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FREE VIBRATION ANALYSIS OF RECTANGULAR PLATE BY SPLIT-DEFLECTION METHOD

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Abstract

This paper presents free vibration analysis of rectangular plate by split-deflection method. In this method, the deflection was split into x and y components of deflection. That is the deflection of the rectangular plate was taken as the product of these two components. Having made this assumption, the study went ahead to formulate total potential energy functional from principles of theory of elasticity based on work-error approach. This energy functional was minimized by direct variation and equation for resonating frequency was obtained. Two illustrative examples were used to test this method. They are plates with all edges simply supported and all edges clamped. The first case used polynomial function for x component of deflection and trigonometric function for y component of. However, the second example used polynomial function for both x and y components of deflection. Fundamental resonating frequencies (in non dimensional forms) of the two plates for aspect ratios ranging from 1.0 to 2.0 (at increment of 0.1) were determined and compared with the values from previous study. From the comparison, it was observed that the maximum percentage difference of 0.06 was recorded for the first example at aspect ratios of 1.1 and 1.3 with non dimensional resonating frequencies of 18.03 and 15.71 respectively. For the

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second example, the maximum percentage difference of 0.09 was recorded at aspect ratio of 1.1 with non dimensional resonating frequencies of 32.96.This small value of percentage differences show that this present method is adequate and reliable for classical plate theory (CPT) free vibration analysis of rectangular plates.

Index Terms—Resonating frequency, split-deflection, work-error, energy functional, polynomial function, trigonometric function

I. INTRODUCTION

Most energy methods used for classical plate theory (CPT) free vibration analysis of rectangular plates include Raleigh, Raleigh-Rit, Ritz, Galerkin, minimum potential energy, work-erroretc (Ugural, 1999, Ventsel and Krauthammer, 2001 and Ibearugbulem et al., 2014). The deflection (displacement normal to the plane of the plate is a single orthogonal function, w.This is apparent in the energy functional for Raleigh, Raleigh-Ritz and Ritz. Typical Ritz energy functional is (Ibearugbulem et al., 2014):

$$\Pi = \frac{D}{2}\int_0^a \int_0^b \left(\left[\frac{\partial^2 w}{\partial x^2}\right]^2 + 2\left[\frac{\partial^2 w}{\partial x \partial y}\right]^2 + \left[\frac{\partial^2 w}{\partial y^2}\right]^2\right) \partial x \partial y - \frac{m\,.\,\lambda^2}{2}\int_0^a \int_0^b w^2\, \partial x \partial y$$

The use of single orthogonal deflection function is also seen in Galerkinand work-error methods. Typical work-error functional is (Ibearugbulem et al., 2014):

$$\Pi = \frac{D}{2}\int_0^a \int_0^b \left(\frac{\partial^4 w}{\partial x^4} w + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} w + \frac{\partial^4 w}{\partial y^4} w\right) \partial \mathrm{x}\, \partial \mathrm{y} - \frac{m\,.\,\lambda^2}{2}\int_0^a \int_0^b w^2\, \partial x \partial y$$

From the literature so far, most scholarly works on CPT analysis of rectangular plates rely on this single orthogonal function (Hutchinson, 1992, Jianqiao, 1994, Ugural, 1999, Ventsel and Krauthammer, 2001, Wang et al., 2002, Taylor and Govindjee, 2004, Szilard,2004, Jiu et al., 2007, Erdem et al., 2007,Ezeh et al., 2013, Ibearugbulem, 2014). Obviously, one can assert that all energy functional in use are based on single orthogonal deflection function andnone has used a deflections function that is typically separated into two independent distinct functions (w = w_x * w_y). In this present study$w_x$and $w_y$may be both polynomials or trigonometric functions or w_x may be polynomialswhilew_y may be trigonometry.The main reason for this modification is to help the analysis who may have difficulty in obtaining orthogonal function for a plate of a particular boundary condition. In this case, the analyst who may have easy access to deflection equations for beams of any boundary condition can find the proposed method quite useful and handy.

II. BASIC ASSUMPTIONS

The assumption here is that the general deflection, w is split into w_x and w_y. That is the split-deflection function is given as:

$$w = w_x . w_y \qquad 1$$

Where the w_x and w_y components of the deflection are defined as:

$$w_x = \sqrt{A}. h_1 \qquad 2$$

$$w_y = \sqrt{A}. h_2 \qquad 3$$

Substituting equations (2) and (3) into equation (1) gives:

$$w = A\, h_1 h_2 \qquad 4$$

III. IN-PLANE DISPLACEMENTS

From the assumption that vertical shear strains are zero for classical plate and making use of split-deflection, we obtain:

$$u = -z\frac{dw}{dx} = -z\frac{dw_x}{dx}w_y \qquad 5$$

$$v = -z\frac{dw}{dy} = -z\frac{dw_y}{dy}w_x \qquad 6$$

IV. STRAIN DEFLECTION RELATIONSHIP

Differentiating equations (5) and (6), the three in-plane strains of CPTare given as:

$$\varepsilon_x = \frac{du}{dx} = -z\frac{d^2w_x}{dx^2}w_y \qquad 7$$

$$\varepsilon_y = \frac{dv}{dy} = -z\frac{d^2w_y}{dy^2}w_x \qquad 8$$

$$\gamma_{xy} = \frac{du}{dy} + \frac{dv}{dx} = -2z\frac{dw_x}{dx}\frac{dw_y}{dy} \qquad 9$$

V. STRESS – STRAIN RELATIONSHIP

The CPT constitutive equations for plane stress plate are given as:

$$\sigma_x = \frac{E}{1-\mu^2}\left[\varepsilon_x + \mu\varepsilon_y\right] \qquad 10$$

$$\sigma_y = \frac{E}{1-\mu^2}\left[\mu\varepsilon_x + \varepsilon_y\right] \qquad 11$$

$$\tau_{xy} = \frac{E(1-\mu)}{2(1-\mu^2)}\gamma_{xy} \qquad 12$$

VI. STRESS – DEFLECTION RELATIONSHIP

Substituting equations (7), (8) and (9) into equations (10), (11) and (12) where appropriate gives the split-deflection stress-deflection equation as:

$$\sigma_x = \frac{-Ez}{1-\mu^2}\left[\frac{d^2w_x}{dx^2}w_y + \mu\frac{d^2w_y}{dy^2}w_x\right] \qquad 13$$

$$\sigma_y = \frac{-Ez}{1-\mu^2}\left[\mu\frac{d^2w_x}{dx^2}w_y + \frac{d^2w_y}{dy^2}w_x\right] \qquad 14$$

$$\tau_{xy} = \frac{-Ez(1-\mu)}{(1-\mu^2)}\frac{dw_x}{dx}\frac{dw_y}{dy} \qquad 15$$

VII. TOTAL POTENTIAL ENERGY

The strain energy is defined as:

$$U = \frac{1}{2}\int_x \int_y \left[\int_{-\frac{t}{2}}^{\frac{t}{2}}[\sigma_x\varepsilon_x + \sigma_x\varepsilon_x + \sigma_x\varepsilon_x]dz\right]dx\,dy \quad 16$$

For pure bending analysis , the external work is given as:

$$V = \int_x \int_y \frac{m\,.\,\lambda^2}{2}{w_x}^2.{w_y}^2 dx\,dy$$

That is

$$V = \frac{m\,.\,\lambda^2}{2}\int_x {w_x}^2\,dx \int_y {w_y}^2\,dy \qquad 17$$

Substituting equations (10) to (15) into equation (16) gives strain energy – deflection relationship as:

$$U = \frac{D}{2}\int_x \int_y \left[\left(\frac{d^2w_x}{dx^2}\right)^2 w_y{}^2 + 2\left(\frac{dw_x}{dx}\right)^2\left(\frac{dw_y}{dy}\right)^2 + \left(\frac{d^2w_y}{dy^2}\right)^2 w_x{}^2\right] dx\, dy$$

That is in the work-error approach, the strain energy becomes:

$$U = \frac{D}{2}\left[\int_x \frac{d^4w_x}{dx^4} w_x\, dx \int_y w_y{}^2\, dy\right] + \frac{2D}{2}\left[\int_x \frac{d^2w_x}{dx^2} w_x\, dx \int_y \frac{d^2w_y}{dy^2} w_y\, dy\right] + \frac{D}{2}\left[\int_x w_x{}^2\, dx \int_y \frac{d^4w_y}{dy^4} w_y\, dy\right] \qquad 18$$

Subtracting equation (17) from Equation (18) gives the total potential energy functional as:

$$\Pi = \frac{D}{2}\left[\int_x \frac{d^4w_x}{dx^4} w_x\, dx \int_y w_y{}^2\, dy\right] + \frac{2D}{2}\left[\int_x \frac{d^2w_x}{dx^2} w_x\, dx \int_y \frac{d^2w_y}{dy^2} w_y\, dy\right] + \frac{D}{2}\left[\int_x w_x{}^2\, dx \int_y \frac{d^4w_y}{dy^4} w_y\, dy\right] - \frac{m\,.\lambda^2}{2}\int_x w_x{}^2 dx \int_y w_y{}^2\, dy \qquad 19$$

Substituting equations (1) and (2) into equation (19) gives:

$$\Pi = \frac{A^2D}{2}\left[\int_x \frac{d^4h_1}{dx^4} h_1\, dx \int_y h_2{}^2\, dy\right] + \frac{2A^2D}{2}\left[\int_x \frac{d^2h_1}{dx^2} h_1\, dx \int_y \frac{d^2h_2}{dy^2} h_2\, dy\right] + \frac{A^2D}{2}\left[\int_x h_1{}^2\, dx \int_y \frac{d^4h_2}{dy^4} h_2\, dy\right] - \frac{m\,.\lambda^2}{2} A^2 \int_x h_1{}^2 dx \int_y h_2{}^2\, dy \qquad 20$$

Now, equation (20) can be written in non dimensional axes R and Q.

$x = aR$ 21

$y = aQ$ 22

$P = b/a$ 23

Where a, b and P are the plate lengths in x and y axes and long span- short span aspect ratio respectively.

Substituting equations (21), (22) and (23) into equation (20) gives:

$$\Pi = \frac{abA^2D}{2a^4}\left[\int_0^1 \frac{d^4h_1}{dR^4} h_1 dR \int_0^1 {h_2}^2 \, dQ\right]$$

$$+2\frac{abA^2D}{2a^4P^2}\left[\int_0^1 \frac{d^2h_1}{dR^2} h_1 dR \int_0^1 \frac{d^2h_2}{dQ^2} h_2 \, dQ\right] + \frac{abA^2D}{2a^4P^4}\left[\int_0^1 {h_1}^2 \, dR \int_0^1 \frac{d^4h_2}{dQ^4} h_2 \, dQ\right]$$

$$- \frac{m\,.\,\lambda^2}{2} A^2 ab \int_0^1 {h_1}^2 dR \int_0^1 {h_2}^2 \; dQ \qquad 24$$

VIII. **DIRECT VARIATION OF TOTAL POTENTIAL ENERGY**

Equation (24) shall be differentiated with respect to the deflection coefficient, A and the result is:

$$\frac{d\Pi}{dA} = \frac{AD}{a^4}\left[\int_0^1 \frac{d^4h_1}{dR^4} h_1 dR \int_0^1 {h_2}^2 \, dQ\right]$$

$$+2\frac{AD}{a^4P^2}\left[\int_0^1 \frac{d^2h_1}{dR^2} h_1 dR \int_0^1 \frac{d^2h_2}{dQ^2} h_2 \, dQ\right]$$

$$+\frac{AD}{a^4P^4}\left[\int_0^1 {h_1}^2 \, dR \int_0^1 \frac{d^4h_2}{dQ^4} h_2 \, dQ\right]$$

$$- m\,.\,\lambda^2 A \int_0^1 {h_1}^2 dR \int_0^1 {h_2}^2 \; dQ \; = 0$$

That is

$$\frac{D}{a^4}\left[\int_0^1 \frac{d^4h_1}{dR^4} h_1 dR \int_0^1 {h_2}^2 \, dQ\right]$$

$$+2\frac{D}{a^4P^2}\left[\int_0^1 \frac{d^2h_1}{dR^2} h_1 dR \int_0^1 \frac{d^2h_2}{dQ^2} h_2 \, dQ\right]$$

$$+\frac{D}{a^4P^4}\left[\int_0^1 {h_1}^2 \, dR \int_0^1 \frac{d^4h_2}{dQ^4} h_2 \, dQ\right] = m\,.\,\lambda^2 \int_0^1 {h_1}^2 dR \int_0^1 {h_2}^2 \; dQ \qquad 25$$

This equation (25) is the direct governing equation of rectangular plate under free vibration using work-error approach from this present method. Rearranging equation (25) and making resonating frequency, λ the subject of the equation gives:

$$\lambda^2 = \left(\frac{k_x + 2\frac{k_{xy}}{P^2} + \frac{k_y}{P^4}}{k_\lambda}\right) * \frac{D}{ma^4} \qquad 26$$

Where

$$k_x = \int_0^1 \frac{d^4 h_1}{dR^4} h_1 dR \int_0^1 {h_2}^2\, dQ \qquad 27$$

$$k_{xy} = \int_0^1 \frac{d^2 h_1}{dR^2} h_1 dR \int_0^1 \frac{d^2 h_2}{dQ^2} h_2\, dQ \qquad 28$$

$$k_y = \int_0^1 {h_1}^2\, dR \int_0^1 \frac{d^4 h_2}{dQ^4} h_2\, dQ \qquad 29$$

$$k_\lambda = \int_0^1 {h_1}^2 dR \int_0^1 {h_2}^2\, dQ \qquad 30$$

IX. NUMERICAL EXAMPLE

Analyze a classical rectangular thin isotropic plate with:

i all the four edge simply supported using polynomial and trigonometry functions respectively for w_x and w_y.

iiall the four edge clamped using only polynomial function for both w_x and w_y

All Simple supported edge rectangular plate

$$w_x = \sqrt{A}\,(R - 2R^3 + R^4) \qquad 31$$

$$w_Y = \sqrt{A}\,\text{Sin}\,\pi Q \qquad 32$$

From equations (31) and (32), h_1 and h_2 are:

$$h_1 = R - 2R^3 + R^4 \qquad 33$$

$$h_2 = \text{Sin}\,\pi Q \qquad 34$$

With these we obtain:

$$\int_0^1 {h_1}^2\, dR = \frac{31}{630}\ and \int_0^1 {h_2}^2\, dQ = 0.5.$$

$$\int_0^1 \frac{d^4 h_1}{dR^4} h_1 dR = 4.8 \text{ and } \int_0^1 \frac{d^4 h_2}{dQ^4} h_2\, dQ = 0.5\pi^4$$

$$\int_0^1 \frac{d^2 h_1}{dR^2} h_1 dR = \frac{17}{35} \text{and } \int_0^1 \frac{d^2 h_2}{dQ^2} h_2\, dQ = 0.5\pi^2.$$

$$k_x = (4.8)(0.5) = 2.4 \qquad 35$$

$$k_{xy} = \left(\frac{17}{35}\right)(0.5\pi^2) = 2.3969 \qquad 36$$

$$k_y = \left(\frac{31}{630}\right)(0.5\pi^4) = 2.3966 \qquad 37$$

$$k_\lambda = \left(\frac{31}{630}\right)(0.5) = 0.02460315 \qquad 38$$

Substituting equations (35) to (38) into equation (26) gives

$$\lambda^2 = \left(\frac{2.4 + \frac{4.7938}{P^2} + \frac{2.3966}{P^4}}{\frac{31}{1260}}\right) * \frac{D}{ma^4}. \; That\ is$$

$$\lambda^2 = \left(97.5484 + \frac{194.8448}{P^2} + \frac{97.4102}{P^4}\right) * \frac{D}{ma^4} \qquad 39$$

All clamped edge rectangular plate

$$\mathrm{w}_x = \sqrt{\mathrm{A}}\,(\mathrm{R}^2 - 2\mathrm{R}^3 + \mathrm{R}^4) \qquad 40$$

$$\mathrm{w}_Y = \sqrt{\mathrm{A}}\,(\mathrm{Q}^2 - 2\mathrm{Q}^3 + \mathrm{Q}^4) \qquad 41$$

From equations (40) and (41), h_1 and h_2 are:

$$\mathrm{h}_1 = \mathrm{R}^2 - 2\mathrm{R}^3 + \mathrm{R}^4 \qquad 42$$

$$\mathrm{h}_2 = \mathrm{Q}^2 - 2\mathrm{Q}^3 + \mathrm{Q}^4 \qquad 43$$

With these we obtain:

$$\int_0^1 {h_1}^2\, dR = \frac{1}{630} \; and \int_0^1 {h_2}^2\, dQ = \frac{1}{630}.$$

$$\int_0^1 \frac{d^4 h_1}{dR^4} h_1 dR = 0.8 \text{ and } \int_0^1 \frac{d^4 h_2}{dQ^4} h_2\, dQ = 0.8$$

$$\int_0^1 \frac{d^2 h_1}{dR^2} h_1 dR = \frac{2}{105} \text{and } \int_0^1 \frac{d^2 h_2}{dQ^2} h_2\, dQ = \frac{2}{105}$$

$$k_x = (0.8)\left(\frac{1}{630}\right) = \frac{2}{1575} \qquad 44$$

$$k_{xy} = \left(\frac{2}{105}\right)\left(\frac{2}{105}\right) = \frac{4}{11025} \qquad 45$$

$$k_y = \left(\frac{1}{630}\right)(0.8) = \frac{2}{1575} \qquad 46$$

$$k_\lambda = \left(\frac{1}{630}\right)\left(\frac{1}{630}\right) = \frac{1}{396900} \qquad 47$$

Substituting equations (44) to (47) into equation (26) gives

$$\lambda^2 = \left(\frac{\frac{2}{1575} + \frac{8}{11025\,P^2} + \frac{2}{1575\,P^4}}{\frac{1}{396900}}\right) * \frac{D}{ma^4}. \ That\ is$$

$$\lambda^2 = \left(504 + \frac{288}{P^2} + \frac{504}{P^4}\right) * \frac{D}{ma^4} \qquad 48$$

RESULTS AND CONCLUSIONS

The non dimensionalform of the resonating frequencies for different aspect ratios for ssss and cccc plates are shown on tables 1 and 2. A close critical examination of the tables reveals that the maximum percentage difference between the values from the present study and those from previous study is 0.09. From statistical point of view, this implies that no difference existed between the two sets of values. Thus, one can infer that the procedure, the deflection function and the energy functional formulated in this present study are reliable and adequate in CPT free vibration analysis rectangular plates. Hence, this method is recommended for stability analysis of CPT plates. It is also recommended that the present method is extended to refined plate theory analysis (RPT).

Table 1: Non dimensional form of resonating frequency of ssss isotropic thin plate

Aspect ratio, P	Resonating frequency, $\lambda\left(\frac{1}{a^2}\sqrt{\frac{D}{m}}\right)$		Percentage difference
	Present	Past (Ibearugbulem et al., 2014)	

1	19.74	19.75	-0.05
1.1	18.03	18.04	-0.06
1.2	16.73	16.73	0.00
1.3	15.71	15.72	-0.06
1.4	14.91	14.91	0.00
1.5	14.26	14.26	0.00
1.6	13.73	13.73	0.00
1.7	13.29	13.29	0.00
1.8	12.92	12.92	0.00
1.9	12.61	12.61	0.00
2	12.34	12.34	0.00

Table 2: Non dimensional form of resonating frequency of cccc isotropic thin plate

Aspect ratio, P	Resonating frequency, $\lambda\left(\frac{1}{a^2}\sqrt{\frac{D}{m}}\right)$		Percentage difference
	Present	Past (Ibearugbulem et al., 2014)	
1	36	35.97	0.08
1.1	32.96	32.93	0.09
1.2	30.77	30.75	0.07
1.3	29.17	29.15	0.07
1.4	27.97	27.95	0.07
1.5	27.05	27.03	0.07
1.6	26.33	26.31	0.08
1.7	25.77	25.75	0.08
1.8	25.32	25.30	0.08
1.9	24.95	24.94	0.04
2	24.65	24.64	0.04

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