OBSERVATIONS ON THE BIQUADRATIC WITH FIVE UNKNOWS

$$x^4 - y^4 - 2xy(x^2 - y^2) = z(X^2 + Y^2)$$

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Abstract:

We obtain infinitely many non-zero integer quintuples (x, y, z, X, Y) satisfying the biquadratic equation with five unknowns $x^4 - y^4 - 2xy(x^2 - y^2) = z(X^2 + Y^2)$. Various interesting properties between the values of x, y, z, X, Y and special number patterns, namely, polygonal numbers, centered pyramidal and polygonal numbers, Jacob-lucas numbers and kynea numbers are presented.

Key words: biquadratic equation with five unknowns, integral solutions, polygonal numbers, centered figurate numbers.

MSC 2000 Mathematics subject classification:11D25.

Notations:

$T_{m,n}$ = Polygonal number of rank n with size m.

P_n^m = Pyramidal number of rank n with size m.

CP_n^m = Centered pyramidal number of rank n whose generating polygon has m sides.

S_n = Star number of rank n

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j_n = Jacobsthal-Lucas number of rank n

ky_n = kynea number of rank n

$F_{4,n,5}$ = Four dimensional pentagonal figurate number of rank n.

Introduction:

Biquadratic Diophantine equations, homogeneous and non-homogeneous, have aroused the interest of numerous Mathematicians since ambiguity as can be seen from [1-7]. Particularly in [8,9] biquadratic diophantine equations with five unknowns are analysed for their non-zero integral solutions. In this paper, another interesting biquadratic equation with five unknowns given by $x^4 - y^4 - 2xy(x^2 - y^2) = z(X^2 + Y^2)$ is considered and five different patterns of integral solutions are illustrated. A few interesting properties between the solutions and special number patterns are exhibited.

Method of analysis:

The biquadratic with five unknown is

$$x^4 - y^4 - 2xy(x^2 - y^2) = z(X^2 + Y^2) \quad (1)$$

It is seen that (1) is satisfied by the quintuple $(u + 2ab, u - 2ab, 16uab, 2a^2 - b^2, 2a^2 + b^2)$.

However, we have other patterns of solutions to (1) which are illustrated as follows.

Introduction of the transformations

$$x = u + v, \quad y = u - v, \quad z = 8uv \quad (2)$$

in (1) leads to $X^2 + Y^2 = 2v^2$ (3)

We present below different methods of solving (3) and thus, in view of (2), one obtains different patterns of solution to (1).

Pattern:1

Let $v = a^2 + b^2$ (4)

Write 2 as

$$2 = (1+i)(1-i) \tag{5}$$

Using (4) and (5) in (3) and applying the methods of factorization, define

$$X + iY = (1+i)(a+ib)^2$$
$$X - iY = (1-i)(a-ib)^2$$

Equating real and imaginary parts, we have

$$\left.\begin{aligned} X &= a^2 - b^2 - 2ab \\ Y &= a^2 - b^2 + 2ab \end{aligned}\right\} \tag{6}$$

Using (4) in (2), it is seen that

$$\left.\begin{aligned} x &= u + a^2 + b^2 \\ y &= u - a^2 - b^2 \\ z &= 8u(a^2 + b^2) \end{aligned}\right\} \tag{7}$$

Thus (6) and (7) represent the non-zero distinct integral solutions to (1).

Properties:

(i) $z(x+y)$ is a sum of two squares.

(ii) $6\left[x(u,a,1)-y(u,a,1)+2X(a,1)+1\right]$ is a nasty number

(iii) $z(1,a,1)\left[x(1,a,1)+y(1,a,1)\right]-t_{34,a}\equiv 1 \pmod{15}$

(iv) $x(1,a,1)+y(1,a,1)+z(1,a,1)-t_{18,a}\equiv 3 \pmod{7}$

(v) $2X(a,b)+2Y(a,b)$ is a difference of two squares.

(vi) $X(2a,1)+Y(2a,1)-t_{3,4a}+2\equiv 0 \pmod{2}$

(vii) $z(a,a,1)-3CP_a^{16}\equiv 0 \pmod{6}$

(viii) $z(\alpha^2,\alpha^2,1)-3CP_{\alpha^2}^{16}$ is a nasty number

(ix) $x(u,n,3)-y(u,n,3)+X(n,3)-2CP_{12,n}+1\equiv 0 \pmod{9}$

(x) $y(-4n,-1,2n)+Y(-1,2n)+CP_{16,n}=1$

Pattern:2

Write (3) as $2v^2-Y^2=X^2*1$ (8)

Let $X=2a^2-b^2$ (9)

Write 1 as

$$1=(\sqrt{2}+1)(\sqrt{2}-1) \quad (10)$$

Substituting (9) and (10) in (8) and employing the factorization method, define

$$\sqrt{2}v+Y=(\sqrt{2}a+b)^2(\sqrt{2}+1)$$

Equating the rational and irrational parts, we get

$$Y = 2a^2 + b^2 + 4ab \qquad (11)$$

$$v = 2a^2 + b^2 + 2ab \qquad (12)$$

Using (12) in (2), we have,

$$\left.\begin{aligned} x &= u + 2a^2 + b^2 + 2ab \\ y &= u - 2a^2 - b^2 - 2ab \\ z &= 8u(2a^2 + b^2 + 2ab) \end{aligned}\right\} \qquad (13)$$

Thus (9), (11) and (13) represent the non-zero distinct integral solutions to (1).

Properties:

(i) $10\,[x(1,a,b) - y(1,a,b) + z(1,a,b) + 10]$ is a sum of two squares.

(ii) $6\,[x(u,a,1) - y(u,a,1) - 1]$ is a nasty number.

(iii) $x(u,1,2^{2n}) - y(u,1,2^{2n}) = 2j_{4n} + j_{2n+2} + 1$

(iv) $x(u,1,2^{2n+1}) - y(u,1,2^{2n+1}) = 2j_{4n+2} + j_{2n+3} + 3$

(v) $[x(n,n,1) - Y(n,1)]X(n,1) - 2CP_n^{12} + 4t_{3,n} + 2t_{6,n} - 4t_{4,n} = 1$

(vi) $z(n,n,1) - 6CP_n^{17} + 2CP_n^{3} - 12t_{21,n} + 2t_{5,n} \equiv 0 \pmod{36}$

(vii) $z(1,1,n) + x(1,1,n) + y(1,1,n) - Y(1,n) - CP_{14,n} \equiv 0 \pmod{5}$

(viii) $[x(u,n,1) - u]X(n,1) + [y(u,n,1) - u]Y(n,1) + 3CP_n^{16} + CP_{24,n} + 2t_{3,n} - t_{4,n} = -1$

Pattern:3

Instead of (10), we write 1 as

$$1=\frac{(5\sqrt{2}+1)(5\sqrt{2}-1)}{49}$$

Following the procedure as presented in pattern 2, the corresponding non zero distinct integral solutions to (1) are obtained as

$$x=u+7(10A^2+5B^2+2AB)$$
$$y=u-7(10A^2+5B^2+2AB)$$
$$z=56u(10A^2+5B^2+2AB)$$
$$X=7(4A^2-7B^2)$$
$$Y=14A^2+7B^2+140AB$$

Properties:

(i) $x(u,A,1)-y(u,A,1)-28t_{3,2A}\equiv 70(\mathrm{mod}\,84)$

(ii) $x(u,A,1)-y(u,A,1)-28t_{3,2A}-14t_{14,A}\equiv 0(\mathrm{mod}\,5)$

(iii) $X(2^{2n},2^n)=49j_{4n+1}-49ky_n+98(J_n-(-1)^n)$

(iv) $[7(x-y)-7(X+Y)]^2$ is a nasty number

(v) $z(u,A,1)-8X(A,1)-x(u,A,1)+y(u,A,1)=14[2t_{3,A}+53]$

(vi) $x(u,n,1)-4CP_{29,n}\equiv 32(\mathrm{mod}\,44)$

(vii) $x(u,1,n)-y(u,1,n)-2X(1,n)+2Y(1,n)-14CP_{28,n}+14CP_{16,n}-28t_{9,n}\equiv -2(\mathrm{mod}\,21)$

(viii) $X(1,-1)+Y(1,-1)-42CP_n^{26}-42CP_n^{28}+21CP_n^{4}+14CP_{15,n}+14t_{3,n}\equiv -8(\mathrm{mod}\,39)$

(ix) $x(1,n,n^2)+Y(1,n^2)-14[4F_{4,n,5}+2CP_n^3-6t_{3,n}]=1$

Pattern:4

Again, choosing 1 as

$$1=\frac{(29\sqrt{2}+1)(29\sqrt{2}-1)}{41^2}$$

and repeating the process similar to pattern 2, the corresponding non-zero distinct integral solutions to (1) are found to be

$$x=u+41(58A^2+29B^2+2AB)$$
$$y=u-41(58A^2+29B^2+2AB)$$
$$z=328u(58A^2+29B^2+2AB)$$
$$X=41(82A^2-41B^2)$$
$$Y=41(58A^2+B^2+116AB)$$

Properties:

(i) $\dfrac{x(u,A,1)-y(u,A,1)-X(A,1)+Y(A,1)-1312t_{3,A}-1640t_{3,A+1}}{164}\equiv 4 \pmod{11}$

(ii) $z(1,A,B)[x(1,A,B)-y(1,A,B)]$ is a perfect square.

(iii) $x(1,2^{2n},1)-y(1,2^{2n},1)=82[29j_{4n+1}+j_{2n+1}+59]$

(iv) $x(1,2^{2n+1},1)-y(1,2^{2n+1},1)=82[29j_{4n+3}+j_{2n+2}+57]$

(v) $z(1,2^{2n},1)=328[29j_{4n+1}+j_{2n+1}+59]$

(vi) $z(1,2^{2n+1},1)=328[29j_{4n+3}+j_{2n+2}+57]$

(vii) $Y(n,-1)-41(t_{16,n}-CP_{24,n}+CP_{14,n})\equiv 41 \pmod{4305}$

(viii) $X(n,1)=41^2(CP_{4,n}+t_{6,n}-2t_{4,n}-2t_{3,n}-2)$

(ix) $x(u,n^2,1)-y(u,n^2,1)-X(n^2,1)-41[t_{23,n}+CP_{26,n}+CP_{20,n}-10t_{4,n}]\equiv 3977$

Pattern:5

Consider the transformations

$$x = u + v, \quad y = u - v, \quad z = 4uv, \quad X = p + q, \; Y = p - q, \qquad (14)$$

Using(14) in (1),we have,

$$p^2 + q^2 = 2v^2$$

Which is satisfied by

$$p = a^2 - b^2 - 2ab$$

$$q = a^2 - b^2 + 2ab$$

$$v = a^2 + b^2$$

Substitute the above values of p, q, v in (14),the non-zero distinct integral solutions to (1) are represented by

$$x = u + a^2 + b^2$$

$$y = u - a^2 - b^2$$

$$z = 4u(a^2 + b^2)$$

$$X = 2a^2 - 2b^2$$

$$Y = -4ab$$

Properties:

(i) $X(1,3) - Y(1,3) + x(u,1,3) - y(u,1,3)$ is a perfect square

(ii) $6[x(u,a,b) - y(u,a,b) + X(a,b)]$is a nasty number

(iii) $X(a,1) - Y(a,1) + x(1,a,1) - y(1,a,1) = 8t_{3,a}$

(iv) $z(u,a,b)\left[x(u,a,b) + y(u,a,b)\right] \equiv 0 \pmod 8$

(v) $x(u,a,1)-y(u,a,1)+X(a,1)+Y(a,1)=2t_{6,a}-2a$

(vi) $Y(-a,a(a+1))=8P_a^5$

(vii) $6Y(-a,a-1)=4S_a-4$

(viii) $Y(a,1)\left[y(u,a,1)-u\right]-6CP_a^8\equiv 0(\bmod 5)$

(ix) $13Y(a,1)\left[y(u,a,1)-u\right]-24CP_a^{13}\equiv 0(\bmod 80)$

(x) $Y(n,1)\left[x(1,n,1)-y(1,n,1)+z(1,n,1)\right]+6CP_n^{24}\equiv 0(\bmod 42)$

(xi) $X(1,n)Y(1,n)-3CP_n^{16}\equiv 0(\bmod 3)$

Conclusion:

One may search for other patterns of integral solutions to (1) and their corresponding properties.

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