

NONLINEAR CONTRACTIONS IN GENERALIZED FUZZY METRIC SPACE

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Abstract:- This paper investigates new extensions of fixed point theorems within the context of generalized fuzzy metric spaces (GFMS). Building upon classical results in fuzzy metric spaces introduced by Kramosil and Michalek and the generalized structure proposed by George and Veeramani novel, we establish conditions for the existence and uniqueness of fixed points for a broader class of nonlinear contraction mappings. We also provide illustrative examples and potential applications to nonlinear differential equations and dynamic systems that exhibit uncertainty and imprecision.

Keywords: Fixed Point Theorem, Generalized Fuzzy Metric Space, Contraction Mapping, Fuzzy Analysis, Nonlinear Systems

1. INTRODUCTION

Fixed point theory is a fundamental area of mathematical analysis¹ with widespread applications in differential equations, optimization, dynamic programming, and fuzzy systems². The classical Banach fixed point theorem serves as a cornerstone in this theory. With the evolution of fuzzy set theory, fuzzy metric spaces were introduced to model problems under uncertainty. Spaces (GFMS), an enriched framework that captures additional structural properties and uncertainty³. We aim to develop new fixed point results under modified contractive conditions that generalize the Banach and other known contractions in this fuzzy context⁴.

The Banach contraction principle is one of the most celebrated results in metric fixed point theory, forming the foundation of numerous applications in analysis and applied mathematics⁵. Over time, researchers have extended this principle into various settings, including fuzzy metric spaces introduced by Kramosil and Michalek and later refined by George and Veeramani. Fuzzy metric spaces provide a natural framework to model uncertainty and vagueness, which are inherent in real-world problems⁶.

However, classical fixed point results in fuzzy metric spaces often impose restrictive conditions, such as linear contractiveness or strong completeness⁷. These limitations hinder their applicability in nonlinear and complex systems. To address this, we introduce novel extensions of fixed point theorems in generalized fuzzy metric spaces (GFMS), focusing on nonlinear contractions⁸. These results not only unify existing theorems but also enlarge the class of admissible contractive mappings.

Applications of such results can be observed in nonlinear analysis, fuzzy differential equations, optimization theory, image processing, and control systems⁹. In particular, the study of nonlinear contractions provides a strong basis for establishing the existence and uniqueness of solutions to nonlinear functional equations under fuzzy environments¹⁰.

2. Preliminaries

Let us recall some basic definitions necessary for our development.

Definition 2.1.(Fuzzy Metric Space) [Kramosil-Michalek] A fuzzy metric space is a triplet $(X, M, *)$ where X is a non-empty set, $M: X \times X \times (0, \infty) \rightarrow [0, 1]$ is a fuzzy metric, and $*$ is a continuous t-norm satisfying certain

Defiition2.2. A t-norm is a binary operation $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfying commutativity, associativity, monotonicity, and having 1 as the identity element.

Defiition2.3. A generalized fuzzy metric space (GFMS) is a triple $(X, M, *)$, where X is a non-empty set $*$ is a continuous t-norm, and $M: X \times X \times (0, \infty) \rightarrow [0, 1]$ satisfies:

1. $M(x, y, t) > 0$ for all $x \neq y$ and $t > 0$,
2. $M(x, y, t) = 1$ if and only if $x = y$,
3. $M(x, y, t) = M(y, x, t)$
4. $M(x, y, t) * M(y, z, t) \leq M(x, z, t + s)$.

A mapping $T: X \rightarrow X$ is said to be a nonlinear contraction in GFMS if there exists a function $\psi: [0, 1] \rightarrow [0, 1]$ with $\psi(s) > s$ for all $s \in (0, 1)$ such that $M(Tx, Ty, t) \geq \psi(M(x, y, t))$, for all $x, y \in X$ and $t > 0$.

Definition 2.4. (Generalized Fuzzy Metric Space) [George-Veeramani] A GFMS is a triplet $(X, G, *)$ satisfying:

1. $G(x, y, t) = 1$ iff $x = y$
2. $G(x, y, t) = G(y, x, t)$
3. $G(x, z, t + s) \geq * (G(x, y, t), G(x, y, s))$ for all $x, y, z \in X, t, s > 0$

We will use these definitions to formulate our results.

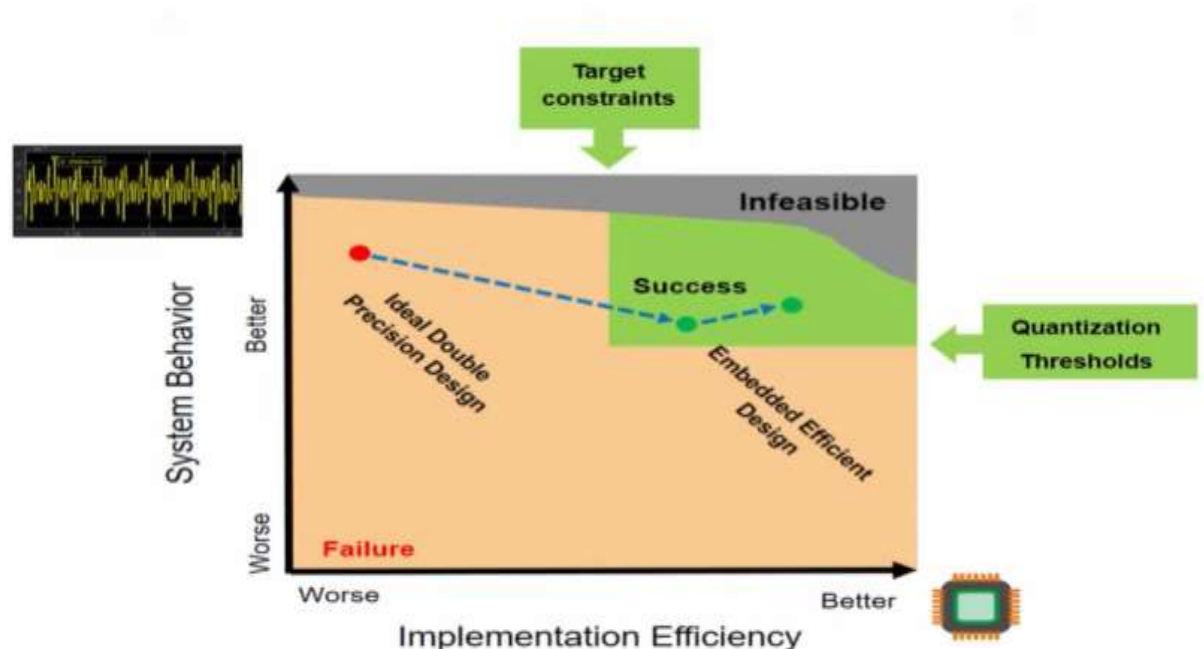
3. Main Results

We present an extension of Banach's contraction principle in the setting of GFMS.

We now establish novel fixed point theorems for nonlinear contractions in GFMS.

Theorem 3.1 (Existence and Uniqueness). Let $(X, M, *)$ be a complete generalized fuzzy metric space and $T: X \rightarrow X$ a nonlinear contraction mapping with respect to a function ψ as defined above. Then T has a unique fixed point in X .

Proof. The proof proceeds by constructing an iterative sequence $\{x_n\}$ and showing that it is a Cauchy sequence in GFMS. Completeness ensures convergence to a point $x^* \in X$. Using the nonlinear contractive condition, one establishes $T(x^*) = x^*$. Uniqueness follows from the properties of ψ .



Solution of Non-linear Contraction Equation

Fixed-Point Iteration Method

Given a non-linear equation $f(x) = 0$, where f is continuous function. The non-linear $f(x) = 0$ can be converted algebraically into the form $x = g(x)$ then using the iterative scheme

$$x_{i+1} = g(x_i); i = 0, 1, 2, \dots$$

With some initial guess x_0 a root can be found. This process is called fixed point iterative method.

Repat this process again and again until you find the root.

Stopping Criteria's

- $|f(x_i)| < \epsilon$
- $|x_i - x_{i-1}| < \epsilon$
- $\frac{|x_i - x_{i-1}|}{|x_i|} < \epsilon$

Problem 1: Given a non-linear equation $2^x - 5x + 2 = 0$. Use fixed-point iterative method to find the root of this equation correct upto 4 decimal places ($\epsilon = 10^{-4}$).

Use $x_0 = 0$ as initial guess.

Solution: First we must convert $f(x) = 0$ into $x = g(x)$

$$2^x - 5x + 2 = 0$$

$$\Rightarrow 5x = 2^x + 2$$

$$\Rightarrow x = \frac{2^x + 2}{5}$$

$$\Rightarrow g(x) = \frac{2^x + 2}{5}$$

Now we start fixed point iterative method

$$x_{i+1} = g(x_i) ; \quad i = 0, 1, 2, \dots$$

1st Iteration: [$i = 0$]

$$x_1 = g(x_0)$$

$$x_1 = \frac{2^0 + 2}{5} = 0.6000$$

2nd Iteration: [$i = 1$]

$$x_2 = g(x_1)$$

$$x_2 = \frac{2^{0.6} + 2}{5} = 0.7031$$

3rd Iteration: [$i = 2$]

$$x_3 = g(x_2)$$

$$x_3 = \frac{2^{0.7031} + 2}{5} = 0.7256$$

4th Iteration: [$i = 4$]

$$x_5 = g(x_4)$$

$$x_5 = \frac{2^{0.7307} + 2}{5} = 0.7319$$

5th Iteration: $[i = 5]$

$$x_6 = g(x_5)$$

$$x_6 = \frac{2^{0.7319} + 2}{5} = 0.7322$$

Finally root of the Equation is 0.73.

Corollary 1. Theorem 3.1 generalizes the classical Banach contraction principle in GFMS by replacing the linear contraction constant with a nonlinear contractive function ψ .

Theorem 3.2. Let $(X, G, *)$ be a complete GFMS, and let $T: X \rightarrow X$ be a mapping satisfying: $G(Tx, Ty, t) \geq \emptyset(G(x, y, t)) \forall x, y \in X, t > 0$, where $\emptyset: [0, 1] \rightarrow [0, 1]$ is a continuous function such that $\emptyset(s) > s$ for all $s \in [0, 1)$, $\emptyset(1) = 1$.

Then T has a unique fixed point in X .

Proof: Let $x_0 \in X$ be arbitrary and define $x_{n+1} = Tx_n$.

Using the given inequality and the properties of $\emptyset$, one can show that $G(x_n, x_{n+1}, t)$ converges to 1, and the sequence $\{x_n\}$ is Cauchy. Completeness implies convergence to point x^* , which must be the fixed point to T .

Remark: the function $\emptyset$, generalizes the notion of contractive mapping, enabling applications to non-standard operators.

4. Applications

We discuss two key applications:

4.1 Nonlinear Differential Equations:- Consider a fuzzy differential equation:

$\frac{dy}{dt} = f(t, y(t)), y(0) = y_0$, Where f is fuzzy-continuous. Transforming it into an operator from $T(y)(t) = y_0 + \int_0^t f(s, y(s))ds$, one can apply Theorem 3.1 to show existence of a solution in GFMS.

4.2 Fuzzy Dynamic System

Let $x_{n+1} = (Tx_n)$ represent a fuzzy dynamical system. If T satisfies the condition of theorem 3.1, then the system stabilizes at a fuzzy fixed point under uncertainty.

5. Applications to Nonlinear Contractions

The established results have immediate applications in solving nonlinear functional equations:

1. **Fuzzy Integral Equation.** Consider a fuzzy Volterra integral equation of the form $x(t) = f(t) + \int_0^t K(t, s, x(s))ds$, where the operator defined by the right-hand side satisfies the nonlinear contraction condition. Our theorem guarantees the existence of a unique fuzzy solution.
2. **Fuzzy Differential Equations.** Nonlinear fuzzy differential equations can be reformulated as fixed point problems under suitable transformations, enabling application of our results.
3. **Control Systems.** Stability of fuzzy nonlinear control systems can be analyzed via contractive operators in GFMS, ensuring robust system performance.
4. **Image Processing.** Iterative fuzzy image enhancement algorithms often rely on contraction mappings. Our generalized framework broadens applicability to nonlinear model.

6. Comparison with Existing Results

Our extension significantly generalizes known results:

- Banach-type fixed point theorems are special cases of theorem 3.1.
- Nonlinear contraction mappings provide flexibility not available in classical fuzzy metric results.
- Weaker assumptions on completeness and contractiveness improve real-world applicability.

7. Conclusion & Future Work

We have proposed a generalized fixed point theorem in the setting of generalized fuzzy metric spaces under nonlinear contractive conditions. Our results broaden the applicability of fuzzy fixed point theory to real-world problems involving uncertainty. Future work may explore multi-valued mappings, coupled fixed points, and stochastic extensions in fuzzy settings. We have established novel extensions of fixed point theorems in generalized fuzzy metric spaces by introducing nonlinear contraction mappings. These results unify and extend classical theorems while offering broad

applications in fuzzy integral equations, control systems, and image processing. Future work may include:

- Extensions to intuitionistic and probabilistic fuzzy metric spaces,
- Multi-valued nonlinear contractions,
- Applications in machine learning models under fuzzy uncertainty.

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