SOME NEW FAMILIES OF STRONGLY PRIME GRAPHS

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ABSTRACT:

A graph $G=(V,E)$ with $n$ vertices is said to admit prime labeling if its vertices can be labeled with distinct positive integers not exceeding $n$ such that the label of each pair of adjacent vertices are relatively prime. A graph $G$ which admits prime labeling is called a prime graph and a graph $G$ is said to be a strongly prime graph if for any vertex $v$ of $G$ there exists a prime labeling $f$ satisfying $f(v) = 1$. In the present work we investigate some classes of graphs and subdivision of some classes of graphs which admit strongly prime labeling.

Keywords: Graph labeling, prime labeling, prime graph, strongly prime graph.

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1 Introduction:

In this paper, we consider only simple, finite, undirected and non-trivial graph $G = (V(G), E(G))$ with vertex set $V(G)$ and edge set $E(G)$. The set of vertices adjacent to a vertex $u$ of $G$ is denoted by $N(u)$. For notations and terminology we refer to Bondy and Murthy [1].

Two integers $a$ and $b$ are said to be relatively prime if their greatest common divisor is $1$. Relatively prime numbers play an important role in both analytic and algebraic number theory. The notion of a prime labeling was introduced by Roger Etringer and was discussed in a paper by Tout.A [8].

Many researchers have studied prime graph. For example Fu.H [3] have proved that path $P_n$ on $n$ vertices is a prime graph. Deresky.T [2] have proved that the cycle $C_n$ with $n$ vertices is a prime graph. Lee.S [5] have proved that wheel $W_n$ is a prime graph iff $n$ is even. Around 1980 Roger Etringer conjectured that all trees having prime labeling which is not settled till today. In [6] S.Meena and K.Vaithiligam have investigated the Prime labeling for some helm related graph.

In [9] S.K.Vaidya and Udayan M.Prajapati have introduced Strongly prime graph and has proved $C_n$, $P_n$, and $K_{1,n}$ are strongly prime graphs and $W_n$ is a strongly prime graph for every even integer $n \geq 4$, in Some new results on prime graph. In [7] Sharon Philomena. V and K. Thirusangu have investigated Square and cube difference labeling of cycle cactus, special tree and a new key graphs. Graph labeling can also be applied in areas such as communication network, mobile telecommunications and medical field. Latest Dynamic survey on graph labeling we refer to Gallian [4].

A vast amount of literature is available on different types of graph labeling. More than 1000 research papers have been published so far in last four decades. We give a brief summary of definitions which are useful for this paper.

**Definition 1.1:** If the vertices of the graph are assigned values subject to certain conditions then it is known as graph labeling.

**Definition 1.2.** Let $G = (V(G), E(G))$ be a graph with $p$ vertices. A bijection $f : V(G) \rightarrow [1, 2, \ldots, p]$ is called a prime labeling if for each edge $e = uv$, $\gcd\{f(u), f(v)\} = 1$. A graph which admits prime labeling is called a prime graph.
Definition 1.3. A graph G is said to be a strongly prime graph if for any vertex v of G there exists a prime labeling f satisfying f(v) = 1.

Definition 1.4. A key graph is a graph obtained from K2 by appending one vertex of C5 to one end point and Hoffman tree Pn ∥ K1 to the other end point of K2.

Definition 1.5. Let e = uv be an edge of a graph G and w is not a vertex of G. Then edge e is said to be subdivided when it is replaced by edges e' = uw and e'' = wv.

Definition 1.6. If every edge of graph G is subdivided, then the resulting graph is called barycentric subdivision of graph G.

Definition 1.7. The Crown graph Cn* is obtained from a cycle Cn by attaching a pendent edge at each vertex of the n-cycle.

Definition 1.8. The graph Pn ∥ K1 is called a comb Cbn.

2. Strongly Prime Labeling Of Some Graphs

Theorem 2.1:
The key graph C5 ∥ Pn is a strongly prime graph for all integer n ≥ 1.

Proof:
Let G be the key graph C5 ∥ Pn with vertex set V(G) = {w1, w2, w3, v1, v2, ..., vn, u1, u2, ..., un} and the edge set
E(G) = \{(wi, wi+1) / 1 ≤ i ≤ 4\} ∪ \{(v1, vi) / 1 ≤ i ≤ n-1\} ∪ \{(vi, u1) / 1 ≤ i ≤ n\} ∪ \{w1, w5\} ∪ \{w1v1\}. Here |V(G)| = 2n + 5.

Let a be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

Case (i): When a is any arbitrary vertex of C5.

Let a = wj for some j ∈ {1, 2, ..., 5} then the function f : V(G) → {1, 2, ..., 2n+5} defined by

f(wi) = \begin{cases} 
5 + i - j + 1 & \text{if } i = 1, 2, ..., j - 1; \\
i - j + 1 & \text{if } i = j, j + 1, ..., 5;
\end{cases}

f(vi) = 5 + 2i & \text{if } i = 1, 2, ..., n;
\( f(u_i) = 5 + 2i - 1 \) if \( i = 1, 2, ..., n; \)

is a prime labeling for \( C_5 \diamond P_n \) with \( f(a) = f(w_j) = 1. \)

Thus \( f \) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of \( a = w_j \) in \( C_5 \diamond P_n \).

**Case (ii):** When \( a \) is any arbitrary vertex of Hoffman graph.

**Subcase (i):**

Let \( a = v_j \) for some \( j \in \{1, 2, ..., n\} \) then the function \( f : V(G) \rightarrow \{1, 2, ..., 2n+5\} \) defined by

\[
\begin{align*}
  f(w_i) &= i + 2 \quad \text{if } i = 3, 4, 5; \\
  f(w_1) &= 4, f(w_2) = 3; \\
  f(v_j) &= \begin{cases} 
  2(n+i-j)+7 & \text{if } i = 1, 2, ..., j-1; \\
  2(i-j)+7 & \text{if } i = j+1, j+2, ..., n; 
\end{cases} \\
  f(v_j) &= 1; \\
  f(u_i) &= \begin{cases} 
  2(n+i-j)+6 & \text{if } i = 1, 2, ..., j-1; \\
  2(i-j)+6 & \text{if } i = j+1, j+2, ..., n; 
\end{cases} \\
  f(u_j) &= 2;
\end{align*}
\]

is a prime labeling for \( C_5 \diamond P_n \) with \( f(a) = f(v_j) = 1. \) Thus \( f \) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of \( a = v_j \) in \( C_5 \diamond P_n \).

**Subcase (ii):**

Let \( a = u_j \) for some \( j \in \{1, 2, ..., n\} \) then define a labeling \( f_2 \) using the labeling \( f \) defined in subcase (i) as follows: \( f_2(u_j) = f(v_j), f_2(v_j) = f(u_j) \) for \( j \in \{1, 2, ..., n\} \) and \( f_2(v) = f(v) \) for all the remaining vertices. Then the resulting labeling \( f_2 \) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex \( a = u_j \) in \( C_5 \diamond P_n \) graph. Thus from all the cases described above \( G \) is a strongly prime graph.

![Figure 1. A prime labeling of \( C_5 \diamond P_n \) having \( w_4 \) as label 1](image-url)
Theorem 2.2:

The graph $G$ obtained by attaching $K_{1,3}$ at each vertex of a cycle $C_n$ is a strongly prime graph for all integers $n \geq 3$.

Proof:

Let $C_n$ be the cycle $u_1, u_2, ..., u_n, u_1$. Let $v_i, x_i, y_i$ be the vertices of $i^{th}$ copy of $K_{1,3}$ in which $v_i$ is the central vertex. Identify $z_i$ with $u_i, 1 \leq i \leq n$. Let the resultant graph be $G$. Now the vertex set $V(G) = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n, x_1, x_2, ..., x_n, y_1, y_2, ..., y_n\}$ and the edge set $E(G) = \{u_iu_{i+1} / 1 \leq i \leq n-1 \} \cup \{u_iu_i\} \cup \{u_iv_i, x_iv_i, y_iv_i / 1 \leq i \leq n\}$, here $|V(G)| = 4n$.

Let $a$ be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

Case (i):

Let $a = u_j$ for some $j \in \{1, 2, ..., n\}$ then the function $f : V(G) \rightarrow \{1, 2, ..., 4n\}$ defined by

$$f(u_i) = \begin{cases} 4n + 4i - 4j + 1 & \text{if } i = 1, 2, ..., j - 1; \\ 4i - 4j + 1 & \text{if } i = j, j + 1, ..., n; \end{cases}$$

$$f(v_i) = \begin{cases} 4n + 4i - 4j + 3 & \text{if } i = 1, 2, ..., j - 1; \\ 4i - 4j + 3 & \text{if } i = j, j + 1, ..., n; \end{cases}$$

$$f(x_i) = \begin{cases} 4n + 4i - 4j + 3 & \text{if } i = 1, 2, ..., j - 1; \\ 4i - 4j + 2 & \text{if } i = j, j + 1, ..., n; \end{cases}$$

$$f(y_i) = \begin{cases} 4n + 4i - 4j + 4 & \text{if } i = 1, 2, ..., j - 1; \\ 4i - 4j + 4 & \text{if } i = j, j + 1, ..., n; \end{cases}$$

is a prime labeling for $G$ with $f(a) = f(u_j) = 1$. Thus $f$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $a = u_j$ in $G$.

Case (ii):

Let $a = x_j$ for some $j \in \{1, 2, ..., n\}$ then define a labeling $f_2$ using the labeling $f$ defined in case (i) as follows: $f_2(u_j) = f(x_j), f_2(x_j) = f(u_j)$ for $j = 1, 2, ..., n$ and $f_2(v) = f(v)$ for all the remaining vertices. Then the resulting labeling $f_2$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = x_j$ in $G$.

Case (iii):

...
Let $a = y_j$ for some $j \in \{1, 2, ..., n\}$ then define a labeling $f_3$ using the labeling $f$ defined in case (i) as follows: $f_3(u_j) = f(y_j), f_3(y_j) = f(u_j)$ for $j = 1, 2, ..., n$ and $f_3(v) = f(v)$ for all the remaining vertices. Then the resulting labeling $f_3$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = y_j$ in $G$.

Case (iv):

Let $a = v_j$ for some $j \in \{1, 2, ..., n\}$ then define a labeling $f_4$ using the labeling $f$ defined in case (ii) as follows: $f_4(v_j) = f_2(x_j), f_4(x_j) = f_2(v_j)$ for $j = 1, 2, ..., n$ and $f_4(v) = f_2(v)$ for all the remaining vertices. Then the resulting labeling $f_4$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = v_j$ in $G$. Thus from all the cases described above $G$ is a strongly prime graph.

3. Strongly Prime Labeling Of Some Subdivision Of Graphs:

**Theorem 3.1:**

The graph $G$ obtained from subdividing the pendant edges of the crown graph $C_n^e$ is a strongly prime graph for all integers $n \geq 2$.

**Proof:**
Let $C_n^*$ be the crown graph with vertices $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. Let $w_1, w_2, \ldots, w_n$ be the corresponding new vertices which is subdividing the pendant edges of $C_n^*$. Then the resulting graph be $G$. Now the vertex set $V(G) = \{u_i, v_i, w_i \mid 1 \leq i \leq n\}$ and the edge set $E(G) = \{u_i w_i, w_i v_i \mid 1 \leq i \leq n\} \cup \{u_i u_{i+1} \mid 1 \leq i \leq n-1\} \cup \{u_n u_1\}$. Here $|V(G)| = 3n$.

Let $a$ be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

**Case (i):**

Let $a = u_j$ for some $j \in \{1, 2, \ldots, n\}$ then define the function $f : V(G) \rightarrow \{1, 2, \ldots, 3n\}$ defined by

$$
\begin{align*}
  f(u_i) &= \begin{cases} 
    3n + 3i - 3j + 1 & \text{if } i = 1, 2, \ldots, j-1; \\
    3i - 3j + 1 & \text{if } i = j, j+1, \ldots, n;
  \end{cases} \\
  f(v_i) &= \begin{cases} 
    3n + 3i - 3j + 3 & \text{if } i = 1, 2, \ldots, j-1; \\
    3i - 3j + 3 & \text{if } i = j, j+1, \ldots, n;
  \end{cases} \\
  f(w_i) &= \begin{cases} 
    3n + 3i - 3j + 2 & \text{if } i = 1, 2, \ldots, j-1; \\
    3i - 3j + 2 & \text{if } i = j, j+1, \ldots, n;
  \end{cases}
\end{align*}
$$

is a prime labeling for $G$ with $f(a) = f(u_j) = 1$. Thus $f$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $a = u_j$ in $G$.

**Case (ii):**

Let $a = v_j$ for some $j \in \{1, 2, \ldots, n\}$ then define a labeling $f_2$ using the labeling $f$ defined in case (i) as follows: $f_2(u_i) = f(v_i), f_2(v_j) = f(u_j)$ for $j = 1, 2, \ldots, n$ and $f_2(v) = f(v)$ for all the remaining vertices. Then the resulting labeling $f_2$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = v_j$ in $G$.

**Case (iii):**

Let $a = w_j$ for some $j \in \{1, 2, \ldots, n\}$ then define a labeling $f_3$ using the labeling $f_2$ defined in case (ii) as follows: $f_3(w_j) = f_2(v_j), f_3(v_j) = f_2(w_j)$ for $j = 1, 2, \ldots, n$ and $f_3(v) = f_2(v)$ for all the remaining vertices. Then the resulting labeling $f_3$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = w_j$ in $G$. Thus from all the cases described above $G$ is a strongly prime graph.
Theorem 3.2: 
The graph $G$ obtained from subdividing the edges of the path $P_n$ of the comb graph $C_{bn}$ is a strongly prime graph for all integers $n \geq 2$.

Proof: 
Let $C_{bn}$ be the comb graph with vertices $v_1, v_2, ..., v_n, u_1, u_2, ..., u_n$. Let $w_1, w_2, ..., w_{n-1}$ be the corresponding new vertices which subdivides the edges of the path $P_n$ in $C_{bn}$. Then the resulting graph be $G$. Now the vertex set $V(G) = \{u_i, v_i, w_i / 1 \leq i \leq n, 1 \leq k \leq n - 1\}$ and the edge set of $G$ is $E(G) = \{u_i, v_i, w_i, w_i u_{k+1} / 1 \leq i \leq n, 1 \leq k \leq n - 1\}$. Here $|V(G)| = 3n - 1$.

Let $a$ be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

Case (i): When $n$ is odd.

Subcase (i): 
Let $a = u_j$ for some $j \in \{1, 2, ..., n\}$ then the function $f : V(G) \rightarrow \{1, 2, ..., 3n - 1\}$ defined by

$$f(w_i) = \begin{cases} 3n + 3i - 3j + 2 & \text{if } i = 1, 2, ..., j - 1; \\ 3i - 3j + 3 & \text{if } i = j, j + 1, ..., n - 1; \end{cases}$$

$$f(u_i) = 3i - 3j + 2$$

$$f(v_i) = 3i - 3j + 1$$

if $i = \begin{cases} j + 1, j + 3, ..., n & \text{for } j \text{ is even}; \\ j + 1, j + 3, ..., n - 1 & \text{for } j \text{ is odd}; \end{cases}$
is a prime labeling for $G$ with $f(a) = f(u_j) = 1$. Thus $f$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $a = u_j$ in $G$.

Subcase (ii):

Let $a = w_j$ for some $j \in \{1, 2, \ldots, n-1\}$ then define a labeling $f_2$ using the labeling $f$ defined in subcase (i) of case (i) as follows: $f_2(u_j) = f(w_j), f_2(w_j) = f(u_j)$ for $j = 1, 2, \ldots, n-1$ and $f_2(v) = f(v)$ for all the remaining vertices. Then the resulting labeling $f_2$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = w_j$ in $G$.

Subcase (iii):

Let $a = v_j$ for some $j \in \{1, 2, \ldots, n\}$ then define a labeling $f_3$ using the labeling $f$ defined in subcase (ii) of case (i) as follows: $f_3(v_j) = f_2(w_j), f_3(w_j) = f_2(v_j)$ for $j = 1, 2, \ldots, n$ and $f_3(v) = f_2(v)$ for all the remaining vertices. Then the resulting labeling $f_3$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = v_j$ in $G$.

Case (ii): When $n$ is even.

Subcase (i):

Let $a = u_j$ for some $j \in \{1, 2, \ldots, n\}$ then the function $f: V(G) \rightarrow \{1, 2, \ldots, 3n-1\}$ defined by

$$f(u_j) = \begin{cases} 3i - 3j + 1 & \text{if } i = j, j + 2, \ldots, n - 1, \\ 3i - 3j + 2 & \text{if } i = j, j + 1, \ldots, n - 1, \end{cases}$$

$$f(v_j) = \begin{cases} 3i - 3j + 2 & \text{if } i = j, j + 2, \ldots, n - 1, \\ 3i - 3j + 3 & \text{if } i = j, j + 1, \ldots, n - 1, \end{cases}$$

where $j = 1, 2, \ldots, n$.
\[ f(u_i) = 3n + 3i - 3j \]
\[ f(v_j) = 3n + 3i - 3j + 1 \]
\[
\begin{align*}
&\text{if } i = \begin{cases} 
1,3,5,... & \text{for } j \text{ is even;}
1,3,5,... & \text{for } j \text{ is odd;}
\end{cases} \\
&\text{if } i = \begin{cases} 
2,4,... & \text{for } j \text{ is even;}
2,4,... & \text{for } j \text{ is odd;}
\end{cases}
\end{align*}
\]

is a prime labeling for \( G \) with \( f(a) = f(u_i) = 1 \). Thus \( f \) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of \( a = u_j \) in \( G \).

**Subcase (ii):**

Let \( a = w_j \) for some \( j \in \{1,2,...,n-1\} \) then define a labeling \( f_4 \) using the labeling \( f \) defined in sub case (i) of case (ii) as follows: \( f_4(u_j) = f(w_j), \ f_4(w_j) = f(v_j) \) for \( j=1,2,...,n-1 \) and \( f_4(v) = f(v) \) for all the remaining vertices. Then the resulting labeling \( f_4 \) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex \( a = w_j \) in \( G \).

**Subcase (ii):**

Let \( a = v_j \) for some \( j \in \{1,2,...,n\} \) then define a labeling \( f_5 \) using the labeling \( f_4 \) defined in subcase (ii) of case (ii) as follows: \( f_5(v_j) = f_4(w_j), \ f_5(w_j) = f_4(v_j) \) for \( j=1,2,...,n \) and \( f_5(v) = f_4(v) \) for all the remaining vertices. Then the resulting labeling \( f_5 \) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex \( a = v_j \) in \( G \). Thus from all the cases described above \( G \) is a strongly prime graph.

![Figure 4. A prime labeling of a graph \( G \) obtained from subdividing the edges of the path \( P_n \) of the comb graph \( C_{mn} \) having \( u_i \) as label 1 ( \( n, j \text{ is odd} \) )](image-url)
Figure 5. A prime labeling of a graph $G$ obtained from subdividing the edges of the path $P_n$ of the comb graph $C_{bn}$ having $u_k$ as label 1 (n is odd, j is even)

**Theorem 3.3:**

The graph $G$ obtained from subdividing the pendant edges of the comb graph $C_{bn}$ is a strongly prime graph for all integers $n \geq 2$.

**Proof:**

Let $C_{bn}$ be the comb graph with vertices $v_1, v_2, ..., v_n, u_1, u_2, ..., u_n$. Let $w_1, w_2, ..., w_n$ be the corresponding new vertices which is subdividing the pendant edges of $C_{bn}$. Then the resulting graph be $G$. Now the vertex set $V(G) = \{u_i, v_i, w_i \mid 1 \leq i \leq n\}$ and the edge set $E(G) = \{u_iw_i, w_iv_i \mid 1 \leq i \leq n\} \cup \{u_iu_{i+1} \mid 1 \leq i \leq n-1\}$. Here $|V(G)| = 3n$.

Let $a$ be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

**Case (i):**

Let $a = u_j$ for some $j \in \{1, 2, ..., n\}$ then the function $f : V(G) \rightarrow \{1, 2, ..., 3n\}$ defined by

$$f(u_i) = \begin{cases} 3n + 3i - 3j + 1 & \text{if } i = 1, 2, ..., j - 1; \\ 3i - 3j + 1 & \text{if } i = j, j + 1, ..., n; \end{cases}$$

$$f(v_i) = \begin{cases} 3n + 3i - 3j + 3 & \text{if } i = 1, 2, ..., j - 1; \\ 3i - 3j + 3 & \text{if } i = j, j + 1, ..., n; \end{cases}$$

$$f(w_i) = \begin{cases} 3n + 3i - 3j + 2 & \text{if } i = 1, 2, ..., j - 1; \\ 3i - 3j + 2 & \text{if } i = j, j + 1, ..., n; \end{cases}$$

is a prime labeling for $G$ with $f(a) = f(u_j) = 1$. Thus $f$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $a = u_j$ in $G$.

**Case (ii):**

Let $a = v_j$ for some $j \in \{1, 2, ..., n\}$ then define a labeling $f_2$ using the labeling $f$ defined in case (i) as follows: $f_2(u_j) = f(v_j), f_2(v_j) = f(u_j)$ for $j = 1, 2, ..., n$ and $f_2(v) = f(v)$ for all the remaining vertices. Then the resulting labeling $f_2$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = v_j$ in $G$. 
Case (iii):
Let $a = w_j$ for some $j \in \{1, 2, ..., n\}$ then define a labeling $f_3$ using the labeling $f_2$ defined in case (ii) as follows: $f_3(w_j) = f_2(v_j)$, $f_3(v_j) = f_2(w_j)$ for $j = 1, 2, ..., n$ and $f_3(v) = f_2(v)$ for all the remaining vertices. Then the resulting labeling $f_3$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex $a = w_j$ in $G$. Thus from all the cases described above $G$ is a strongly prime graph.

![Figure 6. A prime labeling of a graph $G$ obtained from subdividing the pendant edges of the comb graph $C_{4u}$ having $u_1$ as label 1](image)

**Theorem 3.4:**
The graph $G$ obtained from subdividing the edges of the cycle $C_5$ in the key graph $C_5 \square P_n$ is a strongly prime graph for all integers $n \geq 2$.

**Proof:**
Let $C_5 \square P_n$ be the key graph with vertices $w_1, w_2, ..., w_5, u_1, u_2, ..., u_n, v_1, v_2, ..., v_n$. Let $w_1, w_2, ..., w_5$ be the corresponding new vertices which is subdividing the edges of the cycle $C_5$ in key graph.

Then the resulting graph be $G$. Now the vertex set $V(G) = \{v_1, u_i / 1 \leq i \leq n, w_i, w_j / 1 \leq i \leq 5\}$ and the edge set $E(G) = \{v_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{w_i w_{i+1} / 1 \leq i \leq n-1\} \cup \{w_i v_1\}$

$\cup (w_i w_j / 1 \leq i \leq n) \cup (w_i w_{i+1} / 1 \leq i \leq n-1) \cup \{w_5 w_j\}$. Here $|V(G)| = 2n+10$.

Let $a$ be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

**Case (i):** When $a$ is any arbitrary vertex of a cycle.

**Subcase (i):**
Let $a = w_j$ for some $j \in \{1, 2, ..., 5\}$ then define the function $f : V(G) \rightarrow \{1, 2, ..., 2n+10\}$ as

$f(w_j) = \begin{cases} 10 + 2(i - j) + 1 & \text{if } i = 1, 2, ..., j - 1; \\ 2(i - j) + 1 & \text{if } i = j, j + 1, ..., 5; \end{cases}$
\[
\begin{align*}
    f(w_i) &= \begin{cases} 
        10 + 2(i - j) + 2 & \text{if } i = 1,2,...,j-1; \\
        2(i - j) + 2 & \text{if } i = j, j+1,...,5;
    \end{cases} \\
    f(v_i) &= 10 + 2i - 1 & \text{if } i = 1,2,...,n; \\
    f(u_i) &= 10 + 2i & \text{if } i = 1,2,...,n;
\end{align*}
\]

is a prime labeling for \( G \) with \( f(a) = f(w_j) = 1 \). Thus \( f \) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of \( a = w_j \) in \( G \).

**Subcase (ii):**

Let \( a = w_j \) for some \( j \in \{1,2,...,5\} \) then the function \( f_i \) using the labeling \( f \) defined in subcase (i) as follows:

\[
\begin{align*}
    f_i(w_j) &= \begin{cases} 
        10 + 2(i - j) & \text{if } i = 1,2,...,j; \\
        2(i - j) & \text{if } i = j+1, j+2,...,5;
    \end{cases} \\
    f_i(w_j') &= \begin{cases} 
        10 + 2(i - j) + 1 & \text{if } i = 1,2,...,j-1; \\
        2(i - j) + 1 & \text{if } i = j, j+1,...,5;
    \end{cases}
\end{align*}
\]

and \( f_i(v) = f(v) \) for all the remaining vertices. Then the resulting labeling \( f_i \) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex \( a = w_j \) in \( G \).

**Case (ii):** When \( a \) is any arbitrary vertex of path \( P_n \).

Let \( a = v_j \) for some \( j \in \{1,2,...,n\} \) then define the function \( f : V(G) \rightarrow \{1,2,...,2n+10\} \) as

\[
\begin{align*}
    f(w_i) &= 2i + 2 & \text{if } i = 1,2,...,5; \\
    f(w_i) &= 2i + 1 & \text{if } i = 2,3,...,5; \\
    f(w_i) &= 13; \\
    f(v_i) &= \begin{cases} 
        2(n+i - j + 5) + 1 & \text{if } i = 1,2,...,j-1; \\
        2(i - j + 5) + 1 & \text{if } i = j+3, j+4,...,n; 
    \end{cases} \\
    f(v_j) &= 1; \\
    f(v_{j+1}) &= 3; \\
    f(v_{j+2}) &= f(w_i) + 3; \\
    f(u_i) &= \begin{cases} 
        2(n+i - j + 5) + 2 & \text{if } i = 1,2,...,j-1; \\
        2(i - j + 5) + 2 & \text{if } i = j+3, j+4,...,n; 
    \end{cases}
\end{align*}
\]
\[ f(u_j) = 2; \]
\[ f(u_{j+1}) = f(w_i) + 1; \]
\[ f(u_{j+2}) = f(w_i) + 2; \]

is a prime labeling for \( G \) with \( f(a) = f(v_j) = 1 \). Thus \( f \) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of \( a = v_j \) in \( G \).

[In this case, if \( f(v_j) \) is a multiple of 13 then keep the above labeling \( f \) defined in case (ii) as same and change the labels \((f(w_i), f(w_i'))\) as \( f(w_i) = 2i + 2 \) if \( i = 1, 2, \ldots, 5; \]
\[ f(w_i') = 2i + 3 \] if \( i = 1, 2, \ldots, 5; \)]

Case (iii):
Let \( a = u_j \) for some \( j \in \{1, 2, \ldots, n\} \) then define a labeling \( f_2 \) using the labeling \( f \) defined in case (ii) as follows: \( f_2(u_j) = f(v_j), f_2(v_j) = f(u_j) \) for \( j \in \{1, 2, \ldots, n\} \) and \( f_2(v) = f(v) \) for all the remaining vertices. Then the resulting labeling \( f_2 \) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex \( a = u_j \) in \( G \). Thus from all the cases described above \( G \) is a strongly prime graph.

Figure 7. A prime labeling of a graph \( G \) obtained from subdividing the edges of the cycle \( C_5 \) in \( C_5 \square P_n \) having \( w_i \) as label 1

Theorem 3.5:
The graph \( G \) obtained from subdividing the edges of the cycle \( C_5 \) and pendant edges in the key graph \( C_5 \square P_n \) is a strongly prime graph for all integers \( n \geq 2 \).

Proof:
Let \( C_5 \square P_n \) be the key graph with vertices \( w_1, w_2, \ldots, w_5, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n \). Let \( w_1', w_2', \ldots, w_5' \) and \( u_1', u_2', \ldots, u_n' \) be the corresponding new vertices which is subdividing the edges of the cycle \( C_5 \) and pendant edges in \( C_5 \square P_n \). Then the resulting graph be \( G \). Now the vertex set
\[ V(G) = \{ u_i, v_i, u'_i / 1 \leq i \leq n, w_i, w'_i / 1 \leq i \leq 5 \} \quad \text{and} \quad \text{the edge set} \]
\[ E(G) = \{ v_i v_{i+1} / 1 \leq i \leq n - 1 \} \cup \{ w_i v_i \} \cup \{ w_i w'_i / 1 \leq i \leq n \} \cup \{ w'_i w_{i+1} / 1 \leq i \leq n - 1 \} \cup \{ w'_i w_i \} \cup \{ u_i u'_i, v_i v'_i / 1 \leq i \leq n \}. \]

Now \( |V(G)| = 3n + 10 \).

Let \( a \) be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

**Case (i):** When \( a \) is any arbitrary vertex of \( C_5 \).

**Subcase (i):**
Let \( a = w_j \) for some \( j \in \{1, 2, ..., 5\} \) then defined the function \( f : V(G) \rightarrow \{1, 2, ..., 3n + 10\} \) as

\[
\begin{align*}
  f(w_i) &= \begin{cases} 
  2(i - j + 5) + 1 & \text{if } i = 1, 2, ..., j - 1; \\
  2(i - j) + 1 & \text{if } i = j, j + 1, ..., 5;
  \end{cases} \\
  f(w'_i) &= \begin{cases} 
  2(i - j + 5) + 2 & \text{if } i = 1, 2, ..., j - 1; \\
  2(i - j) + 2 & \text{if } i = j, j + 1, ..., 5;
  \end{cases} \\
  f(v_i) &= 10 + 3i \quad \text{if } i = 1, 2, ..., n; \\
  f(u_i) &= 10 + 3i - 2 \quad \text{if } i = 1, 2, ..., n; \\
  f(u'_i) &= 10 + 3i - 1 \quad \text{if } i = 1, 2, ..., n;
\end{align*}
\]

is a prime labeling for \( G \) with \( f(a) = f(w_j) = 1 \). Thus \( f \) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of \( a = w_j \) in \( G \).

**Subcase (ii):**
Let \( a = w'_j \) for some \( j \in \{1, 2, ..., 5\} \) then define a labeling \( f_2 \) using the labeling \( f \) defined in subcase (i) of case (i) as follows:

\[
\begin{align*}
  f_2(w_j) &= \begin{cases} 
  2(i - j + 5) & \text{if } i = 1, 2, ..., j; \\
  2(i - j) & \text{if } i = j + 1, j + 2, ..., 5;
  \end{cases} \\
  f_2(w'_j) &= \begin{cases} 
  2(i - j + 5) + 1 & \text{if } i = 1, 2, ..., j - 1; \\
  2(i - j) + 1 & \text{if } i = j, j + 1, ..., 5;
  \end{cases} \\
  f_2(v) &= f(v) \text{ for all the remaining vertices. Then the resulting labeling } f_2 \text{ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex } a = w'_j \text{ in } G.
\end{align*}
\]
Case (ii): When \( a \) is any arbitrary vertex of subdivision of pendant edges in a Hoffman graph.

Subcase (i):

Let \( a = v_j \) for some \( j \in \{2,3,...n\} \) then the function \( f : V(G) \rightarrow \{1,2,...,3n+10\} \) defined by

\[
\begin{align*}
f(v_j) &= \begin{cases} 
2(i+3) + 1 & \text{if } i = 1,2,3; \\
2i - 3 & \text{if } i = 4,5;
\end{cases} \\
f(w'_j) &= \begin{cases} 
2(i+3) + 2 & \text{if } i = 1,2; \\
2i - 2 & \text{if } i = 3,4,5;
\end{cases} \\
f(v_{j'}) &= \begin{cases} 
3(n+i-j+5) - 2 & \text{if } i = 1,2,...,j-1; \\
3(i-j+5) - 2 & \text{if } i = j+1,j+2,...n; 
\end{cases} \\
f(u_{j'}) &= \begin{cases} 
3(n+i-j+5) - 4 & \text{if } i = 1,2,...,j-1; \\
3(i-j+5) - 4 & \text{if } i = j+1,j+2,...n; 
\end{cases} \\
f(u'_j) &= \begin{cases} 
3(n+i-j+5) - 3 & \text{if } i = 1,2,...,j-1; \\
3(i-j+5) - 3 & \text{if } i = j+1,j+2,...n; 
\end{cases} \\
f(v_{j'}) &= 1; \\
f(u_{j'}) &= 2; \\
f(u'_j) &= 3;
\end{align*}
\]

is a prime labeling for \( G \) with \( f(a) = f(v_{j'}) = 1. \) Thus \( f \) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of \( a = v_j \) in \( G \).

Subcase (ii):

Let \( a = u_{j'} \) for some \( j \in \{2,3,...,n\} \) then define a labeling \( f_3 \) using the labeling \( f \) defined in subcase (i) of case (ii) as follows: \( f_3(v_{j'}) = f(u_{j'}) \), \( f_3(u_{j'}) = f(v_{j'}) \) for \( j \in \{1,2,...,n\} \) and \( f_3(v) = f(v) \) for all the remaining vertices. Then the resulting labeling \( f_3 \) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex \( a = u_{j'} \) in \( G \).

Subcase (iii):

Let \( a = u_j \) for some \( j \in \{2,3,...,n\} \) then define a labeling \( f_4 \) using the labeling \( f_3 \) defined in subcase (ii) of case (ii) as follows: \( f_4(u_j) = f_3(u_j) \), \( f_4(u'_j) = f_3(u_j) \) for \( j \in \{1,2,...,n\} \) and
\[ f_i(v) = f_j(v) \] for all the remaining vertices. Then the resulting labeling \( f_i \) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex \( a = u_j \) in \( G \).

Subcase (iv):

Let \( a = v_i \) then define a labeling \( f_5 \) using the labeling \( f \) defined in subcase (i) of case (ii) as follows: \( f_5(w_i) = 2i + 1 \) for \( i = 2, 3, 4, 5 \), \( f_5(w_i) = 2i + 2 \) for \( i = 1, 2, ..., 5 \), \( f_5(w_i) = 13 \) and \( f_5(v) = f(v) \) for all the remaining vertices. Then the resulting labeling \( f_5 \) is a prime labeling and also it is possible to assign label 1 to \( a = v_1 \) in \( G \).

Subcase (iv)a:

Let \( a = u_i \) then in the above labeling \( f_5 \) defined in subcase (iv) interchange the labels of \( v_i \) and \( u_i \). Then the resulting labeling \( f_6 \) is a prime labeling and also it is possible to assign label 1 to \( a = u_i \) in \( G \).

Subcase (iv)b:

Let \( a = u_i \) then in the above labeling \( f_6 \) defined in subcase ((iv)a) interchange the labels of \( u_i \) and \( u_i' \). Then the resulting labeling \( f_7 \) is a prime labeling and also it is possible to assign label 1 to \( a = u_i \) in \( G \). Thus from all the cases described above \( G \) is a strongly prime graph.

Figure 8. A prime labeling of a graph \( G \) obtained from subdividing the edges of the cycle \( C_5 \) and pendant edges in the key graph \( C_5 \uplus P_n \) having \( w_i \) as label 1

Acknowledgment: The authors are highly thankful to the anonymous referees for their kind suggestions and comments.
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