ON HOMOGENEOUS TERNARY QUADRATIC DIOPHANTINE EQUATION

$$2(x^2+y^2)-3xy=16z^2$$

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Abstract:

The ternary quadratic homogeneous equation representing homogeneous cone given by $2(x^2+y^2)-3xy=16z^2$ is analyzed for its non-zero distinct integer points on it. Three different patterns of integer points satisfying the cone under consideration are obtained. A few interesting relations between the solutions and special number patterns namely Polygonal number, Pyramidal number, Centered Polygonal number, Centered Pyramidal number, pronic number and Star number are presented.

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1. INTRODUCTION:

The ternary quadratic Diophantine equations offer an unlimited field for research due to their variety [1, 20]. For an extensive review of various problems, one may refer [2-19]. This communication concerns with yet another interesting ternary quadratic equation $2(x^2+y^2)-3xy=16z^2$ representing a cone for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

NOTATIONS:

$T_{m,n}$ - Polygonal number of rank n with size m.

P_n^m - Pyramidal number of rank n with size m.

$Ct_{m,n}$ - Centered Polygonal number of rank n with size m.

$CP_{m,n}$ - Centered Pyramidal number of rank n with size m.

$\Pr_n$ - Pronic number of rank n.

S_n - Star number of rank n.

2. METHOD OF ANALYSIS:

The ternary quadratic equation to be solved for its non-zero distinct integer solutions is

$$2(x^2+y^2)-3xy=16z^2 \tag{1}$$

The substitution of the linear transformations

$$x=u+v\,,\ y=u-v \tag{2}$$

in (1) leads to

$$u^2+7v^2=16z^2 \tag{3}$$

Assume $z=z(a,b)=a^2+7b^2\,;a,b>0$ (4)

(3) is solved through different approaches and different patterns of solutions thus obtained for (1) are illustrated below:

2.1 PATTERN: 1

Write (3) as

$$u^2 - 9z^2 = 7z^2 - 7v^2 \tag{5}$$

Factorizing (5) we have

$$(u+3z)(u-3z) = 7(z+v)(z-v) \tag{6}$$

which is equivalent to the system of double equations

$$\left.\begin{aligned} bu - av + (3b-a)z &= 0 \\ -au - 7bv + (3a+7b)z &= 0 \end{aligned}\right\} \tag{7}$$

Applying the method of cross multiplication, we get

$$u = -3a^2 + 21b^2 - 14ab$$

$$v = a^2 - 7b^2 - 6ab$$

$$z = -a^2 - 7b^2$$

Employing (2), the values of x, y, z satisfying (1) are given by

$$x = x(a,b) = -2a^2 + 14b^2 - 20ab$$

$$y = y(a,b) = -4a^2 + 28b^2 - 8ab$$

$$z = z(a,b) = -a^2 - 7b^2$$

PROPERTIES:

- $x(b(b+1),b) - 2z(b(b+1),b) - 2T_{30,b} + 40P_b^5 \equiv 0 \pmod{2}$
- $4z(a,7a^2-1) + y(a,7a^2-1) + T_{18,a} + 48CP_{7,a} \equiv 0 \pmod{7}$
- $x(a,1) + y(a,1) + S_a \equiv 9 \pmod{34}$

2.2 PATTERN: 2

One may write (3) as

$$u^2 + 7v^2 = 16z^2 * 1 \tag{8}$$

Write 16 as

$$16 = \left(3 + i\sqrt{7}\right)\left(3 - i\sqrt{7}\right) \tag{9}$$

Also, write 1 as

$$1 = \frac{\left(1 + i3\sqrt{7}\right)\left(1 - i3\sqrt{7}\right)}{64} \tag{10}$$

Substituting (4), (9) and (10) in (8) and employing the method of factorization, define

$$\left(u + i\sqrt{7}v\right)\left(u - i\sqrt{7}v\right) = \left(3 + i\sqrt{7}\right)\left(3 - i\sqrt{7}\right)\frac{\left(1 + i3\sqrt{7}\right)\left(1 - i3\sqrt{7}\right)}{64}\left(a + i\sqrt{7}b\right)^2\left(a - i\sqrt{7}b\right)^2$$

Equating real and imaginary parts, we have

$$\left.\begin{aligned} u &= \frac{1}{4}\left(-9a^2 + 63b^2 - 70ab\right) \\ v &= \frac{1}{4}\left(5a^2 - 35b^2 - 18ab\right) \end{aligned}\right\} \tag{11}$$

The choices a=2A and b=2B in (4) and (11) lead to

$$u = u(A,B) = -9A^2 + 63B^2 - 70AB$$

$$v = v(A,B) = 5A^2 - 35B^2 - 18AB$$

$$z = z(A,B) = 4A^2 + 28B^2 \tag{11A}$$

In view of (2), the integer values of x and y are given by,

$$\left.\begin{aligned} x &= x(A,B) = -4A^2 + 28B^2 - 88AB \\ y &= y(A,B) = -14A^2 + 98B^2 - 52AB \end{aligned}\right\} \tag{11B}$$

Thus (11A) and (11B) represent non-zero distinct integer solutions of (1) in two parameters.

PROPERTIES:

- $y(5B^2+1, B) + z(5B^2+1, B) + 250T^2_{4,B} - 26\Pr_B + 312CP_{5,B} \equiv 0(\text{mod}2)$
- $z(A, A(A+1)) - x(A, A(A+1)) - T_{18,A} - 176P^5_A \equiv 0(\text{mod}7)$

- $2y(A,4A^2-1)-6x(A,4A^2-1)-z(A,4A^2-1)+8\Pr_A-2544CP_{8,A}\equiv 0(\mod 2)$

2.3 PATTERN: 3

Also, instead of (10), write 1 as

$$1=\frac{(3+i4\sqrt{7})(3-i4\sqrt{7})}{121} \tag{12}$$

Following the procedure presented in pattern: 2, the corresponding values of x and y satisfying (1) are

$$x=x(A,B)=-44A^2+308B^2-2728AB$$

$$y=y(A,B)=-374A^2+2618B^2-1892AB$$

$$z=z(A,B)=121A^2+847B^2$$

PROPERTIES:

- $x(1,B)-308\Pr_B\equiv 0(\mod 2)$
- $y(A,A+1)-z(A,A+1)-1276\Pr_A+344Ct_{11,A}\equiv 1(\mod 2)$
- $14x(2B^2+1,B)-y(2B^2+1,B)-2z(2B^2+1,B)-308T_{24,B}+108900CP_{4,B}\equiv 0(\mod 2)$
- $4z(A,7A^2-1)-11x(A,7A^2-1)-121T_{18,A}-180048CP_{7,A}\equiv 0(\mod 7)$

3. REMARKABLE OBSERVATIONS:

Let p, q be any two non-zero distinct positive integers such that p>q>0.

Define $p=x_n+\frac{y_n}{2}$ and $q=\frac{y_n}{2}$. Treat p, q as the generators of the Pythagorean triangle $T(\alpha,\beta,\gamma)$ where $\alpha=2pq,\beta=p^2-q^2,\gamma=p^2+q^2$. Let P, A represent the perimeter and the area of T. Then, each of the following expressions is a perfect square.

a. $6\gamma-2\alpha-4\beta-3\sqrt{2(\gamma-\alpha)(\gamma-\beta)}$

b. $2\gamma+2\alpha-\frac{16A}{P}-3\sqrt{2(\gamma-\alpha)\left(\alpha-\frac{4A}{P}\right)}$

c. $10\gamma-8\beta-6\alpha+\frac{16A}{P}-3\sqrt{2(\gamma-\alpha)\left(2(\gamma-\beta)+\frac{4A}{P}-\alpha\right)}$

4. CONCLUSION:

In this paper, we have presented three different patterns of non-zero distinct integer solutions of the homogeneous cone given by $2(x^2+y^2)-3xy=16z^2$ .To conclude, one may search for other patterns of non-zero integer distinct solution and their corresponding properties

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