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A Comparative Study of Various Contraction Mappings in Metric and Generalised Metric Spaces with Reference to Fixed Point Theorems

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Abstract

Fixed-point theory converts the solution of an equation into the search for a point that is unchanged by an operator. Among its most constructive results is the contraction principle, because it gives existence, uniqueness, and an iterative method simultaneously. This paper presents a comparative study of major contraction mappings in ordinary and generalised metric spaces. Banach, Edelstein, Kannan, Chatterjea, Reich–Hardy–Rogers, Boyd–Wong, Meir–Keeler, Ćirić, and Nadler contractions are formulated and related through their governing inequalities. The role of completeness, compactness, coefficient restrictions, continuity, and the geometry of the distance function is examined. Extensions to b-metric, partial metric, and cone metric spaces are also discussed, with attention to the changes required in convergence estimates. Several examples show that Kannan and Chatterjea mappings may possess fixed points without being Banach contractions, while nonlinear and quasi-contractive conditions replace a constant Lipschitz factor by more flexible control mechanisms. The analysis demonstrates that the essential structure common to successful contraction theorems is the decay of successive distances together with a completeness principle that converts a Cauchy orbit into a fixed point.

Keywords: fixed point; Banach contraction; Kannan mapping; Chatterjea mapping; Meir–Keeler contraction; b-metric; partial metric; cone metric

1. Introduction

A fixed point of a self-mapping $T:X\to X$ is an element $x^*\in X$ satisfying $T(x^*)=x^*$. Many equations can be written in this form. For example, an integral equation $u=f+Ku$ is equivalent to $u=T(u)$, where $T(u)=f+Ku$. Similarly, an initial-value problem may be transformed into a Picard integral operator. Fixed-point theorems are therefore existence tools, uniqueness criteria, and numerical algorithms at the same time.

The Banach contraction principle is the standard starting point of metric fixed-point theory [1]. It assumes a uniform reduction of every pairwise distance. Later results weakened this requirement in different directions: some compare points with their own images, some use cross-distances, some replace the linear constant by a nonlinear control function, and others modify the underlying metric structure. The resulting family is not a simple hierarchy. A Kannan contraction, for instance, need not be continuous and need not satisfy the Banach inequality; nevertheless, on a complete metric space it still has a unique fixed point.

The purpose of this study is to organise these contractions according to (i) the quantities appearing in the contractive inequality, (ii) the assumptions imposed on the space, and (iii) the convergence mechanism of the iteration $x_{n+1}=T(x_n)$. The method is analytical and comparative: classical definitions are stated, representative proofs are reconstructed, and examples are used to identify the scope and limitations of each theorem.

2. Metric Preliminaries and the Picard Process

A metric on a nonempty set X is a function $d:X\times X\to[0,\infty)$ satisfying positivity, symmetry, and the triangle inequality. The pair (X,d) is complete when every Cauchy sequence converges to an element of X. For a self-map T, choose $x_0\in X$ and define its Picard orbit by

$$x_{n+1}=T(x_n)=T^{n+1}(x_0),\quad n=0,1,2,\ldots \quad (1)$$

$$Fix(T)=\{x\in X:T(x)=x\}. \quad (2)$$

The standard strategy is to prove that $d(x_{n+1},x_n)$ tends to zero at a controlled rate, use the triangle inequality to show that $\{x_n\}$ is Cauchy, invoke completeness to obtain $x_n\to x^*$, and finally establish $T(x^*)=x^*$. Uniqueness follows by inserting two hypothetical fixed points into the same contractive condition.

The distinction between $d(x_{n+1},x_n)\to 0$ and the Cauchy property is important. Successive distances can vanish while the sequence still fails to be Cauchy. Effective fixed-point conditions therefore provide either a summable estimate or an additional argument that excludes non-Cauchy behaviour.

3. Classical and Generalised Contraction Mappings

3.1 Banach Contraction

A mapping $T:X\to X$ is a Banach contraction if there exists $q\in[0,1)$ such that

$$d(Tx,Ty)\le q\, d(x,y), \quad x,y\in X. \quad (3)$$

The constant q is a global Lipschitz factor. The theorem gives a unique fixed point and convergence from every starting point. It also supplies an a priori error estimate, making the result directly useful in computation.

Theorem 1 (Banach contraction principle) Let (X,d) be a nonempty complete metric space and let $T:X\to X$ satisfy (3). Then T has a unique fixed point x^*, and $T^n x_0\to x^*$ for every $x_0\in X$.

3.2 Proof and Error Bounds for Banach Iteration

From (3), repeated application gives

$$d(x_{n+1},x_n)\le q^n\, d(x_1,x_0). \quad (4)$$

$$d(x_m,x_n)\le d(x_m,x_{m-1})+\cdots+d(x_{n+1},x_n)\le[q^n/(1-q)]d(x_1,x_0), \quad m>n. \quad (5)$$

$$d(x^*,x_n)\le(q^n/(1-q))d(x_1,x_0). \quad (6)$$

Equation (5) proves that the Picard orbit is Cauchy. Completeness yields $x_n\to x^*$. Since every Lipschitz mapping is continuous, $T(x^*)=\lim T(x_n)=\lim x_{n+1}=x^*$. If u and v are fixed points, then $d(u,v)=d(Tu,Tv)\le q\, d(u,v)$, which forces $d(u,v)=0$. The same calculation produces the computable estimate (6).

3.3 Edelstein Contractive Mapping on Compact Spaces

Edelstein [2] replaced the uniform constant by the strict inequality

$$d(Tx,Ty)<d(x,y) \quad \text{whenever } x\neq y. \quad (7)$$

Condition (7) is weaker than Banach contractivity because the ratio $d(Tx,Ty)/d(x,y)$ may approach one. It does not ensure a fixed point on an arbitrary complete, noncompact space. On a compact metric space, however, strict contractivity is sufficient. Compactness permits minimisation of the continuous displacement function $\varphi(x)=d(x,Tx)$, and the minimum must be zero. Thus, Banach uses quantitative uniform shrinkage plus completeness, whereas Edelstein uses qualitative strict shrinkage plus compactness.

Theorem 2 (Edelstein) If X is compact and $T:X\to X$ satisfies (7), then T has a unique fixed point and every Picard orbit converges to it.

3.4 Kannan and Chatterjea Contractions

Kannan's condition [3,4] measures the motion of each point under T rather than the initial separation of the points:

$$d(Tx,Ty)\le\alpha[d(x,Tx)+d(y,Ty)], \quad 0\le\alpha<1/2. \quad (8)$$

$$d(Tx,Ty)\le\beta[d(x,Ty)+d(y,Tx)], \quad 0\le\beta<1/2. \quad (9)$$

Equation (9) is the Chatterjea condition [10]. Neither condition is generally equivalent to (3). For a Kannan map, applying (8) to x_n and x_{n-1} gives

Theorem 3 (Kannan–Chatterjea principle) On a complete metric space, every self-map satisfying either (8) or (9) has a unique fixed point.

$$d(x_{n+1},x_n)\le\alpha[d(x_n,x_{n+1})+d(x_{n-1},x_n)], \quad (10)$$

$$d(x_{n+1},x_n)\le\lambda\, d(x_n,x_{n-1}), \quad \lambda=\alpha/(1-\alpha)<1. \quad (11)$$

Hence the consecutive differences decrease geometrically, and the Banach-style Cauchy argument applies with λ in place of q. A notable feature is that continuity of T is not required. The coefficient bound 1/2 is natural because it is exactly what ensures $\lambda<1$. The Chatterjea proof similarly produces $\lambda=\beta/(1-\beta)$.

3.5 Reich and Hardy–Rogers Contractions

A useful unification combines the original distance and displacement terms. A Reich-type inequality [8] can be written as

$$d(Tx,Ty)\le a\, d(x,y)+b\, d(x,Tx)+c\, d(y,Ty), \quad (12)$$

$$a+b+c<1, \quad a,b,c\ge0. \quad (13)$$

Setting $b=c=0$ gives Banach, while setting $a=0$ and $b=c=\alpha$ gives Kannan. Hardy and Rogers [11] introduced a still broader five-term form:

$$d(Tx,Ty)\le a\, d(x,y)+b\, d(x,Tx)+c\, d(y,Ty)+e\, d(x,Ty)+f\, d(y,Tx), \quad (14)$$

$$a+b+c+e+f<1. \quad (15)$$

These hybrid inequalities reveal that several named contractions are coordinate choices within a common family. Their proofs estimate $d(x_{n+1},x_n)$ by collecting the terms that contain the same unknown distance on the left. The decisive issue is whether the resulting ratio is strictly below one.

3.6 Nonlinear Contractions: Boyd–Wong and Meir–Keeler

A Boyd–Wong contraction [5] replaces the constant q by a comparison function $\psi:[0,\infty)\to[0,\infty)$ satisfying $\psi(t)<t$ for $t>0$, typically with suitable upper semicontinuity:

$d(Tx,Ty)\le\psi(d(x,y))$. (16)

This permits weak shrinkage at some scales while still excluding a positive limiting distance. The Meir–Keeler condition [6] expresses the same idea through an ε–δ rule: for every $\varepsilon>0$ there exists $\delta>0$ such that

$\varepsilon\le d(x,y)<\varepsilon+\delta \quad\Rightarrow\quad d(Tx,Ty)<\varepsilon$. (17)

Unlike the Banach condition, (17) does not prescribe a single global rate. It guarantees that distances lying just above any positive level are pushed below that level. On complete metric spaces, Meir–Keeler mappings have a unique fixed point and their Picard iterates converge. These nonlinear forms are important when a uniform Lipschitz constant cannot be found even though the operator contracts every positive distance in a controlled manner.

3.7 Ćirić Quasi-Contraction

Ćirić [9] enlarged the comparison set by defining

$M(x,y)=\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$, (18)

$d(Tx,Ty)\le q\, M(x,y), \quad 0\le q<1$. (19)

The inequality can hold even when no individual Banach, Kannan, or Chatterjea condition is valid. The orbit analysis is more delicate because M contains cross-terms, but bounded orbit diameters and geometric decay lead to a Cauchy sequence. Thus the quasi-contraction is a genuine synthesis of several pairwise and displacement-based mechanisms.

3.8 Nadler Multivalued Contraction

For a set-valued mapping $F:X\to CB(X)$, where CB(X) denotes the nonempty closed bounded subsets of X, use the Hausdorff metric

$H(A,B)=\max\{\sup_{a}\in A\ d(a,B), \sup_{\beta}\in B\ d(b,A)\}$. (20)

$H(Fx,Fy)\le q\, d(x,y), \quad 0\le q<1$. (21)

Nadler's theorem [7] states that a multivalued contraction on a complete metric space has $x^*\in X$ with $x^*\in F(x^*)$. The conclusion is inclusion rather than equality because the image is a set. This extension is fundamental for differential inclusions, optimisation, and problems with nonunique responses.

3.9 Comparative Summary

Type	Core control	Principal restriction	Typical setting
Banach	d(Tx,Ty) by d(x,y)	$q<1$	Complete metric
Kannan	Images by two residuals	$\alpha<1/2$	Complete metric
Chatterjea	Images by cross-distances	$\beta<1/2$	Complete metric
Meir–Keeler	Scale-dependent ε–δ decay	Every $\varepsilon>0$ has $\delta>0$	Complete metric
Ćirić	Maximum of five distances	$q<1$	Complete metric
Nadler	Hausdorff distance of images	$q<1$	Complete metric; set-valued

4. Contractions in Generalised Metric Spaces

4.1 b-Metric Spaces

A b-metric, developed in contraction form by Czerwik [12], relaxes the triangle inequality. For a constant $s\geq1$,

$$d(x,z)\leq s[d(x,y)+d(y,z)]. \quad (22)$$

Ordinary metrics correspond to $s=1$. The relaxation changes estimates because repeated use of the b-triangle inequality may introduce powers of s. A common sufficient form for a direct geometric argument is a contraction $d(Tx,Ty)\leq q\, d(x,y)$ with $sq<1$. More refined metrisation and iteration arguments can establish fixed points under broader formulations, but the proof must not blindly copy the ordinary metric series estimate. The main lesson is that the contraction factor and the geometric defect s interact.

4.2 Partial Metric Spaces

A partial metric p permits nonzero self-distance [13]. Its axioms include $p(x,x)\leq p(x,y)$, symmetry, a modified triangle inequality, and the separation rule $x=y$ exactly when $p(x,x)=p(x,y)=p(y,y)$. Matthews associated to p the ordinary metric

$$d_p(x,y)=2p(x,y)-p(x,x)-p(y,y). \quad (23)$$

$$p(Tx,Ty)\leq q\, p(x,y), \quad 0\leq q<1. \quad (24)$$

Under appropriate completeness, (24) yields a unique fixed point x* with p(x*,x*)=0. Partial metrics are relevant in theoretical computer science, where p(x,x) may represent the amount of information still missing from an approximation. Formula (23) shows how many convergence questions can be transferred to an associated metric while preserving the domain-theoretic interpretation.

4.3 Cone Metric Spaces

In a cone metric space [14], distances take values in an ordered Banach space E rather than directly in $\mathbb{R}$. If $P\subset E$ is a cone, then $d(x,y)\in P$ and the triangle inequality is interpreted through the cone order. A typical contraction is

$$d(Tx,Ty)\leq q\, d(x,y), \quad 0\leq q<1, \quad (25)$$

where multiplication and comparison occur in E. Fixed-point proofs use order convergence or scalarisation. Subsequent metrizability results show that broad classes of cone metric spaces admit equivalent ordinary metrics, so many cone-metric contraction theorems can be recovered from classical metric results. The framework nevertheless remains useful when the vector-valued distance records several error components simultaneously.

5. Comparative Examples and Applications

Example 1: A Banach contraction. Let X=[0,1] with $d(x,y)=|x-y|$ and $T(x)=(x+1)/3$. Then $|Tx-Ty|=(1/3)|x-y|$, so q=1/3. The unique fixed point solves $x=(x+1)/3$, hence $x^*=1/2$. Moreover, $|x_n-1/2|\leq(1/3)^n|x_0-1/2|$.

Example 2: A Kannan contraction that is not Banach. Let X=[0,1] with the usual metric and define T(x)=0 for $0\leq x<1$, while T(1)=1/5. For x<1 and y=1, $|Tx-Ty|=1/5$ and $(1/4)(|x-Tx|+|y-Ty|)=(1/4)(x+4/5)\geq1/5$; all other pairs are immediate. Thus T is a Kannan contraction with $\alpha=1/4$ and has the unique fixed point 0. It is not a Banach contraction because $x\to1$ from below gives $|Tx-T1|=1/5$ while $|x-1|\to0$.

Example 3: Integral equation. Consider $u(t)=f(t)+\int_0^t K(t,s,u(s))ds$ on C([0,a]) with the sup metric. If $|K(t,s,r)-K(t,s,v)|\leq L|r-v|$, then the induced operator satisfies $\|Tu-Tv\|_\infty\leq aL\|u-v\|_\infty$. Therefore aL<1 guarantees a unique solution and convergence of successive approximations.

The examples show that a contraction theorem is selected by matching the operator's natural estimate to the correct distance structure. Banach is preferred when a global Lipschitz constant below one is available because it provides explicit error bounds. Kannan and Chatterjea are useful when estimates involve residuals such as d(x,Tx). Meir–Keeler and Boyd–Wong handle scale-dependent contraction. Ćirić accommodates mixed distance expressions, and Nadler treats set-valued dynamics. In generalised spaces, one must verify that the altered triangle or convergence axioms still convert orbital decay into a genuine limit.

6. Discussion

The comparison reveals three levels in a fixed-point argument. The first is local algebra: the contractive inequality must reduce an appropriate measure of separation. The second is orbital geometry: repeated application must make the Picard sequence Cauchy, not merely asymptotically regular. The third is topological closure: completeness, compactness, or an equivalent condition must place the limiting point inside the space.

No single contraction type dominates all others. Banach contractivity is transparent and quantitative but restrictive. Edelstein removes the uniform constant at the price of compactness. Kannan and Chatterjea replace direct pairwise control by displacement or cross-control and may allow discontinuous mappings. Reich, Hardy–Rogers, and Ćirić unify several inequalities but require more elaborate coefficient analysis. Nonlinear contractions provide flexibility without losing uniqueness, although convergence rates may be less explicit. Generalised metric spaces are meaningful when their distance structure reflects the application; otherwise an equivalent ordinary metric may offer a simpler proof.

A useful unified interpretation is to search for an orbital control sequence $r_n \geq 0$ satisfying $d(x_{n+1},x_n) \leq r_n$ and $\sum r_n < \infty$. Banach and Kannan mappings generate geometric r_n. Meir–Keeler and Ćirić conditions establish decay indirectly. Once summability or an equivalent Cauchy criterion is available, the remaining fixed-point proof follows a common pattern.

7. Conclusion

Contraction mapping theory is a collection of related mechanisms rather than a single inequality. The Banach theorem remains the model because it combines a complete metric space, a uniform contraction, a unique fixed point, and a convergent algorithm. Subsequent contractions redistribute these requirements: compactness can replace uniformity; residual and cross-distances can replace direct Lipschitz control; nonlinear functions can replace constants; and generalised metrics can encode weakened or vector-valued notions of distance. Across these settings, the central mathematical principle is invariant: contractive control must force the iterative orbit toward a Cauchy limit, and the geometry of the space must ensure that this limit becomes a fixed point. This perspective provides a practical basis for choosing fixed-point tools in integral equations, differential equations, optimisation, numerical analysis, and computational semantics.

References

[1] Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundamenta Mathematicae, 3, 133–181.

[2] Edelstein, M. (1962). On fixed and periodic points under contractive mappings. Journal of the London Mathematical Society, 37, 74–79. https://doi.org/10.1112/jlms/s1-37.1.74

[3] Kannan, R. (1968). Some results on fixed points. Bulletin of the Calcutta Mathematical Society, 60, 71–76.

[4] Kannan, R. (1969). Some results on fixed points—II. American Mathematical Monthly, 76, 405–408.

[5] Boyd, D. W., & Wong, J. S. W. (1969). On nonlinear contractions. Proceedings of the American Mathematical Society, 20, 458–464. https://doi.org/10.1090/S0002-9939-1969-0239559-9

[6] Meir, A., & Keeler, E. (1969). A theorem on contraction mappings. Journal of Mathematical Analysis and Applications, 28, 326–329. https://doi.org/10.1016/0022-247X(69)90031-6

[7] Nadler, S. B., Jr. (1969). Multi-valued contraction mappings. Pacific Journal of Mathematics, 30, 475–488.

[8] Reich, S. (1971). Kannan's fixed point theorem. Bollettino dell'Unione Matematica Italiana, 4, 1–11.

[9] Ćirić, L. B. (1971). Generalized contractions and fixed-point theorems. Publications de l'Institut Mathématique, 12(26), 19–26.

[10] Chatterjea, S. K. (1972). Fixed-point theorems. Comptes Rendus de l'Académie Bulgare des Sciences, 25, 727–730.

[11] Hardy, G. E., & Rogers, T. D. (1973). A generalization of a fixed point theorem of Reich. Canadian Mathematical Bulletin, 16, 201–206. https://doi.org/10.4153/CMB-1973-036-0

[12] Czerwik, S. (1993). Contraction mappings in b-metric spaces. Acta Mathematica et Informatica Universitatis Ostraviensis, 1, 5–11.

[13] Matthews, S. G. (1994). Partial metric topology. Annals of the New York Academy of Sciences, 728, 183–197. https://doi.org/10.1111/j.1749-6632.1994.tb44144.x

[14] Huang, L.-G., & Zhang, X. (2007). Cone metric spaces and fixed point theorems of contractive mappings. Journal of Mathematical Analysis and Applications, 332, 1468–1476. https://doi.org/10.1016/j.jmaa.2005.03.087

[15] Kirk, W. A., & Khamsi, M. A. (2001). An Introduction to Metric Spaces and Fixed Point Theory. Wiley.