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fτ*-Generalized Closed Sets in Fine Topological Spaces

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Abstract: In this paper, we introduce a new class of sets called fτ*-generalized closed sets and fτ*-generalized open sets in fine topological spaces and study some of their properties**.**

Keywords- fτ*-g-closed set, fτ*-g-open set.

1. Introduction

In 1970, Levine[8][9] introduced the idea of generalized closed sets in topological spaces and explored some of their key features. Later many researchers developed different types of generalized closed sets such as semi-generalized closed sets, generalized semi-closed sets, α-closed sets and several others. Dunham[20] introduced the idea of closure operator cl* by using generalized closed sets and also introduced a new topology τ*, which is weaker than τ.

Powar P. L. and Rajak K.[14] introduced a special type of generalized topological space, which is a fine topological space. In a fine topological space, authors have defined a new class of open sets called fine-open sets which contains all α-open sets, β-open sets, semi-open sets, pre-open sets, regular open sets and some other open sets. Through these fine open sets, authors have defined fine-irresolute mappings, which include pre-continuous functions, semi-continuous functions, α-continuous functions, β-continuous functions, α-irresolute functions, β-irresolute functions, etc.(cf.[12]- [16])

The aim of this paper is to introduce the concept of fτ*- closed sets in a fine topological space (X, τ, τ_f). We introduce a new class of sets called fτ*-generalized closed sets and fτ*-generalized open sets in fine topological spaces and study some of their properties**.**

2. Preliminaries

We recall the following definitions:

Definition 2.1: A subset A of a topological space (X, τ) is called

(i) Generalized closed (briefly g-closed) if cl(A) ⊆ G whenever A ⊆ G and G is open in X.(cf.[8])

(ii) Semi-generalized closed (briefly sg-closed) if scl(A) ⊆ G whenever A ⊆ G and G is semi-open in X.(cf.[10])

(iii) Generalized semi-closed (briefly gs-closed) if scl(A) ⊆ G whenever A ⊆ G and G is open in X.(cf.[19])

(iv) α-closed if cl(int(cl(A))) ⊆ A .(cf.[18])

(v) α-generalized closed (briefly αg-closed)[9] if clα(A)⊆ G whenever A ⊆ G and G is open in X.(cf.[4])

(vi) Generalized α-closed (briefly gα-closed)[10] if spcl (A) ⊆ G whenever A ⊆ G and G is open in X.(cf.[5])

(vii) Generalized semi-preclosed (briefly gsp-closed)[2] if scl(A) ⊆ G whenever A ⊆ G and G is open in X.(cf.[19])

(viii) Strongly generalized closed (briefly strongly g-closed) if cl(A) ⊆ G whenever A ⊆ G and G is g-open in X.(cf. [11])

(ix) Preclosed if cl(int(A)) ⊆ A.(cf.[2])

(x) Semi-closed if int(cl(A)) ⊆ A.(cf.[9])

(xi) Semi-preclosed (briefly sp-closed) if int(cl(int(A))) ⊆ A. (cf. [3])

Remark: The complements of the above-mentioned sets are called their respective open sets.

Definition 2.2: For the subset A of a topological space (X, τ), the generalized closure of A denoted by cl*(A) is defined as the intersection of all g-closed sets containing A. (cf. [20])

Definition 2.3: Let (X, τ) be a topological space. Then the collection $\{G: cl^*(G^C) = (G^C)\}$ is a topology on X and is denoted as τ* that is, $\tau^* = \{G: cl^*(G^C)=(G^C)\}$. (cf. [7])

Definition 2.4: For the subset A of a topological space (X, τ),

(i) The semi-closure of A (briefly scl(A)) is defined as the intersection of all semi-closed sets containing A.(cf. [9])

(ii) The semi-pre-closure of A (briefly spcl(A)) is defined as the intersection of all semi-preclosed sets containing A.(cf. [3])

(iii) The α-closure of A (briefly clα(A)) is defined as the intersection of all α-closed sets containing A.(cf. [18])

Definition 2.5: A subset A of a topological space (X, τ), is called τ^*-generalized closed set (briefly τ^*-g-closed) if $cl^*(A) \subseteq G$ whenever $A \subseteq G$ and G is τ^*-open. The complement of τ^*-generalized closed set is called the τ^*-generalized open set (briefly τ^*-g-open).(cf.[1])

Definition 2.6: Let (X, τ) be a topological space we define

$\tau(A_\alpha) = \tau_\alpha$ (say) $= \{ G_\alpha (\neq X): G_\alpha \cap A_\alpha = \phi$, for $A_\alpha \in \tau$ and $A_\alpha \neq \phi, X$, for some $\alpha \in J$, where J is the index set.$\}$

Now, we define $\tau_f = \{\phi, X, \cup_{(\alpha \in J)} \{\tau_\alpha\}\}$

The above collection τ_f of subsets of X is called the fine collection of subsets of X, and (X, τ, τ_f) is said to be the fine space X generated by the topology τ on X (cf. [7]).

Definition 2.7 A subset U of a fine space X is said to be a fine-open set of X if U belongs to the collection τ_f, and the complement of every fine-open set of X is called the fine-closed sets of X, and we denote the collection by F_f (cf. [7])

Definition 2.8 Let A be a subset of a fine space X. We say that a point $x \in X$ is a fine limit point of A if every fine-open set of X containing x must contain at least one point of A other than x (cf. [7]).

Definition 2.9 Let A be the subset of a fine space X. The fine interior of A is defined as the union of all fine-open sets contained in the set. A i.e. the largest fine-open set contained in the set A, and is denoted by f_{int} (cf. [7]).

Definition 2.10 Let A be the subset of a fine space X. The fine closure of A is defined as the intersection of all fine-closed sets containing the set A, i.e. the smallest fine-closed set containing the set A, and is denoted by f_{cl} (cf. [7])

Definition 2.11 A subset A of a fine-topological space (X, τ, τ_f) is called fine-generalized closed set (briefly, fg-closed) if $f_{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is fine-open in X.(cf.[7])

Definition 2.12: A subset A of a fine - topological space (X, τ, τ_f), the fine-generalized closure of A denoted by $f_{cl^*}(A)$ is defined as the intersection of all fg-closed sets containing A.(cf.[7])

Definition 2.13: Let (X, τ, τ_f) be a topological space. Then the collection $\{G: f_{cl^*}(G^C) = (G^C)$ is a fine-topology on X and is denoted as τ_f^* that is $\tau_f^* = \{G: f_{cl^*}(G^C) = (G^C)$.(cf.[7])

3. fτ^*-Generalized Closed Sets in Fine Topological Space

Definition 3.1: A subset A of a fine topological space (X, τ, τ_f) is called fine semi - generalized closed (briefly fsg-closed) if $f_{scl}(A) \subseteq G$ whenever $A \subseteq G$ and G is fine semi-open in X.

Example 3.1: Let X = {a, b, c} with the topology τ = {ϕ, X, {a}, {a, b}} and the fine topology τ_f = {ϕ, X, {a}, {a, b}, {a, c}} and F_f = {ϕ, X, {b}, {c}, {b, c}}. Then the set {c} is an fsg-closed set in X.

Definition 3.2: A subset A of a fine topological space (X, τ, τ_f) is called fine Generalized semi-closed (briefly fgs-closed) if $f_{scl}(A) \subseteq G$ whenever $A \subseteq G$ and G is fine open in X.

Example 3.2: Let X = {a, b, c} with the topology τ = {ϕ, X, {a}, {a, b}} and the fine topology τ_f = {ϕ, X, {a}, {a, b}, {a, c}} and F_f = {ϕ, X, {b}, {c}, {b, c}}. Then the set {b} is a fgs-closed in X.

Definition 3.3: A subset A of a fine topological space (X, τ, τ_f) is called fα-closed if $f_{cl}(f_{int}(f_{cl}(A))) \subseteq A$.

Example 3.3: Let X = {a, b, c} with the topology τ = {ϕ, X, {a}, {a, b}} and the fine topology τ_f = {ϕ, X, {a}, {a, b}, {a, c}} and F_f = {ϕ, X, {b}, {c}, {b, c}}. Then the set {b, c} is a fα-closed in X.

Definition 3.4: A subset A of a fine topological space (X, τ, τ_f) is called fα-generalized closed (briefly fαg-closed) if $f_{cl\alpha}(A) \subseteq G$ whenever $A \subseteq G$ and G is fine open in X.

Example 3.4: Let X = {a, b, c} with the topology τ = {ϕ, X, {a}, {a, b}} and the fine topology τ_f = {ϕ, X, {a}, {a, b}, {a, c}} and F_f = {ϕ, X, {b}, {c}, {b, c}}. Then the set {b} is a fαg-closed in X.

Definition 3.5: A subset A of a fine topological space (X, τ, τ_f) is called Generalized fα-closed (briefly fgα-closed) if f_{spcl} (A) $\subseteq$ G whenever $A \subseteq G$ and G is fine open in X.

Example 3.5: Let X = {a, b, c} with the topology τ = {ϕ, X, {a}, {a, b}} and the fine topology τ_f = {ϕ, X, {a}, {a, b}, {a, c}} and F_f = {ϕ, X, {b}, {c}, {b, c}}. Then the set {c} is a gfα-closed in X.

Definition 3.6: A subset A of a fine topological space (X, τ, τ_f) is called Fine Generalized semi-preclosed (briefly fgsp-closed) if $f_{scl}(A) \subseteq G$ whenever $A \subseteq G$ and G is fine open in X.

Example 3.6: Let X = {a, b, c} with the topology τ = {ϕ, X, {a}, {a, b}} and the fine topology τ_f = {ϕ, X, {a}, {a, b}, {a, c}} and F_f = {ϕ, X, {b}, {c}, {b, c}}. Then the set {b, c} is an fgsp-closed in X.

Definition 3.7: A subset A of a fine topological space (X, τ, τ_f) is called Strongly fine generalized closed (briefly strongly fg-closed) if $f_{cl}(A) \subseteq G$ whenever $A \subseteq G$ and G is fg-open in X.

Example 3.7: Let X = {a, b, c} with the topology τ = {ϕ, X, {a}, {a, b}} and the fine topology τ_f = {ϕ, X, {a}, {a, b}, {a, c}} and F_f = {ϕ, X, {b}, {c}, {b, c}}. Then the set {c} is a strongly fg-closed in X.

Definition 3.8: A subset A of a fine topological space (X, τ, τ_f) is called Fine Preclosed if $f_{cl}(f_{int}(A)) \subseteq A$.

Example 3.8: Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ and the fine topology $\tau_f = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $F_f = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}$. Then the set $\{b\}$ is a fine preclosed-closed in X.

Definition 3.9: A subset A of a fine topological space (X, τ, τ_f) is called Fine Semi-closed if $f_{int}(f_{cl}(A)) \subseteq A$.

Example 3.9: Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ and the fine topology $\tau_f = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $F_f = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}$. Then the set $\{b\}$ is a fine semi-closed in X.

Definition 3.10: A subset A of a fine topological space (X, τ, τ_f) is called Fine Semi-preclosed (briefly fsp-closed) if $f_{int}(f_{cl}(f_{int}(A))) \subseteq A$.

Example 3.10: Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ and the fine topology $\tau_f = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $F_f = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}$. Then the set $\{b, c\}$ is a fine semi-preclosed set in X.

Definition 3.11: A subset A of a fine topological space (X, τ, τ_f), is called $f\tau^*$-generalized closed set (briefly $f\tau^*$-g-closed) if $f_{cl^*}(A) \subseteq G$ whenever $A \subseteq G$ and G is $f\tau^*$-open. The complement of $f\tau^*$-generalized closed set is called the $f\tau^*$-generalized open set (briefly $f\tau^*$-g-open).

The complements of the above-mentioned sets are called their respective open sets.

Theorem 3.1: Every fine closed set in X is $f\tau^*$-g-closed.

Proof: Let A be a fine closed set. Let $A \subseteq G$. Since A is a fine closed, $f_{cl}(A) = A \subseteq G$. But $f_{cl^*}(A) \subseteq f_{cl}(A)$. Thus, we have $f_{cl^*}(A) \subseteq G$ whenever $A \subseteq G$ and G is $f\tau^{**}$-open. Therefore, A is $f\tau^*$-g-closed.

Theorem 3.2: Every $f\tau^*$-closed set in X is $f\tau^*$-g-closed.

Proof: Let A be a $f\tau^*$-closed set. Let $A \subseteq G$, where G is $f\tau^*$-open. Since A is $f\tau^*$-closed, $f_{cl^*}(A) = A \subseteq G$. Thus, we have $f_{cl^*}(A) \subseteq G$ whenever $A \subseteq G$ and G is $f\tau^*$-open. Therefore, A is $f\tau^*$-g-closed.

Theorem 3.3: Every fg-closed set in X is a $f\tau^*$-g-closed set, but not conversely.

Proof: Let A be an fg-closed set. Assume that $A \subseteq G$ and G is $f\tau^*$-open in X. Then $f_{cl}(A) \subseteq G$, since A is fg-closed. But $f_{cl^*}(A) \subseteq f_{cl}(A)$. Therefore, $f_{cl^*}(A) \subseteq G$. Hence, A is $f\tau^*$-g-closed.

The converse of the above theorem need not be true as seen from the following example.

Example 3.11: Consider the fine topological space $X = \{1, 2, 3\}$ with topology $\tau = \{X, \phi, \{1\}\}$. Then the set $\{1\}$ is fτ*-g-closed but not fg-closed.

Theorem 3.4: For any two sets A and B, $f_{cl^*}(A \cup B) = f_{cl^*}(A) \cup f_{cl^*}(B)$.

Proof: Since $A \subseteq A \cup B$, we have $f_{cl^*}(A) \subseteq f_{cl^*}(A \cup B)$ and since $B \subseteq A \cup B$, we have $f_{cl^*}(B) \subseteq f_{cl^*}(A \cup B)$. Therefore $f_{cl^*}(A) \cup f_{cl^*}(B) \subseteq f_{cl^*}(A \cup B)$. Also , $f_{cl^*}(A)$ and $f_{cl^*}(B)$ are the closed sets. Therefore, $f_{cl^*}(A) \cup f_{cl^*}(B)$ is also a closed set. Again, $A \subseteq f_{cl^*}(A)$ and $B \subseteq f_{cl^*}(B)$ implies $A \cup B \subseteq f_{cl^*}(A) \cup f_{cl^*}(B)$.Thus, $f_{cl^*}(A) \cup f_{cl^*}(B)$ is a closed set containing $A \cup B$. Since $f_{cl^*}(A \cup B)$ is the smallest closed set containing $A \cup B$, we have $f_{cl^*}(A \cup B) \subseteq f_{cl^*}(A) \cup f_{cl^*}(B)$. Thus, $f_{cl^*}(A \cup B) = f_{cl^*}(A) \cup f_{cl^*}(B)$.

Theorem 3.5: Union of two fτ*-g-closed sets in X is a fτ*-g-closed set in X.

Proof: Let A and B be two fτ*-g-closed sets. Let $A \cup B \subseteq G$, where G is fτ*-open. Since A and B are fτ*-g-closed sets, $f_{cl^*}(A) \cup f_{cl^*}(B) \subseteq G$. But by Theorem 3.8., $f_{cl^*}(A) \cup f_{cl^*}(B) = f_{cl^*}(A \cup B)$. Therefore, $f_{cl^*}(A \cup B) \subseteq G$. Hence, $A \cup B$ is a fτ*-g-closed set.

Theorem 3.6: A subset A of X is fτ*-g-closed if and only if $f_{cl^*}(A) - A$ contains no non-empty fτ*-closed set in X.

Proof: Let A be a fτ*-g-closed set. Suppose that F is a non-empty fτ*-closed subset of $f_{cl^*}(A) - A$. Now $F \subseteq f_{cl}(A) - A$. Then $F \subseteq f_{cl^*}(A) \cap A^c$, since $f_{cl^*}(A) - A = f_{cl^*}(A) \cap A^c$. Therefore, $F \subseteq f_{cl^*}(A)$ and $F \subseteq A^c$. Since F^c is a fτ*-open set and A is a fτ**-g-closed, $f_{cl^*}(A) \subseteq F^c$. That is $F \subseteq [f_{cl^*}(A)]^c$. Hence, $F \subseteq f_{cl^*}(A) \cap [f_{cl^*}(A)]^c = \phi$. That is $F = \phi$, a contradiction. Thus, $f_{cl^*}(A) - A$ contain no non-empty fτ*-closed set in X. Conversely, assume that $f_{cl^*}(A) - A$ contains no non-empty fτ*-closed set. Let $A \subseteq G$, where G is fτ*-open. Suppose that $f_{cl^*}(A)$ is not contained in G, then $f_{cl^*}(A) \cap G^c$ is a non-empty fτ*-closed set of $f_{cl^*}(A) - A$, which is a contradiction. Therefore, $f_{cl^*}(A) \subseteq G$ and hence A is fτ*-g-closed.

Corollary: A subset A of X is fτ*-g-closed if and only if $f_{cl^*}(A) - A$ contain no non-empty fine closed set in X.

Corollary: A subset A of X is fτ*-g-closed if and only if $f_{cl^*}(A) - A$ contain no non-empty fine open set in X.

Theorem 3.7: If a subset A of X is fτ*-g-closed and $A \subseteq B \subseteq f_{cl^*}(A)$, then B is a f$\tau$*-g-closed set in X.

Proof: Let A be a fτ*-g-closed set such that $A \subseteq B \subseteq f_{cl^*}(A)$. Let U be a f$\tau$*-open set of X such that $B \subseteq U$. Since A is fτ*-g-closed, we have $f_{cl^*}(A) \subseteq U$. Now $f_{cl^*}(A) \subseteq f_{cl^*}(B) \subseteq f_{cl^*}[f_{cl^*}(A)] = f_{cl^*}(A) \subseteq U$. That is, $f_{cl^*}(B) \subseteq U$, U is fτ*-open. Therefore, B is a fτ*-g-closed set in X.

The converse of the above theorem need not be true as seen from the following example.

Example 3.12: Consider the topological space (X, τ), where X = {1, 2, 3} and the topology τ = {X, ϕ,{1},{1, 2}}. Let A = {3} and B = {1, 3}. Then A and B are fτ*-g-closed sets in (X, τ). But A ⊆ B is not a subset of f_{cl*}(A).

Theorem 3.8: Let A be a fτ*-g-closed in (X, τ). Then A is fg-closed if and only if f_{cl*}(A) – A is fτ*-open.

Proof: Suppose A is fg-closed in X. Then f_{cl*}(A) = A and so f_{cl*}(A) – A = ϕ, which is fτ*-open in X. Conversely, suppose f_{cl*}(A) – A is fτ*-open in X. Since A is fτ*-g-closed, by Theorem 3.10, f_{cl*}(A) – A contains no non-empty fτ*-closed set in X. Then f_{cl*}(A) – A = ϕ. Hence, A is fg-closed.

Theorem 3.9: For x ∈ X, the set X – {x} is fτ*-g-closed or fτ*-open.

Proof: Suppose X – {x} is not fτ*-open. Then X is the only fτ*-open set containing X – {x}. This implies f_{cl*}(X – {x}) ⊆ X. Hence, X – {x} is a fτ*-g-closed set in X.

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