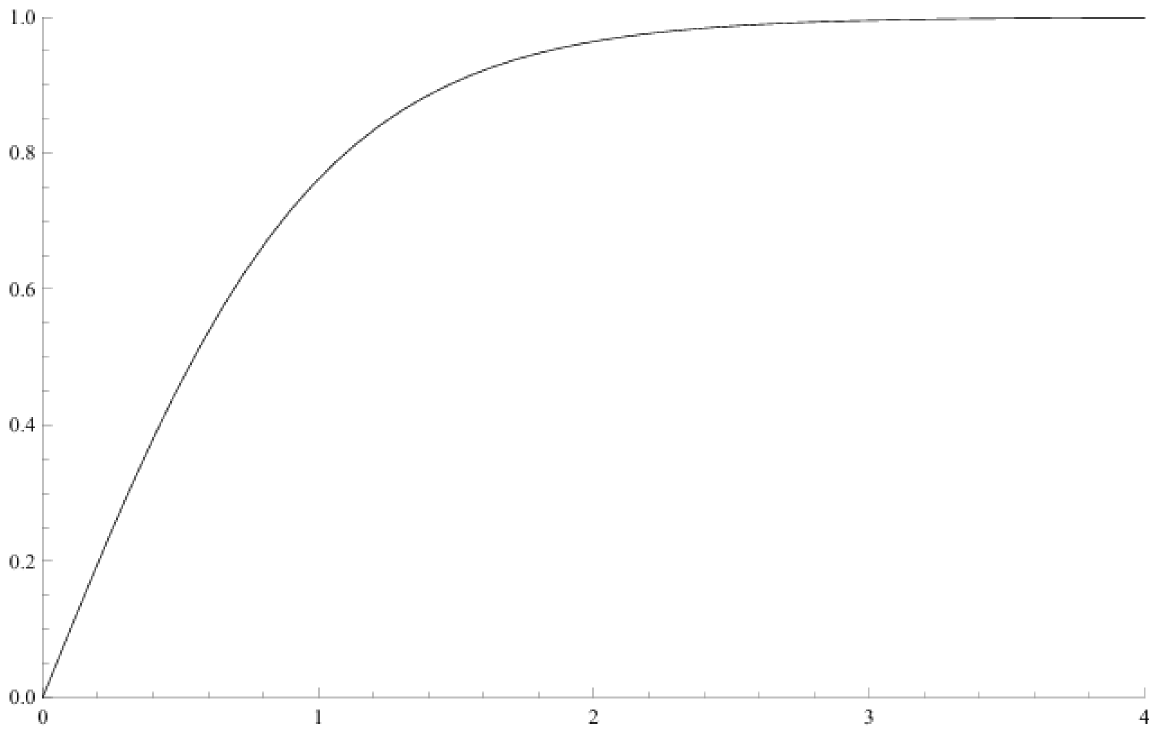

Figure 3. Exact solution.

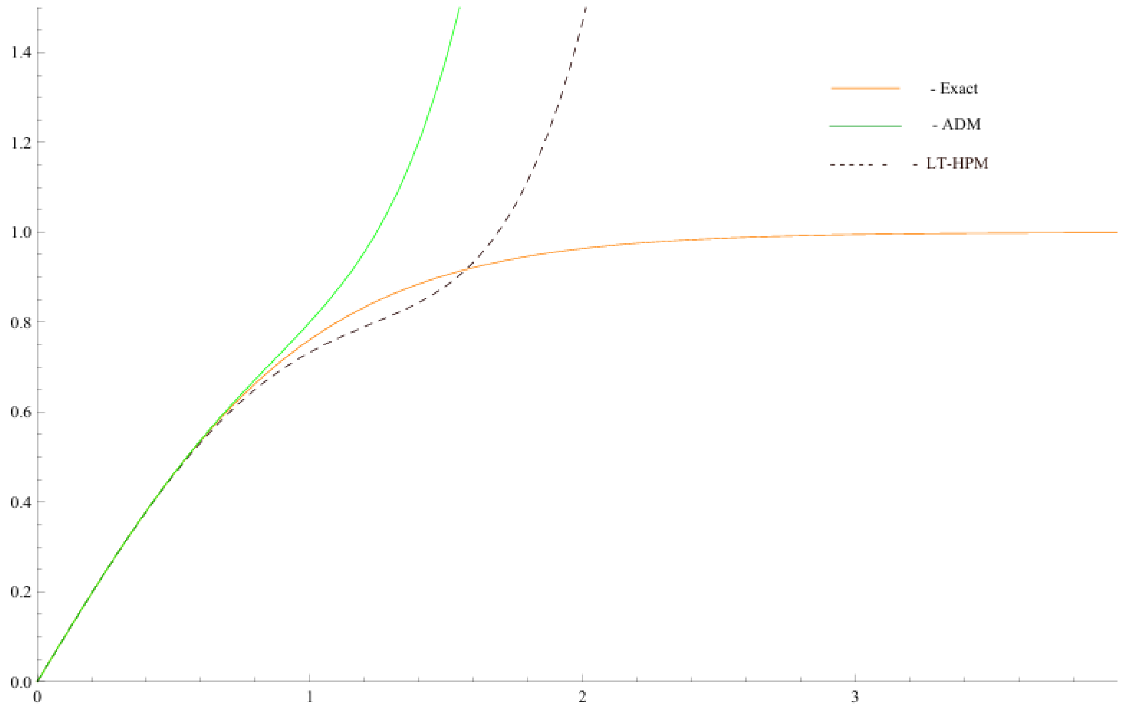

Figure 4. Exact solution, LT-HPM and ADM method solution.

Through Adomian decomposition method [1], solution of (4.10) is given as

$$y(x) = x - \frac{x^3}{3} + \frac{x^5}{15} + O(x^6).$$

(4.14)

In Figure-4 comparison between solution obtained by exact, LT-HPM and ADM is presented.

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