ON TERNARY QUADRATIC DIOPHANTINE EQUATION:

$3x^2 + 2y^2 = 20z^2$

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ABSTRACT:

The Ternary Quadratic Diophantine Equation given by $3x^2 + 2y^2 = 20z^2$ is analysed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

KEY WORDS: Ternary Quadratic, Integral solutions

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INTRODUTION:

The Ternary Quadratic Diophantine Equation offers an unlimited field for research because of their variety [Andreweil,1983;Dickson,1952;Mordell,1969;Nigel,1999;Smith,1953], For an extensive review of various problems, one may refer[Gopalan etal 2000,$2005_{a,b}$,2006, $2007_{a,b,c}$, $2008_{a,b}$,2011, $2012_{a,b}$,2013,2014].Thiscommunication concerns with yet another interesting Ternary Quadratic Diophantine Equation $3x^2 + 2y^2 = 20z^2$ for determining its infinitely many non-zero integral solutions. Also a few interesting relations among the solutions have been presented.

NOTATIONS USED:

$T_{m,n}$ = Polygonal Number of rank n with sides m

p_n^m = Pyramidal number of rank n with sides m

j_n = Jacobsthal-Lucas number of rank n

J_n = Jacobsthal number of rank n

Method of analysis:

The ternary quadratic equation to be solved for it is non-zero solution is

$$3x^2 + 2y^2 = 20z^2 \quad \text{--------------------->} \ (1)$$

We present below different patterns of solutions to (1)

Pattern: I

Introducing the linear transformations

$$\left.\begin{aligned} x &= X + 2T \\ y &= X - 3T \\ z &= 5W \end{aligned}\right\} \quad \text{--------------------->} \ (2)$$

in (1),we have

$$X^2 + 6T^2 = 100W^2 \quad \text{--------------------->} (3)$$

Assume $$W = W(a,b) = a^2 + 6b^2 \quad \text{---------------->} (4)$$

where a, b>0

Write 100 as $100 = (2 + i4\sqrt{6})(2 - i4\sqrt{6}) \quad \text{--------------------->} (5)$

Substituting $(4), (5)$ in (3) and employing the method of factorizations, define

$$(X + i\sqrt{6}T) = (a + i\sqrt{6}b)^2 (2 + i4\sqrt{6})$$

Equating the real and imaginary parts, we have

$$X = 2(a^2 - 6b^2 - 24ab)$$

$$T = 4(a^2 - 6b^2 + ab)$$

Substituting the above values of X, T and (4) in (2), the corresponding non-zero distinct integer solutions to(1) are

$$x = x(a,b) = 10a^2 - 60b^2 - 40ab$$

$$y = y(a,b) = -10a^2 + 60b^2 - 60ab$$

$$z = z(a,b) = 5a^2 + 30b^2$$

Properties:

(i) $x(a,1) - t_{14,a} - t_{10,a} \equiv -28(mod\ 32)$

(ii) $x(a,1) + z(a,1) - t_{22,a} - t_{12,a} \equiv 0(mod\ 3)$

(iii) $x(2^n,1) = 10[(j_{2n} - 1) - 4(j_n - (-1)^n) - 6]$

(iv) $z(2^n,1) = 5[(j_n - (-1)^n) + 6]$

Note:

Instead of (5), if we write 100 as

$$100 = (-2 + i4\sqrt{6})(-2 - i4\sqrt{6}) \quad \text{------------------------>} (6)$$

then , the corresponding integer solutions to (1) are obtained as

$$x = x(a,b) = 6a^2 - 36b^2 - 56ab$$

$$y = y(a,b) = -14a^2 + 84b^2 - 36ab$$

$$z = z(a,b) = 5a^2 + 30b^2$$

Properties:

(i) $x(a,1) - y(a,1) - 40t_{3,a-1} + 120 = 0$

(ii) $x(2^n,1) = 6(j_{2n} - 1) - 56(j_n - (-1)^n) - 36$

(iii) $y(2^n,1) = -14(j_{2n} - 1) - 2(18\ 2^n - 42)$

(iv) $x(a,1) + z(a,1) - t_{22,a} \equiv 0(mod\ 6)$

Pattern: II

Replace x by 2X in (1) ,it is written as

$$6X^2 + y^2 = 10z^2 \quad \text{-------------------------> } (7)$$

Write 10 as $10 = (2 + i\sqrt{6})(2 - i\sqrt{6})$ -------------------------> (8)

Assume $z = z(a,b) = a^2 + 6b^2$ -------------------------> (9)

where a , b > 0

Substituting (8), (9) in (7) and employing the method of factorizations, define

$$(y + i\sqrt{6}X) = (2 + i\sqrt{6})\left(a + ib\sqrt{6}\right)^2$$

Equating the real and imaginary parts , we get

$$X = a^2 - 6b^2 + 4ab$$

$$y = 2a^2 - 12b^2 - 12ab$$

The corresponding integer solution to (7) are

$x = x(a,b) = 2X = 2a^2 - 12b^2 + 8ab$

$y = y(a,b) = 2a^2 - 12b^2 - 12ab$

$z = z(a,b) = a^2 + 6b^2$

Properties:

(i) $x(a,1) - t_{6,a} \equiv 0 (mod\ 3)$

(ii) $y(a,1) - t_{6,a} \equiv -1 (mod\ 11)$

(iii) $x(2a,1) - t_{18,a} \equiv 11 (mod\ 23)$

(iv) $x(2a,1) + z(a,1) - t_{20,a} \equiv 0 (mod\ 6)$

(v) $y(a,1) + z(a,1) - t_{8,a} \equiv 0 (mod\ 2)$

Pattern: III

(7) is written as

$$y^2 = 10z^2 - 6X^2 \quad \text{---------------------> (10)}$$

Introduing the linear transformations

$$\left.\begin{array}{r} X = \alpha + 10T \\ z = \alpha + 6T \end{array}\right\} \quad \text{------------------> (11)}$$

in (1),we have

$$(2\alpha)^2 = 240T^2 + y^2$$

which is satisfied by

T=4rS --> (12)

$\alpha = 120r^2 + 2S^2$ --------------------------------------> (13)

$y = 240r^2 - 4S^2$

Substituting α, T in (11), we have

$$X = 120r^2 + 2S^2 + 40rS$$

$$x = 2X = x(r,S) = 240rS^2 + 4S^2 + 80rS \text{ ------------> (14)}$$

$$z = z(r,S) = 120r^2 + 2S^2 + 24rS \quad \text{--------------->} \ (15)$$

Thus, (12), (13) & (14),(15) represent the corresponding integer solutions to (1).

$$x = 2X = x(r,S) = 240rS^2 + 4S^2 + 80rS$$

$$y = y(r,S) = 240r^2 - S^2$$

$$z = z(r,S) = 120r^2 + 2S^2 + 24rS$$

Properties:

(i) $x(1,S) - t_{10,s} \equiv 74(mod\ 83)$

(ii) $x(1,S) + z(1,S) - t_{10,S} - t_{6,S} \equiv 0(mod\ 6)$

(iii) $x(r,1) - t_{202,r} - t_{282,r} \equiv 0(mod\ 2)$

(iv) $z(r,1) - t_{102,r} - t_{142,r} \equiv 0(mod\ 2)$

(v) $6[x(r,r)]$ is nasty number

Note:

Instead of (11), if we consider

$$\left.\begin{matrix} X = \alpha - 10T \\ z = \alpha - 6T \end{matrix}\right\} \quad \text{------------->} \ (16)$$

then , the corresponding integer solutions to (1) are given by

$$x(r,S) = 240r^2 + 4S^2 - 80rS$$

$$y(r,S) = 240r^2 - 4S^2$$

$$z(r,S) = 120r^2 + 2S^2 - 24rS$$

Properties

(i) $x(r,2) - t_{442,r} - t_{42,r} \equiv 0(mod\ 2)$

(ii) $x(1,S) - z(1,S) - t_{6,S} \equiv 0(mod\ 5)$

(iii)$z(1,S)-t_{6,S}\equiv 5(mod\ 23)$

(iv)$x(r,1)-y(r,1)+z(r,1)-t_{112,r}-t_{132,r}\equiv 0(mod\ 2)$

Remarkable observations:

I. If the non-zero integer triple (x_0,y_0,z_0) is any solution of (1), then each of the following two triples also satisfies (1)

Triple: 1

$$(x_n,y_n,z_n)$$

where

$$x_n=3^n x_0$$

$$y_n=\frac{1}{6}\{3(3)^n(20-18(-1)^n y_0)+180(3)^n((-1)+(-1)^n z_0)\}$$

$$z_n=\frac{1}{6}\{18(3)^n(1-(-1)^n y_0)+3(3)^n(-18+20(-1)^n z_0)\}$$

Triple 2:

$$(x_n,y_n,z_n)$$

where

$$x_n=\frac{1}{4}\{4(2)^n(16-15(-1)^n x_0)+160(2)^n((-1)+(-1)^n z_0)\}$$

$$y_n=2^n y_0$$

$$Z_n=\frac{1}{4}\{24(2)^n(1-(-1)^n x_0)+4(2)^n(-15+16(-1)^n z_0)\}$$

II. Employing the solutions (x, y, z) of (1) each of following expressions among the special polygonal & pyramidal number is a nasty number

1. $30\left\{3\left[\frac{p_x^5}{t_{3,x}}\right]^2+2\left[\frac{3p_{y-2}^2}{t_{3,y-2}}\right]^2\right\}$

2. $30\left\{3\left[\frac{p_x^5}{t_{3,x}}\right]^2+2\left[\frac{3p_y^3}{t_{3,y+1}}\right]^2\right\}$

3. $30\left\{3\left[\frac{p_x^5}{t_{3,x}}\right]^2+2\left[\frac{2p_{y-1}^5}{t_{4,y-1}}\right]^2\right\}$

Conclusion:

To conclude, one may search for other patterns of solutions and their corresponding properties.

Reference:

Andre Weil, Number theory: An approach through history: from hammurapi to legendre/ Andre Weil: Boston (Birkahasuser boston),1983

L.E.Dickson, History of Theory of numbers, Vol.2, Chelsea Publishing Company, New York,1952.

L.J. Mordell, Diophantine Equations, Academic press, London,1969.

Nigel P. Smart, The algorithmic Resolutions of Diophantine equations, Cambridge university Press,1999.

Smith D.E History of mathematics vol.I and II, Dover publications, New York 1953.

M.A. Gopalan, Note on the Diophantine equation $x^2+axy+by^2=z^2$ Acta Ciencia Indica, Vol.XXVIM, No:2,2000,105-106.

M.A. Gopalan and R.Anbuselvi, On ternary Quadratic Homogeneous Diophantine Equation $x^2+Pxy+y^2=z^2$, Bulletin of Pure and Applied Sciences Vol 24E, No:2, (2005_a), 405- 408.

M.A.Gopalan, S. Vidhyalakshmi and A. Krishnamoorthy, Integral solutions Ternary Quadratic $ax^2+by^2=c(a+b)z^2$, Bulletin of Pure and Applied Sciences , Vol.24E, No: 2,(2005_b) 443-446.

M.A. Gopalan, S. Vidhyalakshmi and S. Devibala, Integral solutions of

$ka(x^2+y^2)+bxy=4k\alpha^2 z^2$, Bulletin of Pure and Applied Sciences,

Vol.25E, No:2, (2006), 401-406.

M.A. Gopalan, S. Vidhyalakshmi and S.Devibala, Intergral solutions of

$7x^2+8y^2=9z^2$, Pure and Applied Mathematika sciences,

Vol.LXVI, No: 1-2, 2007_a,, 83-86.

M.A Gopalan and S.Vidhyalakshmi, An observation on $kax^2+by^2=cz^2$,

Acta Cienica Indica Vol.XXXIIIM, No:1, 2007_b,, 97-99.

M.A.Gopalan, Manju somanath and A.Vanitha, Integral solutions of

$kxy+m(x+y)=z^2$, Acta Cienica Indica Vol.XXXIIIM, No:4, 2007_c,, 1287-1290.

M.A.Gopalan, and J.Kaliga Rani, Observation on the Diophantine Equation

$y^2=Dx^2+z^2$, Impact J.Sci. Tech, Vol(2), No:2, 2008_a,, 91-95.

M.A.Gopalan, and V.Pondichelvi, On Ternary Quadratic Equation

$x^2+y^2=z^2+1$, Impact J.Sci. Tech, Vol(2), No:2, 2008_b,8, 55-58.

M.A.Gopalan, and V.Pondichelvi, Integral solution of ternary quadratic equation

$z(x-y)=4xy$, Impact J.Sci. Tech, Vol(5), No:1, 01.06.2011

M.A.Gopalan,Manjusomanath, G.Sangeetha, Observations on ternary quadratic equation

$y^2=3x^2+z^2$, Bessel J.Math 2012_a,2(2) ,101-105.

M.A.Gopalan, and R.Vijayalakshmi., On the ternary quadratic equation

$x^2=(\alpha^2-1)(y^2-z^2), \alpha>1$, Bessel J.Math 2012_b,,2(2) ,147-151.

M.A.Gopalan, S. Vidhyalakshmi and K.Lakshmi "On the Non Homogeneous

Quadratic Equation $x^2+y^2+z^2=t^2-1$; International Journal of Applied Mathematics

Science , Vol 6(1),2013,Page No: 1-6.

k. Meena, S. Vidhyalakshmi, M.A.Gopalan , S.Nivetha(2014); "On the Ternary Quadratic Equation $29x^2 + y^2 = z^2$; International Journal of Engineering Research Online, Vol .2 , Issue 1,2014,Page No: 67-71.