ON THE BINARY QUADRATIC DIOPHANTINE EQUATION

$$x^2 - 3xy + y^2 + 10x = 0$$

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Abstract:

The binary quadratic equation $x^2 - 3xy + y^2 + 10x = 0$ represents a hyperbola. In this paper we obtain a sequence of its integral solutions and present a few interesting relations among them.

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INTRODUCTION

The binary quadratic Diophantine equations (both homogeneous and non homogeneous) are rich in variety [1 – 6]. In [7 – 16] the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. These results have motivated us to search for infinitely many non-zero integral solutions of an another interesting binary quadratic equation given by $x^2 - 3xy + y^2 + 10x = 0$. The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited.

METHOD OF ANALYSIS:

The Diophantine equation representing the binary quadratic equation to be solved for its non-zero distinct integral solution is

$$x^2 - 3xy + y^2 + 10x = 0 \tag{1}$$

Note that (1) is satisfied by the following non-zero distinct integer pairs

(-2,2), (10,10), (40,20), (-90,-30), (-32,-8).

However, we have other patterns of solutions for (1), which are illustrated below:

PATTERN :1

Solving (1) for x , we've

$$x = \frac{3y - 10 \pm \sqrt{5y^2 - 60y + 100}}{2} \tag{2}$$

Let $5y^2 - 60y + 100 = \alpha^2$

which is written as , $(5y - 30)^2 = 5\alpha^2 + 400$

$$\Rightarrow Y^2 = 5\alpha^2 + 20^2 \tag{3}$$

where $Y = 5y - 30$ (4)

The least positive integer solution of (3) is

$$\alpha_0 = 10, Y_0 = 30$$

Now to find the other solution of (3), consider the pellian equation

$$Y^2 = 5\alpha^2 + 1 \quad (5)$$

whose fundamental solution is $(\tilde{\alpha}_0, \tilde{Y}_0) = (4,9)$.

The other solutions of (5) can be derived from the relations

$$\tilde{Y}_n = \frac{f_n}{2} \qquad \alpha_n = \frac{g_n}{2\sqrt{5}}$$

where

$$f_n = [(9+4\sqrt{5})^{n+1} + (9-4\sqrt{5})^{n+1}]$$

$$g_n = [(9+4\sqrt{5})^{n+1} - (9-4\sqrt{5})^{n+1}]$$, n = -1,0,1,2…..

Applying the lemma of Brahmagupta between (α_0, Y_0) and $(\tilde{\alpha}_n, \tilde{Y}_n)$

the other solutions of (3) can be obtained from the relations

$$\alpha_{n+1} = 5f_n + \frac{15g_n}{\sqrt{5}}$$

$$Y_{n+1} = 15f_n + \frac{25g_n}{\sqrt{5}} \quad (6)$$

Taking positive sign on the R.H.S of (2) and using (4) and (6) the non-zero distinct integer solution of the hyperbola (1) are obtained as follows ,

$$x_{n+1} = 7f_n + 3\sqrt{5}g_n + 4 \qquad (7)$$

$$y_{n+1} = 3f_n + \sqrt{5}g_n + 6 \qquad (8)$$

Some numerical examples are presented as below ,

n	x_{n+1}	y_{n+1}
-1	18	12
0	250	100
1	4418	1692
2	79210	30260
3	1421298	542892
4	25504090	9741700

The recurrence relations satisfied by x_{n+1} , y_{n+1} are respectively

$$x_{n+3} - 18x_{n+2} + x_{n+1} = -64$$

$$y_{n+3} - 18y_{n+2} + y_{n+1} = -96$$

PROPERTIES :

- $x_{n+1} + x_{n+2} \equiv 0 (\bmod 389)$
- $x_{n+3} + y_{n+2} + x_{n+1} \equiv 0 (\bmod 317)$
- $x_{n+1} + y_{n+3} + x_{n+2} \equiv 0 (\bmod 16)$
- $y_{n+3} + x_{n+3} \equiv 0 (\bmod 30)$

Note : Taking negative sign on the R.H.S of (2) , the corresponding values of x are given by

$$x_{n+1} = 4 f_n + 8.$$

PATTERN :2

Solving (1) for y , we get

$$y = \frac{3x \pm \sqrt{9x^2 - 4(x^2 + 10x)}}{2} \quad (9)$$

Replacing x by 2X in the above equation, we have (10)

$$y = 3X \pm \sqrt{5X^2 - 20X} \quad (11)$$

$$Let \;\; 5X^2 - 20X = \beta^2 \quad (12)$$

which is be written as

$$(5X - 10)^2 = 5\beta^2 + 10 \quad (13)$$

and (11) becomes $y = 3X \pm \beta$ (14)

$$\Rightarrow S^2 = 5\beta^2 + 10^2 \quad (15)$$

$$where\; S = 5X - 10 \quad (15a)$$

Now, consider the pellian equation of (15)

$$S^2 = 5\beta^2 + 1 \quad (16)$$

whose least positive integer solutions are $\tilde{\beta}_0 = 4, \tilde{S}_0 = 9$

The general solution $(\tilde{S}_n, \tilde{\beta}_n)$ of (16) is given by ,

$$\tilde{S}_n = \frac{1}{2}\left[(9+4\sqrt{5})^{n+1} + (9-4\sqrt{5})^{n+1}\right] \quad (17)$$

$$\tilde{\beta}_n = \frac{1}{2\sqrt{5}}\left[(9+4\sqrt{5})^{n+1} - (9-4\sqrt{5})^{n+1}\right] \quad (18)$$

where n=0,1,2…..

Thus the general solutions of (15) are obtained by

$$S_n = 10\tilde{S}_n = \frac{10}{2}\left[(9+4\sqrt{5})^{n+1} + (9-4\sqrt{5})^{n+1}\right] \quad (19)$$

$$\beta_n = 10\tilde{\beta}_n = \frac{10}{2\sqrt{5}}\left[(9+4\sqrt{5})^{n+1} - (9-4\sqrt{5})^{n+1}\right] \quad (20)$$

where n=0,1,2…..

From (10) , (15a) and (17) we've

$$x_n = 4\left[\frac{f}{2}+1\right] \quad (21)$$

where,

$$f = [(9+4\sqrt{5})^{n+1} + (9-4\sqrt{5})^{n+1}]$$

From (10) ,(14) and (18) we've

$$y_n = 6\left[\frac{f}{2}+1\right] + 10\left[\frac{g}{2\sqrt{5}}\right]$$, (by taking the positive sign of (11)) (22)

where

$$g = [(9+4\sqrt{5})^{n+1} - (9-4\sqrt{5})^{n+1}]$$

Our aim is to get integer solution to (1) which is obtained for n=0,2,4….

in (21) and (22)

$$x_{2n} = 4\left[\frac{F}{2}+1\right] \quad (23)$$

$$y_{2n} = 6\left[\frac{F}{2}+1\right]+10\left[\frac{G}{2\sqrt{5}}\right] \quad (24)$$

Where

$$F = [(9+4\sqrt{5})^{n+1} + (9-4\sqrt{5})^{n+1}] \quad (25)$$

$$G = [(9+4\sqrt{5})^{n+1} - (9-4\sqrt{5})^{n+1}] \quad (26)$$

Equations (23) and (24) together will give the distinct integral

solutions of (1).

$$x_{2n+2} = 4\left[\frac{9F+4\sqrt{5}G}{2}+1\right] \quad (27)$$

$$x_{2n+4} = 4\left[\frac{2889F+1292\sqrt{5}G}{2}+1\right] \quad (28)$$

$$y_{2n+2} = 6\left[\frac{9F+4\sqrt{5}G}{2}+1\right]+10\left[\frac{9G+4\sqrt{5}F}{2\sqrt{5}}\right] \quad (29)$$

$$y_{2n+4} = 6\left[\frac{2889F+1292\sqrt{5}G}{2}+1\right]+10\left[\frac{2889G+1292\sqrt{5}F}{2\sqrt{5}}\right] \quad (30)$$

The above values of x_{2n} and y_{2n} satisfy the following recurrence relations.

$$x_{2n+4} - 323x_{2n+2} + 18x_{2n} = -1216 \quad (31)$$

$$y_{2n+4} - 323y_{2n+2} + 18y_{2n} = -1824 \quad (32)$$

We give some the numerical values for n =0,1,2…. in x_{2n} and y_{2n}

n	x_{2n}	y_{2n}
-1	8	12
0	40	100
1	11560	30260
2	3721000	9741700
3	1201673704	31462022596

PROPERTIES :

$\rightarrow y_{2n} - x_{2n} \equiv 0(\mathrm{mod}\,24)$

$\rightarrow y_n - x_n \equiv 0(\mathrm{mod}\,24)$

$\rightarrow x_{2n+2} + y_{2n+2} \equiv 0(\mathrm{mod}\,20)$

$\rightarrow x_{2n+2} + y_{2n+4} \equiv 0(\mathrm{mod}\,30)$

$\rightarrow x_{2n+2} + y_{2n+2} + y_{2n+4} = 0(\mathrm{mod}\,12)$

CONCLUSION:

In this paper, we have made an attempt to obtain a complete set of non-trivial distinct solutions for the non-homogeneous binary quadratic equation. To conclude , one may search for other choices of solutions to the considered binary equation and further , quadratic equations with multi-variables.

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