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OPERATOR THEORY ON DIFFERENTIAL RIEMANNIAN GEOMETRY

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Abstract .

This paper develop the Riemannian geometry of classical gauge theories , on compact dimensional manifolds, some important properties of fields , the manifold structure of the configuration space , we study the problem of differentially projection mapping parameterization system by constructing rank k on surfaces $n-k$ dimensional is sub manifold space R^n .

Index Terms- basic notion on differential geometry – differential between surfaces $M, N \subset R$ is called the differential manifolds- tangent and cotangent space- differentiable injective manifold- Operator geometric on Riemannian manifolds.

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I. Introduction

The object of this paper is to familiarize the reader with the basic language of and some fundamental theorem in Riemannian Geometry. To avoid referring to previous knowledge of differentiable manifolds, we include surfaces , which contains those concepts and result on differentiable manifolds which are used in an essential way in the rest of the paper . The first section II present the basic concepts of Riemannian Geometry (Riemannian metrics , Riemannian connections , geodesics and curvature) . A good part of the study of Riemannian Geometry consists of understanding the relationship between geodesics and curvature , Jacobi fields an essential tool for this understanding, are introduced in III we introduce the second fundamental from associated with an isometric immersion and prove a generalization of the theorem Egregium of Riemannian Geometry this allows us to real the notion of curvature in Riemannian manifolds to the classical concept of Gaussian curvature for surfaces. Starting we begin the study of global questions we emphasize techniques if the calculus of variations which we present without assuming a previous knowledge of the subject . Among other things we prove that theorems of Riemannian Geometry one the most remarkable applications of these techniques of calculus of variations the sphere theorem is presented in paper .In addition, we include a uniformization theorem for manifolds of constant curvature and a study of the fundamental group of compact manifolds of negative curvature .It is Euclidean in E^n in that every point has a neighbored, called a chart homeomorphism to an open subset of R^n , the coordinates on a chart allow one to carry out computations as though in a Euclidean space , so that many concepts from R^n , such as differentiability, point derivations , tangents , cotangents spaces , and differential forms carry over to a manifold. In this paper we given the basic definitions and properties of a smooth manifold and smooth maps between manifolds , initially the only way we have to verify that a space , we describe a set of sufficient conditions under which a quotient topological space becomes a manifold is exhibit a collection of C^∞ compatible charts covering the space becomes a manifold , giving us a second way to construct manifolds , a topological manifolds C^∞ analytic manifolds , stating with topological manifolds , which are Hausdorff second countable is locally Euclidean space We introduce the concept of maximal C^∞ atlas , which makes a topological manifold into a smooth manifold , a topological manifold is a Hausdorff , second countable is local Euclidean of dimension n .If every point p in M has a neighborhood U such that there is a homeomorphism φ from U onto a open subset of R^n. We call the pair a coordinate map or coordinate system on U. We said chart (U,φ) is centered at $p \in U$, $\varphi(p) = 0$, and we define the smooth maps $f : M \to N$ where M, N are differential manifolds we will say that f is smooth if there are atlases (U_α, h_α) on M and (V_β, g_β) on N.

II. A BASIC NOTIONS ON DIFFERENTIAL GEOMETRY

In this section is review of basic notions on differential geometry :

2.1 First principles

Hausdrff 2.1.1

A topological space M is called (Hausdorff) if for all $x, y \in M$ there exist open sets such that $x \in U$ and $y \in V$ and $U \cap V = \phi$

Second countable 2.1.2

A topological space M is second countable if there exists a countable basis for the topology on M.

Locally Euclidean of dimension n 2.1.3

A topological space M is locally Euclidean of dimension n if for every point $x \in M$ there exists on open set $U \in M$ and open set $w \subset R^n$ so that U and W are (homeomorphic).

Definition 2.1.3

A topological manifold of dimension n is a topological space that is Hausdorff, second countable and locally Euclidean of dimension n .

Definition 2.1.4

A smooth atlas A of a topological space M is given by : (i) An open covering $\{U_i\}_{i\in I}$ where $U_i \subset M$

Open and $M = \cup_{i\in I} U_i$ (ii) A family $\{\phi_i : U_i \to W_i\}_{i\in I}$ of homeomorphism ϕ_i onto open subsets $W_i \subset R^n$ so that if $U_i \cap U_j \neq \phi$ then the map $\phi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is (a diffoemorphism)

Definition 2.1.5

If $(U_i \cap U_j) \neq \phi$ then the diffeomorphism $\phi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is known as the (transitition map).

Definition 2.1.6

A smooth structure on a Hausdorff topological space is an equivalence class of atlases, with two atlases A and B being equivalent if for $(U_i, \varphi_i) \in A$ and $(V_j, \Psi_j) \in B$ with $U_i \cap V_j \neq \varphi$ then the transition map $\phi_i(U_i \cap V_j) \to \Psi_j(U_i \cap V_j)$ is a diffeomorphism (as a map between open sets of R^n).

Definition 2.1.7

A smooth manifold M of dimension n is a topological manifold of dimension n together with a smooth structure .

Definition 2.1.8

Let M and N be two manifolds of dimension m,n respectively a map $F: M \to N$ is called smooth at $p \in M$ if there exist charts $(U,\phi),(V,\Psi)$ with $p \in U \subset M$ and $F(p) \in V \subset N$ with $F(U) \subset V$ and the composition $\Psi \circ F \circ \phi^{-1} : \phi(U) \to \Psi(V)$ is a smooth (as map between open sets of R^n is called smooth if it smooth at every $p \in M$.

Definition 2.1.9

A map $F: M \to N$ is called a diffeomorphism if it is smooth bijective and inverse $F^{-1}: N \to M$ is also smooth.

Definition 2.1.10

A map F is called an embedding if F is an immersion and homeomorphic onto its image

Definition 2.1.11

If $F: M \to N$ is an embedding then $F(M)$ is an immersed submanifolds of N .

2.2 Tangent space and vector fields

Let $C^\infty(M,N)$ be smooth maps from M and N and let $C^\infty(M)$ smooth functions on M is given a point $p \in M$ denote, $C^\infty(p)$ is functions defined on some open neighbourhood of p and smooth at p.

Definition 2.2.1

(i) The tangent vector X to the curve $c:(-\varepsilon,\varepsilon) \to M$ at $t=0$ is the map $c(0): C^\infty(c(0)) \to R$ given by the formula .

(1) $$X(f) = c(0)(f) = \left(\frac{d(f \circ c)}{dt}\right)_{t=0} \quad \forall f \in C^\infty c(0)$$

(ii) A tangent vector X at $p \in M$ is the tangent vector at $t=0$ of some curve $\alpha:(-\varepsilon,\varepsilon) \to M$ with $\alpha(0) = p$ this is $X = \alpha'(0): C^\infty(p) \to R$.

Remark 2.2.2

A tangent vector at p is known as a liner function defined on $C^\infty(p)$ which satisfies the (Leibniz property)

(2) $$X(f\,g) = X(f)g + f\,X(g) \quad ,\forall\, f,g \in C^\infty(p).$$

2.3 Differential

Given $F \in C^\infty(M,N)$ and $p \in M$ and $X \in T_pM$ choose a curve $\alpha:(-\varepsilon,\varepsilon) \to M$ with $\alpha(0) = p$ and $\alpha'(0) = X$ this is possible due to the theorem about existence of solutions of liner first order ODEs , then consider the map $F_{*p}: T_pM \to T_{F(p)}N$ mapping $X \to F_{*p}(X) = (F \circ \alpha)'(0)$, this is liner map between two vector spaces and it is independent of the choice of α .

Definition 2.3.1

The liner map F_{*p} defined above is called the derivative or differential of F at p while the image $F_{*p}(X)$ is called the push forward X at $p \in M$.

Definition 2.3.2

Given a smooth manifold M a vector field V is a map $V: M \to TM$ mapping $p \to V(p) \equiv V_p$ and V is called smooth if it is smooth as a map from M to TM .

(Not) $X(M)$ is an R vector space for $Y,Z\in X(M)$, $p\in M$ and $a,b\in R$, $(aY+bZ)_p=aV_p+bZ_p$ and for $f\in C^{\infty}(M)$, $Y\in X(M)$ define $fY:M\to TM$ mapping $p\to(fY)_p=f(p)Y_p$

2.4 Cotangent space and Vector Bundles and Tensor Fields

Let M be a smooth n-manifolds and $p\in M$. We define cotangent space at p denoted by T_p^*M to be the dual space of the tangent space at $p: T_p^*(M)=\{f:T_pM\to R\}$, f smooth Element of T_p^*M are called cotangent vectors or tangent convectors at p. **(i)** For $f:M\to R$ smooth the composition $T_p^*M\to T_{f(p)}R\cong R$ is called df_p and referred to the differential of f. Not that $df_p\in T_p^*M$ so it is a cotangent vector at p **(ii)** For a chart (U,ϕ,x^i) of M and $p\in U$ then $\{dx^i\}_{i=1}^n$ is a basis of T_p^*M in fact $\{dx^i\}$ is the dual basis of $\left\{\frac{d}{dx^i}\right\}_{i=1}^n$.

Definition 2.4.1

The elements in the tensor product $V_s^r=V\otimes\ldots\otimes V\otimes V^*\otimes\ldots\otimes V^*$ are called (r,s) tensors or r-contravariant , s- contravariant tensor .

Remark 2.4.2

The Tensor product is bilinear and associative however it is in general not commutative that is $(T_1\otimes T_2)\neq(T_2\otimes T_1)$ in general .

Definition 2.4.3

$T\in V_s^r$ is called reducible if it can be written in the form $T=V_1\otimes\ldots\otimes V_r\otimes L^1\otimes\ldots\otimes L^s$ for.

(3) $$V_i\otimes V_r,\ L^j\in V^* \text{ for } 1\le i\le r,\ 1\le j\le s.$$

Definition 2.4.4

Choose two indices (i,j) where $1\le i\le r$, $1\le j\le s$ for any reducible tensor $T=V_1\otimes\ldots\otimes V_r\otimes L^1\otimes\ldots\otimes L^2$ let $C_i^r(T)\in V_{s-1}^{r-1}$ We extend this linearly to get a linear map $C_i^j:V_s^r\to V_{s-1}^{r-1}$ which is called tensor-contraction.

Remark 2.4.4

An ant symmetric (or alternating $(0,k)$ tensor) $T\in V_k^0$ is called a k-form on V and the space of all k-forms on V is denoted $\wedge^k V^*=\{T\in V_k^0 : T$ alternating$\}$.

Definition 2.4.5

A smooth real vector boundle of rank k denoted (E,M,π) is a smooth manifold E of dimension $n+1$

The total space a smooth manifold M of dimension n the manifold dimension $n+k$ and a smooth subjective map $\pi:E\to M$ (projection map) with the following properties :

(i) There exists an open cover $\{V_\alpha\}_{\alpha\in I}$ of M and diffeomorphisms $\Psi_\alpha:\pi^{-1}(V_\alpha)\to V_\alpha\times R^k$.

(ii) For any point $p\in M$, $\Psi_\alpha(\pi^{-1}(p))=\{p\}\times R^k\cong R^k$ and we get a commutative diagram (in this case $\pi_1:V_\alpha\times R^k\to V_\alpha$ is projection onto the first component .

(iii) whenever $V_\alpha\cap V_\beta\neq\phi$ the diffeomorphism.

(4) $$\psi_\alpha\circ\Psi_\beta^{-1}:(V_\alpha\cap V_\beta)\times R^k\to(V_\alpha\cap V_\beta)\times R^k$$

takes the form $\Psi_\alpha\circ\Psi_\beta^{-1}(p,a)=(p,A_{\alpha\beta}(p)(a))$, $a\in R^k$ where $A_{\alpha\beta}:V_\alpha\cap V_\beta\to GL(k,R)$ is called transition maps.

2.5 Bundle Maps and isomorphisms

Suppose (E, M, π) and $(\tilde{E}, \tilde{M}, \tilde{\pi})$ are two vector bundles a smooth map $F: E \to \tilde{E}$ is called a smooth bundle map from (E, M, π) to $(\tilde{E}, \tilde{M}, \tilde{\pi})$.

(i) There exists a smooth map $f: M \to \tilde{M}$ such that the following diagram commutes that $\tilde{\pi}(F(q)) = f(\pi(q))$ for all $p \in M$

(ii) F induces a linear map from E_p to $\tilde{E}_{f(p)}$ for any $p \in M$.

Definition 2.5.1 Dual Bundle

Take a vector bundle (E, M, π) where $E : \cup_{p \in M} E_p$ replace E_p with its dual $E^*{}_p$ and consider $E^* : \cup_{p \in M} E^*{}_p$. Let $V_\alpha, \psi_\alpha, A_{\alpha\beta}$ by an in the transition maps for the dial bundle E^* are denoted $(A^{dual})_{\alpha\beta} = (A^{-1}_{\alpha\beta})^T$ observe that $(A^{dual})_{\alpha\beta} = (A^{dual})_{\beta\gamma}$.

Definition 2.5.2 Tensor product of vector Bundles

Suppose (E, M, π) is vector bundle of rank k and $(\tilde{E}, \tilde{M}, \tilde{\pi})$ is vector bundle of rank l over the same base manifold M then define $E \otimes \tilde{E} = \cup_{p \in M} E_p \otimes \tilde{E}_p$, this is well defined because E_p and $\tilde{E}_p$ are vector spaces . Let be an open cover of $M, \Psi_\alpha, \tilde{\Psi}_\alpha, A_{\alpha\beta}, \tilde{A}_{\alpha\beta}$ be the local trivializations and transition maps to E and $\tilde{E}$ respectively then the transudation maps and local trivializations for $E \otimes \tilde{E}$ are given .

(5) $$a \otimes \tilde{a} \to A_{\alpha\beta} a \otimes \tilde{A}_{\alpha\beta} \tilde{a} \in R^k \otimes R^l \cong R^{k+1} \text{ , } \forall a \in R^k, \tilde{a} \in R^l$$

Definition 2.5.3

Let $F: M \to N$ be a smooth map between two smooth manifolds and $w \in \Gamma(T^0_k N)$ be a k covariant tensor field we define a k covariant tensor field $F^* w$ over M by .

(6) $$(F^* w)_p(v_1, \ldots, v_k) = w_{F(p)}(F_{*p}(v_1), \ldots, F_{*p}(v_k)), \forall v_1, \ldots, v_k \in T_p M$$

In this case $F^* w$ is called the pullback of w by F .

Proposition 2.5.4

Suppose $F: M \to N$ is a smooth map and $G: N \to Q$ a smooth map for M, N, Q smooth manifolds and $w \in T(T^0_k N), \eta \in T(T^0_l N)$ and $f \in C^\infty(N)$ then .

(i) $(G \circ F)^* = F^* \circ G^*$.

(ii) $F^*(w \otimes \eta) = F^* w \otimes F^* \eta$ in particular , $F^*(f \circ w) = (f \circ F) F^* w$.

(iii) $F^*(df) = d(f \circ F)$ (iv) if $p \in M$ and (y^i) are local coordinates in a chart containing the point $F(p) \in N$ then .

$$F^*(w_{j1,\ldots,jk}\, dy^{i1} \otimes \ldots \otimes dy^{ik}) = (w_{j1,\ldots,jk} \circ F)\, d(y^{j1} \circ F) \otimes \ldots \otimes d(y^{ik} \circ F).$$

2.6 Exterior derivative

The exterior derivative is a map $d: \Omega^k(M) \to \Omega^{k+1}(M)$ which is R linear such that $d \circ d = 0$ and if f is a k vector field on M then $(df)(X) = Xf$.

2.7 Integration of differential forms

$\int_M w$ is well defined only if M is orient able $\dim(M) = n$ and has a partition of unity and w has compact support and is a differential n-form on M .

2.8 Riemannian Manifolds

An inner product (or scalar product) on a vector space V is a function $\langle \cdot,\cdot \rangle : V\times V \to R$ that is :

(i)symmetric $\langle u,v \rangle = \langle v,u \rangle$ for all $u,v\in V$.

(ii) Bilinear $\langle au+bv,w \rangle = a\langle u,w\rangle + b\langle v,w\rangle$ and $\langle u,av+bw \rangle = a\langle u,v\rangle + b\langle u,w\rangle$ for all $a,b\in R$ and $u,v,w,\in V$.

(iii) positive definite $\langle u,v\rangle \succ 0$ for all $u\neq 0$.

Definition 2.8.1

A pair (M,g) of a manifold M equipped with a Riemannian metric g is called a Riemannian manifold.

2.9 Length and Angle between tangent vectors

Suppose (M,g) is a Riemannian manifold and $p\in M$ we define the length (or norm) of a tangent vector $v\in T_pM$ to be $|v| = \sqrt{\langle v,v \rangle_p}$ Recall $g(\cdot,\cdot) = \langle \cdot,\cdot \rangle$ and the angle v,w between $v,w\in T_pM$ $(v\neq 0\neq w)$ by $\cos(v,w) = \dfrac{\langle v,w\rangle_p}{|v||w|}$.

Examples of Riemannian metrics 2.9.1

1. Euclidean metric (canonical metric) g_{Eucl} on R^n .

(7)

$$g_{Eucl} = \delta_{ij}\, dx^i \otimes dx^j = dx^1 \otimes dx^1 + \ldots + dx^n \otimes dx^n = dx^1 dx^1 + \ldots + dx^n dx^n$$

2. Induced metric

Let (M,g) be a Riemannian manifold and $f : N \to (M,g)$ an immersion where N is a smooth manifold (that is f is a smooth map and f is injective) then induced metric on N is defined .

(8) $$(f^* g)_p (v,w) = g_{f(p)} (f_*(v), f_*(w)) \quad ,\forall v,w\in T_pN, p\in N$$

3. Induced metric $i^* g_{Eucl}$ on S^n

The induced metric S^n sometimes denoted $g_{Eucl}\,|_{S^n}$ from the Euclidean space R^{n+1} and g_{Eucl} by the inclusion $i : S^2 \to R^{n+1}$ is called the standard (or round) metric on S^n clearly i is an immersion .Consider stereographic projection $S^2 \to R^3$ and denote the inverse map $u : R^2 \to S^2$ then $u^* g_{Eucl}$

Given the Riemannian metric for R^2 .

4. Product metric

If (M_1,g_1), (M_2,g_2) are two Riemannian manifolds then the product $M_1\times M_2$ admits a Riemannian metric $g = g_1 \oplus g_2$ is called the product metric defined by .

(9) $$g(u_1\oplus u_2, v_1\oplus v_2) = g_1(u_1,v_1) \oplus g_2(u_2,v_2)$$

Where $u_i, v_i \in T_{pi}M_i$ for $i=1,2,\ldots.$ we use the fact that $T_{p_1,p_2}(M_1\times M_2) \cong T_{p_1}M_1 \oplus T_{p_1}M_2$.

5. Warped product

Suppose (M_1,g_1), (M_2,g_2) are two Riemannian manifolds then $(M_1\times M_2, g_1\oplus f^2 g_2)$ is the warped product of g_1,g_2 or denoted $(M_1,g_1)\times_f (M_2,g_2)$ where $f : M_1 \to R$ is a smooth positive function.

(10) $$(g_1 \oplus f^2 g_2)_{p_1p_2} (u_2\oplus u_2, v_1\oplus v_2) = g_{1_{p_1}}(u_1,v_1) \oplus f(p_1)\, g_{2_{p_1}}(u_2,w_2)$$

2.10 Conformal map and Isometric

A smooth map $f:(M,g)\to(N,h)$ between two Riemannian manifolds is called a conformal map with conformal factor $\lambda:M\to R^{+}$ if $(f^{*}h)=\lambda^{2}g$.

(Not)A conformal map preserves angles that is $(v,w)=(f_{*}(v),f_{*}(w))$ for all $u,v\in T_{p}M$ and $p\in M$.

Example 2.10.1

$S^{2}\subset R^{3}$ we consider stereographi projection $S^{2}/p_{n}\to R^{2}$. As stereographic projection is a diffeomorphism its inverse $u:R\to S/p_{n}$ is a conformal map . It follows from an exercise sheet that u is a conformal map with conformal factor $\rho(x,y)=2/(1+x^{2}+y^{2})$.

Definition 2.10.2

A Riemannian manifold (M,g) is locally flat if for every point $p\in M$ there exist a conformal diffeomorphism $f:U\to V$ between an open neighbourhoods U of p and $V\subset R^{n}$ of $f(p)$.

Definition 2.10.3

Given two Riemannian manifold (M,g) and (N,h) they are called isometric of there is a diffeomorphism $f:M\to N$ such that $f^{*}h=g$ such that a differomorphism f is called an isometric.

Remark 2.10.4

In particular an isometrics $f:(M,g)\to(M,g)$ is called an isometric of (M,g). All isometrics on a Riemannian manifold from a group .

Definition 2.10.5

$(M,g),(N,h)$ are called locally isometric if for every point $p\in M$ there is an isometric $f:U\to V$ from an open neighbourhood U of p in M and an open neighbourhood V of $f(p)$ in N.

Definition 2.10.6

Suppose $f:(M,g)\to(N,h)$ is an immersion then f is isometric if $f^{*}h=g$.

Definition 2.10.7

Let (M,g) be an oriented Riemannian n-manifold with its Riemannian volume from dV_{g} if f is a compactly supported smooth function on M then $f\,dV_{g}$ is a new n-form which is compactly supported we can define the integral of f over M as .

(11) $$\int_{M} f=\int_{M} f\,dV_{g}$$

Recall the integration of n-forms over n-manifolds.

2.11 Bundle metrics

The recall from liner algebra on a vector space V a bilinear from $B:V\times V\to R$ can be considered as an element $B\in E^{*}\otimes E^{*}$ given a vector bundle (E,M,π) a bundle metric is a map that assigns each fiber E_{p} an inner product $\langle\cdot,\cdot\rangle_{p}$ which depends smoothly on $p\in M$.

Definition 2.11.1

A bundle metric h on the vector bundle (E,M,π) is an element of $\Gamma(E^{*}\otimes E^{*})$ which is stmmetric and positive definite.

Remark 2.11.2

Given a vector bundle (E, M, π) with a bundle metric h we can define an isomorphism $E \to E^*$ we can extend h to any (r, s) tensor products of E and E^*.

2.12 Differentiable injective manifold

the basically an m-dimensional topological manifold is a topological space M which is locally homeomorphic to R^m, definition is a topological space M is called an m-dimensional (topological manifold) if the following conditions hold.

(i) M is a hausdorff space.

(ii) for any $p \in M$ there exists a neighborhood U of P which is homeomorphic to an open subset $V \subset R^m$. (iii) M has a countable basis of open sets , coordinate charts (U, φ) Axiom (iv) is equivalent to saying that $p \in M$ has a open neighborhood $U \in P$ homeomorphic to open disc D^m in R^m , axiom (v) says that M can covered by countable many of such neighborhoods , the coordinate chart (U, φ) where U are coordinate neighborhoods or charts and φ are coordinate .

A homeomorphisms , transitions between different choices of coordinates are called transitions maps $\varphi_{ij} = \varphi_j \circ \varphi_i$, which are again homeomorphisms by definition , we usually write $p = \varphi^{-1}(x), \varphi : U \to V \subset R^n$ as coordinates for U and $p = \varphi^{-1}(x), \varphi^{-1} : V \to U \subset M$ as coordinates for U , the coordinate charts (U, φ) are coordinate neighborhoods, or charts , and φ are coordinate homeomorphisms , transitions between different choices of coordinates are called transitions maps $\varphi_{ij} = \varphi_j \circ \varphi_i$ which are again homeomorphisms by definition , we usually $x = \varphi(p), \varphi : U \to V \subset R^n$ as a parameterization U .A collection $A = \{\varphi_i, U_i\}_{i \in I}$ of coordinate chart with $M = \cup_i U_i$ is called atlas for M . The transition maps φ_{ij} a topological space M is called (hausdorff) if for any pair $p, q \in M$, there exist open neighborhoods $p \in U$ and $q \in U'$ such that $U \cap U' \neq \phi$ for a topological space M with topology $\tau \in U$ can be written as union of sets in β , a basis is called a countable basis β is a countable set .

Definition 2.12.1

A topological space M is called an m-dimensional topological manifold with boundary $\partial M \subset M$ if the following conditions .

(i) M is hausdorff space.

(ii) for any point $p \in M$ there exists a neighborhood U of p which is homeomorphism to an open subset $V \subset H^m$.

(iii) M has a countable basis of open sets , can be rephrased as follows any point $p \in U$ is contained in neighborhood U to $D^m \cap H^m$ the set M is a locally homeomorphism to R^m or H^m the boundary $\partial M \subset M$ is subset of M which consists of points p .

Definition 2.12.2

Let X be a set a topology U for X is collection of X satisfying .

(i) ϕ and X are in U (ii) the intersection of two members of U is in U .

(iii) the union of any number of members U is in U . The set X with U is called a topological space the members $U \in u$ are called the open sets . let X be a topological space a subset $N \subseteq X$ with $x \in N$ is called a neighborhood of x if there is an open set U with $x \in U \subseteq N$, for example if X a metric space then the closed ball $D_\varepsilon(x)$ and the open ball $D_\varepsilon(x)$ are neighborhoods of x a subset C is said to closed if $X \setminus C$ is open

Definition 2.12.3

A function $f : X \to Y$ between two topological spaces is said to be continuous if for every open set U of Y the pre-image $f^{-1}(U)$ is open in X .

Definition 2.12.4

Let X and Y be topological spaces we say that X and Y are homeomorphic if there exist continuous function such that $f \circ g = id_y$ and $g \circ f = id_X$ we write $X \cong Y$ and say that f and g are homeomorphisms between X and Y, by the definition a function $f : X \to Y$ is a homeomorphisms if and only if .(i) f is a bijective .(ii) f is continuous (iii) f^{-1} is also continuous.

2.5 Differentiable manifolds

A differentiable manifolds is necessary for extending the methods of differential calculus to spaces more general R^n a subset $S \subset R^3$ is regular surface if for every point $p \in S$ the a neighborhood V of P is R^3 and mapping $x : u \subset R^2 \to V \cap S$ open set $U \subset R^2$ such that.

(i) x is differentiable homomorphism. (ii) the differentiable $(dx)_q : R^2 \to R^3$, the mapping x is called a parametnzation of S at P the important consequence of differentiable of regular surface is the fact that the transition also example below if $x_\alpha : U_\alpha \to S^1$ and $x_\beta : U_\beta \to S^1$ are $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = w \neq \phi$, the mappings $x_\beta^{-1} \circ x_\alpha : x^{-1}(w) \to R^2$ and .

(12) $$x_\alpha^{-1} \circ x_\beta = x_\beta^{-1}(w) \to R$$

Are differentiable . A differentiable structure on a set M induces a natural topology on M it suffices to $A \subset M$ to be an open set in M if and only if $x_\alpha^{-1}(A \cap x_\alpha(U_\alpha))$ is an open set in R^n for all α it is easy to verify that M and the empty set are open sets that a union of open sets is again set and that the finite intersection of open sets remains an open set. Manifold is necessary for the methods of differential calculus to spaces more general than de R^n , a differential structure on a manifolds M induces a differential structure on every open subset of M , in particular writing the entries of an $n \times k$ matrix in succession identifies the set of all matrices with $R^{n,k}$, an $n \times k$ matrix of rank k can be viewed as a k-frame that is set of k linearly independent vectors in R^n , $V_{n,k} K \leq n$ is called the steels manifold ,the general linear group $GL(n)$ by the foregoing $V_{n,k}$ is differential structure on the group n of orthogonal matrices, we define the smooth maps function $f : M \to N$ where M, N are differential manifolds we will say that f is smooth if there are atlases (U_α, h_α) on M , (V_B, g_B) on N , such that the maps $g_B f h_\alpha^{-1}$ are smooth wherever they are defined f is a homeomorphism if is smooth and a smooth inverse. A differentiable structures is topological is a manifold it an open covering U_α where each set U_α is homeoomorphic, via some homeomorphism h_α to an open subset of Euclidean space R^n , let M be a topological space , a chart in M consists of an open subset $U \subset M$ and a homeomorphism h of U onto an open subset of R^m , a C^r atlas on M is a collection (U_α, h_α) of charts such that the U_α cover M and h_B, h_α^{-1} the differentiable

2.6 Definition (Differentiable injective manifold)

A differentiable manifold of dimension N is a set M and a family of injective mapping $x_\alpha \subset R^n \to M$ of open sets $u_\alpha \in R^n$ into M such that.

(i) $u_\alpha x_\alpha(u_\alpha) = M$ (ii) for any α, β with $x_\alpha(u_\alpha) \cap x_\beta(u_\beta)$

(iii) the family (u_α, x_α) is maximal relative to conditions (I),(II) the pair (u_α, x_α) or the mapping x_α with $p \in x_\alpha(u_\alpha)$ is called a parameterization , or system of coordinates of M , $u_\alpha x_\alpha(u_\alpha) = M$ the coordinate charts (U, φ) where U are coordinate neighborhoods or charts , and φ are coordinate homeomorphisms transitions are between different choices of coordinates are called transitions maps.

(13) $$\varphi_{i,j} : (\varphi_j \circ \varphi_i^{-1})$$

Which are anise homeomorphisms by definition , we usually write $x = \varphi(p), \varphi : U \to V \subset R^n$ collection U and $p = \varphi^{-1}(x), \varphi^{-1} : V \to U \subset M$ for coordinate charts with is $M = \cup U_i$ called an atlas for M of topological manifolds. A topological manifold M for which the transition maps $\varphi_{i,j} = (\varphi_j \circ \varphi_i)$ for all pairs φ_i, φ_j in the atlas are homeomorphisms is called a differentiable , or smooth manifold , the transition maps are mapping between open subset of R^m, homeomorphisms between open subsets of R^m are C^∞ maps whose inverses are also C^∞ maps , for two charts U_i and U_j the transitions mapping.

(14) $$\varphi_{i,j} = (\varphi_j \circ \varphi_i^{-1}) : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

And as such are homeomorphisms between these open of R^m, for example the differentiability $(\varphi'' \circ \varphi^{-1})$ is achieved the mapping $(\varphi'' \circ (\tilde{\varphi})^{-1})$ and $(\tilde{\varphi} \circ \varphi^{-1})$ which are homeomorphisms since $(A \approx A'')$ by assumption this establishes the equivalence $(A \approx A'')$, for example let N and M be smooth manifolds n and m respecpectively , and let $f : N \to M$ be smooth mapping in local coordinates $f' = (\psi \circ f \circ \varphi^{-1}) : \varphi(U) \to \psi(V)$ Figurer (5) ,with respects charts (U, φ) and (V, ψ), the rank of f at $p \in N$ is defined as the rank of f' at $\varphi(p)$ (i.e) $rk(f)_p = rk(J f')_{\varphi(p)}$ is the Jacobean of f at p this definition is independent of the chosen chart , the commutative diagram in that.

(15) $$f'' = (\psi' \circ \psi^{-1}) \circ \tilde{f} \circ (\varphi' \circ \varphi^{-1})^{-1}$$

Since $(\psi' \circ \psi^{-1})$ and $(\varphi' \circ \varphi^{-1})$ are homeomorphisms it easily follows that which show that our notion of rank is well defined $(J f'')_{x^j} = J(\psi' \circ \psi^{-1})_{y^j} J f' (\varphi' \circ \varphi^{-1})^{-1}$, if a map has constant rank for all $p \in N$ we simply write $rk(f)$, these are called constant rank mapping. The product two manifolds M_1 and M_2 be two C^k -manifolds of dimension n_1 and n_2 respectively the topological space $M_1 \times M_2$ are arbitral unions of sets of the form $U \times V$ where U is open in M_1 and V is open in M_2, can be given the structure C^k manifolds of dimension n_1, n_2 by defining charts as follows for any charts M_1 on (V_j, ψ_j) on M_2 we declare that $(U_i \times V_j\ ,\ \varphi_i \times \psi_j)$ is chart on $M_1 \times M_2$ where $\varphi_i \times \psi_j : U_i \times V_j \to R^{(n_1+n_2)}$ is defined so that.

(16) $$\varphi_i \times \psi_j (p, q) = (\varphi_i(p), \psi_j(q)) \text{ for all } (p, q) \in U_i \times V_j$$

A given a C^k n-atlas, A on M for any other chart (U,φ) we say that (U,φ) is compatible with the atlas A if every map $(\varphi_i \circ \varphi^{-1})$ and $(\varphi \circ \varphi_i^{-1})$ is C^k whenever $U \cap U_i \neq 0$ the two atlases A and $\tilde{A}$ is compatible if every chart of one is compatible with other atlas . A sub manifolds of others of R^n for instance S^2 is sub manifolds of R^3 it can be obtained as the image of map into R^3 or as the level set of function with domain R^3 we shall examine both methods below first to develop the basic concepts of the theory of Riemannian sub manifolds and then to use these concepts to derive a equantitive interpretation of curvature tensor , some basic definitions and terminology concerning sub manifolds, we define a tensor field called the second fundamental form which measures the way a sub manifold curves with the ambient manifold , for example X be a sub manifold of Y of $\pi : E \to X$ and $g : E_1 \to Y$ be two vector brindled and assume that E is compressible , let $f : E \to Y$ and $g : E_1 \to Y$ be two tubular neighborhoods of X in Y then there exists a C^{p-1} . The smooth manifold , an n-dimensional manifolds is a set that looks like R^n . It is a union of subsets each of which may be equipped with a coordinate system with coordinates running over an open subset of R^n . Here is a precise definition.

Definition 2.6 .1

Let M be a metric space we now define what is meant by the statement that M is an n-dimensional C^∞ manifold.

(i). A chart on M is a pair (U,φ) with U an open subset of M and φ a homeomorphism a (1-1) onto, continuous function with continuous inverse from U to an open subset of R^n , think of φ as assigning coordinates to each point of U .

(ii) Two charts (U,φ) and (V,ψ) are said to be compatible if the transition functions .

(17)

$$(\psi \circ \varphi^{-1}) : \varphi(U \cap V) \subset R^n \to \psi(U \cap V) \subset R^n \quad , \quad (\varphi \circ \psi^{-1}) : \psi(U \cap V) \subset R^n \to \varphi(U \cap V) \subset R^n$$

Are C^∞ that is all partial derivatives of all orders of $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ exist and are continuous.

(ii) An atlas for M is a family $A = \{(U_i,\varphi_i) : i \in I\}$ of charts on M such that $\{U_i\}_{i \in I}$ is an open cover of M and such that every pair of charts in A are compatible . The index set I is completely arbitrary . It could consist of just a single index. It could consist of uncountable many indices . An atlas A is called maximal if every chart (U,φ) on M that is compatible with every chat of A .

(iii) An n-dimensional manifold consists of a metric space M together with a maximal atlas A

Example 2.6.2 (open subset of R^n)

Let I_n be the identity map on R^n , then $\{(R^n, I_n)\}$ is an atlas for R^n indeed , if U is any nonempty open subset of R^n , then $\{(U, I_n)\}$ is an atlas for U so every open subset of R^n is naturally a C^∞ manifold.

Example 2.6.3 (The n-sphere)

The n-space $S^n = \{x = (x_1,\ldots,x_{n+1}) \in R^{n+1}, |x_1^2,\ldots,x_{n+1}^2 = 1|\}$ is a manifold of dimension n when equipped with the atlas $A_1 = \{(U_i,\varphi_i),(V_i,\psi_i),|1 \le i \le n+1|\}$ where for each $1 \le i \le n+1$.

(18) $$U_i = \{(x_1,\ldots,x_{n+1}) \in S^n , x_1 \ge 0\} \quad \varphi_i(x_1,\ldots,x_{n+1}) = (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+1})$$

$V_i = \{(x_1,\ldots,x_{n+1}) \in S^n, x_1 \le 0\}$ $\psi_i(x_1,\ldots,x_{n+1}) = (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+1})$
So both φ_i and ψ_i just discard the coordinate x_i they project onto R^n viewed as the hyper plane $x_i = 0$. A another possible atlas , compatible with A_1 is $A_2 = \{(U,\varphi),(V,\psi)\}$ where the domains that
$U = S^m \setminus \{(0,\ldots,0,1)\}$ and $V = S^n \setminus \{(0,\ldots,0,-1)\}$ are the stereographic projection from the north and south poles, respectively , Both φ and ψ have range R^n plus an additional single point at infinity

Example 2.6.4 (A Torus)

The torus T^2 is the two dimensional surface $T^2 = \left\{ (x,y,z) \in R^3, (\sqrt{x^2+y^2}-1)^2 + z^2 = 1/4 \right\}$ in R^3 in cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta, z = 0$ the equation of the torus is $(r-1)^2 + z^2 = 1/4$ fix any $\theta, say\ \theta_0$. Recall that the set of all points in R^n that have $\theta = \theta_0$ is an open book , it is a hal-plane that starts at the z axis . The intersection of the tours with that half plane is circle of radius 1/2 centered on $r = 1, z = 0$ as φ runs form $0 to 2\pi$, the point $r = 1 + 1/2\cos\varphi$ and $\theta = \theta_0$ runs over that circle. If we now run θ from $0 to 2\pi$ the point $(x,y,z) = ((1+1/2\cos\varphi)\cos\theta, (1+1/2\sin\varphi)$ Runs over the whole torus . So we may build coordinate patches for T^2 using θ and φ with ranges $(0,2\pi)$ or $(-\pi,\pi)$ as coordinates).

III. OERATOR GEOMETRIC ON RIEMANNIAN MANIFOLDS

3.1 Vector Analysis one Method Lengths]

Classical vector analysis describes one method of measuring lengths of smooth paths in R^3 if $v : [0,1] \to R^3$ is such a paths ,then its length is given by length $v = v(t)\,dt$. Where $[v]$ is the Euclidean length of the tangent vector (t) , we want to do the same thing on an abstract manifold ,and we are clearly faced with one problem , how do we make sense of the length $|v(t)|$,obviously , this problem can be solved if we assume that there is a procedure of measuring lengths of tangent vectors at any point on our manifold The simplest way to do achieve this is to assume that each tangent space is endowed with an inner product (which can vary point in a smooth).

Definition 3.1.1

A Riemannian manifold is a pair $(M.g)$ consisting of a smooth manifold M and a metric g on the tangent bundle ,i.e a smooth symmetric positive definite tensor field on M . The tensor g is called a Riemannian metric on M . Two Riemannian manifold are said to be isometric if there exists a diffeomorphism $\phi : M_1 \to M_2$ such that $\phi^* : g_1 = g_2$ If $(M.g)$ is a Riemannian manifold then , for any $x \in M$ the restriction $g_x : T_x(M_1) \times T_x(M_2) \to R$. Is an inner product on the tangent space $T_x(M)$ we will frequently use thee alternative notation $(\cdot,\cdot)_x = g_x(\cdot,\cdot)$ the length of a tangent vector $v \in T_x(M)$ is defined as usual $|v|_x = g_x(v,v)^{1/2}$. If $v : [a,b] \to M$ is a piecewise smooth path , then we defined is length by $L(v) = \int_a^b |v(t)|\,dt$. If we choose local coordinates $(x^1,\ldots,x^n)$ on M ,then we get a local description of g as.

(19) $$g = g_{ij}(dx^i, dx^j), g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$$

Proposition 3.1.2

Let be a smooth manifold , and denote by R_M the set of Riemannian metrics on M ,then R_M is a non – empty convex cone in the linear of symmetric tensor

Proof :

The only thing that is not obvious is that R_M is non-empty we will use again partitions of unity . Cover M by coordinate neighborhoods $(U_\alpha)_{\alpha\in A}$. Let x^j be a collection of local coordinates on U_α . Using these local coordinates we can construct by hand the metric g_α on U_α by $g_\alpha = (dx^1)^2 + \ldots + (dx^n)^2$

now , pick a partition of unity $B \subset C_0^\infty(M)$ subordinated to cover $(U_\alpha)_{\alpha\in A}$ (i.e) there exists a map $\phi: B \to A$ such that $\forall \beta \in B \subset U_{\phi\in B}$ then define $g = \sum_{\beta\in B} \beta g\,\phi(B)$ The reader can check easily g is well defined ,and it is indeed a Riemann metric on M .

Example 3.1.3 The Euclidean Space

The space R^n has a natural Riemann metric $g_0 = dx^1, \ldots, dx^n$ The geometry of (R^n, g_0) is the classical Euclidean geometry

Example 3.1.4 Induced Metrics On Sub manifolds

Let (M, g) be Riemann manifold and $S \subset M$ a sub manifold if $S \to M$, denotes the natural inclusion then we obtain by pull back a metric on $S, g^S = i g^* = g/S$. For example , any invertible symmetric $n \times n$ matrix defines a quadratic hyper surface in R^n by $H_A = \{x \in R^n, (A_x, x) = 1\}$ where $[\cdot,\cdot]$ denotes the Euclidean inner on R^n , H_A has a natural .

Remark 3.1.5

On any manifold there exist many Riemannian metrics , and there is not natural way of selecting on of them . One can visualize a Riemannian structure as defining " shape " of the manifold . For example , the unit sphere $x^2 + y^2 + z^2 = 1$, is diffeomorphic to the ellipsoid $(x^2/1^2) + (y^2/2^2) + (z^2/3^3) = 1$,but they look "different" by However , appearances may be deceiving in is illustrated the deformation of a cylinder they look different ,but the metric structures are the same since we have not change length of curves on our sheep . the conclusion to be drawn from these two examples is that we have to be very careful when we use the attribute "different".

Example 3.1.5 The Hyperbolic Plane

The Poincare model of the hyperbolic plane is the Riemannian manifold (D, g) where D is the unit open disk in the plan R^n and the metric g is given by .

(20) $$g = \frac{1}{1 - x^2 - y^2}(dx^2 + dy^2)$$

Example 3.1.6 Left Invariant Metrics on lie groups

Consider a lie group G ,and denote by L_G its lie algebra then any inner product $\langle \cdot,\cdot \rangle$ on L_G induces a Riemannian metric $h = \langle \cdot,\cdot \rangle_g$ on G defined by.

(21) $$h_g(x, y) = \langle x, y \rangle_g = \langle L_g^{-1} \rangle * X, (L_g^{-1}) * Y \quad , \forall : g \in G, X \; y \in T_g(G)$$

Where $(L_g^{-1})_* : T_g(G) \to T_1(G)$ is the differential at $g \in G$ of the left translation map L_g^{-1}. One checks easily that check easily that the correspondence $G \in g \to \langle \cdot, \cdot \rangle$ is a smooth tensor field , and it is left invariant (i,e) $L_g^* h = h \quad \forall g \in G$. If G is also compact ,we can use the averaging technician to produce metrics which are both left and right invariant .

3.2 The Levi-Cavite Connection

To continue our study of Riemannian manifolds we will try to follow a close parallel with classical Euclidean geometry the first question one may ask is whether there is a notion of " straight line " on a Riemannian manifold .In the Euclidean space R^3 there are at least ways to define a line segment a line segment is the shortest path connecting two given points a line segment is a smooth path $v : [0,1] \to R^3$ satisfying $\ddot{v}(t) = 0$. Since we have not said anything about calculus of variations which deals precisely with problems of type. (i) we will use the second interpretation as our starting point ,we will soon see however that both points of view yield the same conclusion . Let us first reformulate as know the tangent bundle of R^3 is equipped with a natural trivialization , and as such it has a natural trivial connection ∇^0 defined by. $\nabla_i^0(\partial_j) = 0 \ \forall : i, j$ where $\nabla^0 \partial_j = 0, \forall i, j \quad \partial_j = \frac{\partial}{\partial x_i}, \nabla_i = \nabla_{\partial_i}$

all the Christ off symbols vanish ,moreover , if g_0 denotes the Euclidean metric ,then .

(22)
$$(\nabla^0{}_i g_0)(\partial_j, \partial_k) = \nabla^0_{ijk} - g_0(\nabla^0_i \partial_j, \partial_k) - g_0(\partial_j, \nabla^0 \partial_k) = 0$$
$$\nabla^0_{V(t)} V(t) = 0$$

So that the problem of defining " lines " in a Riemannian manifold reduces to choosing a " natural " connection on the tangent bundle of course , we would like this connection to be compatible with the metric but even so , there infinitely many connections to choose from . the following fundamental result will solve this dilemma .

Proposition 3.2.1 Levi-Cavite Connection

Consider a Riemannian manifold (M, g) ,then there exists a unique symmetric connection ∇ on $T(M)$ compatible with the metric g $T(\nabla) = 0$, $\nabla_g = 0$ the connection ∇ is usually called the Levi-Civita connection associated to the metric g .

Proof

Uniqueness we will achieve this by producing an explicit description of a connection with the above two m properties let ∇ be such a connection , i.e $\nabla_g = 0$ and

$\nabla_x Y - \nabla_y X = [X, Y] \forall X, Y \in (M)$ for any $X, Y, Z \in (M)$ we have

$Z_g(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$ Since.

(23)
$$(X, Y) - Y_g(Z, X) + X_g(Y, Z) = g(\nabla_Z X, Y) - g(\nabla_Y Z, X) + g(\nabla_X Y, Z) + g(X, \nabla_Z Y) - g(Z, \nabla_Y Z)$$
$$= g([Z, Y]X) + g([X, Y]Y) + g([Z, X]Y) + 2g(\nabla_X, Z, Y)$$

We conclude that.

(24)
$$g(\nabla_Z X, Y) = \frac{1}{2}(X_g(Y, Z) - Y_g(Z, X) - g([X, Y]Y) + g([Y, Z]X) - g([Z, X]Y))$$

The above equality establishes the uniqueness of ∇ using local coordinates $x^1, \ldots, x^n$ on M we deduce from (24) with.

(25) $$\left(X=\partial_i=\frac{\partial}{\partial x_i}\right),\left(Y=\partial_k=\frac{\partial}{\partial x_k}\right),\left(Z=\partial_j=\frac{\partial}{\partial x_j}\right)$$

That $g(\nabla_i,\partial_j,\partial_k)=g_{K l_{i,j}}=\frac{1}{2}(\partial_i g_{kl}-\partial_k g_{ij}-\partial_i g_{ikl})$. Above ,the scalars ℓ_{ij} denote the christoffel symbols of ∇ in these coordinates i.e $\nabla_{\partial_i}\partial_j=l_{ij}\partial_j$, , denotes the inverse of (g^{kl}) we deduce .

(26) $$l_{ij}=\frac{1}{2}g^{kl}(\partial_i g_{ik}+\partial_k g_{ij}+\partial g_{ik})$$

Definition 3.2.2 Riemannian Manifold Is Smooth

A geodesic on a Riemannian manifold i (M,g) s a smooth path , $v:(a,b)\to M$ satisfying

(27) $$\nabla_{V(t)}V(t)=0$$

Where ∇ is the " Levi - Civita " connection. Using local coordinates $x^1,\ldots,x^n$ with respect to which the Christopher simples are k_{ij} and the path v is described $v(t)=(x^1(t),\ldots,x^n(t))$ we can rewrite the geodesic equation as a second , order nonlinear system of ordinary differential equations .

(28) $$\frac{d}{dt}=V(t)=X^i\partial_i$$

set $\nabla\frac{d}{dt}V(t)=\ddot{X}\partial_i+\dot{X}^i\nabla_{\frac{d}{dt}}\partial_i=\ddot{X}\partial_i+\dot{X}\ddot{X}\nabla_j\partial_i=\ddot{X}\partial_k+k_{ij}\dot{X}^i\dot{X}^j\partial_k(k_{ij}=k_{ji})$ So the geodesic equation is equivalent to $\ddot{X}^k+^k_{ij}\dot{X}^i\dot{X}^j=0,\forall k=1,\ldots,n$ Since the coefficients $(k_{ij}(X)=k_{ji}(X))$ depend smooth up on , x we can use the classical Banach-Picard

Proposition 3.2.3 Riemannian for any Compact subset

Let (M,g) be a Riemannian manifold for any compact subset $\subset TM$ there exists $\varepsilon\geq 0$ such that for any $(x,X)\in k$ there exists a unique geodesic $V=V_xX:(-\varepsilon,\varepsilon)\to M$ such that $V(0)=x,V(0)=X$

One can think of a geodesic as defining a path in the tangent bundle $t\to(V(t),V(t))$ .The above proposition shows that the geodesics define a local flow ϕ on $T(M)$ by $\phi^t(x.X)=(V(t),V(t)),V_xX$

Definition 3.2.4 Geodesic Low

The local flow defined above is called the geodesic flow the Riemannian manifold (M,g) when the geodesic low is global flow i,e any V_xX is defined at each moment of t for any $(x,X)\in T(M)$, then the Riemannian manifold is call geodetically complete .

Proposition 3.2.5 Conservation of energy

If the path $V(t)$ is a geodesic , then length of $V(t)$ is independent of time

Proof : we have

(29) $$\frac{d}{dt}|V(t)|^2=\frac{d}{dt}g(V(t),V(t))=2g(\nabla_{V(t)},V(t))=0$$

Thus , if we consider the sphere bundles $S_r(M)=\{X\in T(M),|X|=r\}$ We deduce that $S_r(M)$ are invariant subset of geodesic flow .

Definition 3.2.6 Lie algebra Group

Let G be a connected lie group ,and let L_G be its lie algebra any $X \in L_G$ defines an endomorphism of X of L_G by $(X)Y = [X,Y]$. The Jacobi identity implies that adjoin $[X,Y] = [(X),(Y)]$ where the bracket in the right hand side is the usual commutator of two endomorphism. Assume that there exists an inner product $\langle \cdot,\cdot \rangle$ on L_G such that for any $X \in L_G$ the operator adjoin (X) is skew-adjoins i.e

(30) $$\langle [X,Y],Z \rangle = \langle Y, [X,Y] \rangle$$

We can now extend this inner product to a left invariant metric h on G . We want to describe its geodesic first ,we have to determine associated " Levi-civita " connection .using (30) we get.

(31)

$$h(\nabla_x Z,Y) = \frac{1}{2}\{Xh(Y,Z) - Y(Z,X) + Zh(X,Y) - h([X,Y],Y) + h([Y,Z],X) - h([Z,X],Y)\}$$

If we take $X,Y,Z \in L_G$ these vector fields are left invariant ,then $h(Y,Z) = const$, $h(Z,X) = const$, $h(X,Y) = const$, so that the first three terms in the above formula vanish we obtain the following equality at $1 \in G$

(32) $$\langle \nabla_x Z,Y \rangle = \frac{1}{2}\{\langle [X,Z],Z \rangle + \langle -[Y,Z],X \rangle - \langle -[Z,X],X \rangle\}$$

Using the skew-symmetry of ad(X) and ad(Z) we deduce $\langle \nabla_x Z,Y \rangle = \frac{1}{2}\langle [X,Z],Y \rangle$ so that $1 \in G$, $\nabla_x Z = \frac{1}{2}[X,Z] \forall : X,Z \in L_G$. This formula correctly defines a connection since any $X \in vector(G)$ can be written as a linear combination $X = \sum \alpha_i X_i$, $\alpha_i \in C_0^\infty(G), X_i \in L_G$. If $V(t)$ is a geodesic ,we can write $V(t) = \sum V_i X_i$, so that .

(33) $$0 = \nabla_{V(t)}(t) = \sum_i V_i X_i + \frac{1}{2}\sum_{i,J} V_i V_j [X_i, X_j]$$

Since $[X_i, X_j] = -[X_j, X_i]$ we deduce $V_i = 0$ i,e $V(t) = \sum V_i(0) X_i = X$. This means that V is an integral curve of the left invariant vector field X ,so that the geodesics through the origin with initial direction $X \in T_1(G)$ are $V_x(t) = \exp(X(t))$.

Example 2.3.7 (Surfaces)

Any smooth n-dimensional R^{n+1} is an n-dimensional manifold. Roughly speaking a subset of R^{n+m} a an n-dimensional surface if , locally m of the $m+n$ coordinates of points on the surface are determined by the other n coordinates in a C^∞ way , For example , the unit circle S^1 is a one dimensional surface in R^2 . Near (0.1) a point $(x,y) \in R^2$ is on S^1 if and only if $y = \sqrt{1-x^2}$ and near (-1.0) , (x,y) is on S^1 if and only if $y = -\sqrt{1-x^2}$. The precise definition is that M is an n-dimensional surface in R^{n+m} if M is a subset of R^{n+m} with the property that for each $z = (z_1,\ldots,z_{n+m}) \in M$ there are a neighborhood U_z of z in R^{n+m} , and n integers $1 \le J_1 \le j_2 \le \ldots \le j_{n+m}$, C^∞ function $f_k(x_{j1},\ldots,x_{jn})$, $k \in \{1,\ldots,n+m\} \setminus \{1,\ldots,j_n\}$ such that the point $x = (x_1,\ldots,x_{n+m}) \in U_z$.

That is we may express the part of M that is near z as $x_{i1} = f_{i1}(x_{j1}, x_{j2},\ldots,x_{jn})$, $x_{i2} = f_{i2}(x_{j1}, x_{j2},\ldots,x_{jn}),\ldots, x_{im} = f_{im}(x_{j1}, x_{j2},\ldots,x_{jn})$

Where there for some C^∞ function $f_1,\ldots,f_m$. We many use $x_{j1},x_{j2},\ldots,x_{jn}$ as coordinates for R^2 in $M\cap U_z$ .Of course an atlas is $A=\{U_z\cap M,\varphi_z,|\ z\in M\}$, with $\varphi_z(x)=(x_{j1},\ldots,x_{jn})$ Equivalently, M is an n-dimensional surface in R^{n+m} if for each $z\in M$, there are a neighborhood U_z of z in R^{n+m}, and mC^∞ functions $g_k:U_z\to R$ with the vector $\{\nabla_{gz}(z)|,1\le k\le m\}$ linearly independent such that the point $x\in U_z$ is in M if and only if $g_k(x)=0$ for all $1\le k\le m$. To get from the implicit equations for M given by the g_k to the explicit equations for M given by the f_k one need only invoke (possible after renumbering of x) .

Definition 3.2.8 [Killing Paring]

Let L be a finite dimensional real lie algebra ,the killing paring or form is the bilinear map.

(34) $$K:L\times L\to R,K(X,Y)=-tr(ad(X).ad(Y))\quad\forall:X,Y\in L$$

The lie algebra L is said to be semi simple if killing paring is a duality ,a lie group G is called semi simple if its lie algebra is semi simple .

3.3 The Exponential Map Normal Coordinates

We have already seen that there are many difference between the classical Euclidean geometry and the general Riemannian geometry in the large . In particular we have seen examples in which one of basic axioms of Euclidean geometry no longer holds .Two distinct geodesic (real lines) may intersect in more than one point . The global topology of the manifold is responsible for this " failure " . In this we will define using the metric some special collections to being Euclidean . Let (M,g) be Riemannian manifold and U ,an open coordinate neighborhood with coordinate $x^1,\ldots,x^n$ .We will try to find a local change in coordinate $(x^i\to y^i)$ in which the expression of the metric is as close are to the Euclidean metric $g_0=i,j\,dy^i\,dy^j$. Let $q\in u$,be the point with coordinate $(0,\ldots,0)$ via a linear we may as well assume that $g_{ij}(q)=i,j$. We would like "spread" the above equality to an entire neighborhood of q . To achieve this we try to find local coordinates y^J near q such that in these new coordinates the metric is Euclidean up to order one i,e .

(35) $$g_{i,j}(q)=\frac{\partial g_{ij}}{\partial y^k}(q)=\frac{\partial_{i,J}}{\partial y^k}=\frac{\partial_{i,j}}{\partial y^k}(q)=0\ ,\ \forall:i,j,k\in g$$

We now describe a geometric way of producing such coordinates using the geodesic flow .Denote as usual the geodesic from q with initial direction $X\in T_q(M)$. By $X_q(t)$ Not the following simple fact $L\ \ X\in V$. Hence , there exists a small neighborhood V of $T_q(M)$, Such that , for any $X\in V$,the geodesic $X_q(t)$ is defined for all $|t|\le 1$ .we define the exponential map at q

(36) $$\exp_q:V\subset T_q(M)\to M\ \ ,\ X\to X_q(1)$$

The tangent space $T_q(M)$ is a Euclidean space , and we can define $D_q(r)\subset T_q(M)$, the open "disk" of radius r centered at the origin we have the following result centered at the origin .we have the following result

Proposition 3.3.1 Radii

Let (M,g) and $q\in M$ as above .Then there exists $r\ge 0$ such that the exponential map.

(37) $$\exp_q:D_q(r)\to M$$

Is a diffeomorphism on to .The supermom of all radii r with this property is denoted $P_M(q)$.

Definition 3.3.2 Injectivity Radius of M
The positive real number $P_M(q)$ is called the infectivity radius of M at q the in femur .

(38) $$P_M = \inf_q P_M(q)$$

Is called the infectivity radius of M

Lemma 3.3.2 Freshet Differential
The Freshet differential at $0 \in T_q(M)$ of the exponential map ,

$D_0 \exp_q : T_q(M) \to T \exp_q(0) M = T_q(M)$. Is the identity $T_q(M) \to T_q(M)$

Proposition 3.3.3 Metric Tensor
Let (x^i) be normal coordinates at $\in M$, and denote by g_{ij},the expression of the metric tensor in these coordinates then we have $g_{i,j}(q) = i, j$ and $\frac{\partial_{i,j}}{\partial x^k}(q) = 0 \quad \forall : i, j \in q$

Thus ,the normal coordinates provide a first order contact between g ,and the Euclidean metric .

Lemma 3.3.4
In normal coordinates (x^i) at (q) the christoffel symbols ik^i vanish at q

3.4 The Length Minimizing Property Of Geodesics
For each $q \in M$,there exists $0 \le r \le P_M(q)$ and $\varepsilon \ge 0$ such that $\forall m \in B_r(q)$, we have $\varepsilon \le P_M(M)$ and $B_\varepsilon(M) \subset B_r(q)$ in particular , any two of $B_r(q)$ can be joined by a unique geodesic of length $\le \varepsilon$. We must warn the reader the above result does not guarantee that the postulated connecting geodesic lies entirely in $B_r(q)$ .This is a different ball game .

Theorem 3.3.5 Unique Geodesic
Let q, r and ε as in the previous and consider the unique geodesic $r : [0,1] \to M$ of length $\le \varepsilon$,joining two points $B_r(q)$ .if $w : [0,1] \to M$ is a a piecewise smooth path with the same endpoint as then .

(39) $$\int_0^1 |(t)| \, dt \le \int_0^1 | w(t) | \, dt$$

With equality if and only if $w([0,1]) = ([0,1])$ .Thus is the shortest path, joining its endpoints .

3.4 Riemannian Geometry

Definition 3.4.1 Riemannian Metrics
Differential forms and the exterior derivative provide one piece of analysis on manifolds which , as we have seen , links in with global topological questions . There is much more on can do when on introduces a Riemannian metric . Since the whole subject of Riemannian geometry is a huge to the use of differential forms . The study of harmonic from and of geodesics in particular , we ignore completely hare questions related to curvature.

Definition 3.4.2 Metric Tensor
In informal terms a Riemannian metric on a manifold M is a smooth varying positive definite inner product on tangent space T_x . To make global sense of this note that an inner product is a bilinear form so at each point x , we want a vector in tensor product . $T_x^* \otimes T_x^*$ We can put , just as we did for exterior forms a vector bundle striation on. $T^*M \otimes T^*M = \bigcup_{x \in M} T_x^* \otimes T_x^*$. The conditions we need to satisfy for a vector bundle are provided two

facts we used for the bundle of p-forms each coordinate system $x_1,\ldots\ldots,x_n$ defiance a basis $dx_1,\ldots\ldots\ldots,dx_n$ for each T_x^* in the coordinate neighborhood and the n^2 element .

(40) $$dx_i \otimes dx_j \qquad 1 \le i, j \le n$$

Given a corresponding basis for $T_x^* \otimes T_x^*$. The Jacobean of a change of coordinates defines an invertible linear transformation . $J : T_x^* \to T_x^*$ and we have a corresponding .

(41) $$J \otimes J = T_x^* \otimes T_x^* \to T_x^* \otimes T_x^*$$

Definition 3.4.3 Local Coordinate System

A Riemannian metric on manifold M is a section g of $T_x^* \otimes T_x^*$ which at each point is symmetric and positive definite . In a local coordinate system we can write.

(42) $$g = \sum_{i,j} g_{ij}(x)\, dx_i dx_j$$

Where $g_{i,j}(x) = g_{j,i}(x)$ and is a smooth function , with $g_{i,j}(x)$ positive definite . Often the tensor product symbol is omitted and one simply writes. $g = \sum_{i,j} g_{ij}(x)\, dx_i dx_j$

Definition 3.4.4 Two Riemannian Manifold Is an Isometric

A diffehomorphism $F : M \to N$, between two Riemannian manifold is an isometric if $F^* g_N = g_N$

Definition 3.4.5 Upper half-plan

Let $M = \{(x, y) \in R^2 : y \ge 0\}$, and $g = \dfrac{dx^2 + dy^2}{y^2}$, if $Z = x + iy$ and

$F^* Y = Y \circ F = \dfrac{1}{i}\left(\dfrac{az+b}{cz+d} - \dfrac{a\bar{z}+b}{c\bar{z}+d}\right)$ then.

(43) $$FY = (ad-bc)^2 \frac{dx^2+dy^2}{\left|(cz+d)^2\right|^2} \cdot \frac{|xz+d|}{|ad-bc|^2 y} = \frac{dx^2+dy^2}{y^2} = g$$

S0 these Movies transformation are isometrics of Riemannian metric on the upper half-plan.

Definition 3.4.6 Smooth Curve in M

Let M be a Riemannian manifold and $\gamma : [0,1] \to M$ a smooth map i,e a smooth curve in M .

The length of curve is $L(\gamma)$ and $F(Z) = \dfrac{az+b}{cz+d}$

With a, b, c, d and $ad - bc \ge 0$, then $Fdz = (ad-bc)\dfrac{dz}{(cz+d)^2}$ and .

(44) $$Fg = (ad-bc)^2 \frac{dx+dy}{\left|(cz+d)^2\right|^2} \cdot \frac{|cz+d|^2}{(ad-bc)^2 y^2} = \frac{dx^2+dy^2}{y^2} = g$$

So these Movies transformation are isometrics of Riemannian metric on the upper half-plan.

Definition 3.4.7 A smooth Curve

Let M a Riemannian manifold and $\gamma : [0,1] \to M$ a smooth map i,e a smooth curve in M . The length of curve is $L(\gamma) = \int_0^1 \sqrt{g(\gamma, \gamma)}\, dt$. Where $\gamma'(t) = D_{\gamma_t}\left(\dfrac{d}{dt}\right)$, with this definition , any Riemannian manifold is metric space define .

(45) $$d(x, y) = \inf\{(\gamma) \in R : \gamma(t) = y\}$$

are Riemannian an manifold space.

Proposition 3.4.8 Manifold admits a Riemannian Metric

Any manifold a demits a Riemannian metric

Proof :

Take a converging by coordinate neighborhoods and a partition of unit subordinate to covering on each open set U_α we have a metric $g_\alpha = \sum_i dx_i^2$. In the local coordinates , define $g = \sum_i \varphi_i g_{\alpha(i)}$ this sum is well-defined because the support of φ_i. Are locally finite . Since $\varphi_i \geq 0$ at each point every term in the sum is positive definite or zero, but at least one is positive definite so that sum is positive definite.

Proposition 3.4.9 The Geodesic Flow

Consider any manifold M and its cotangent bundle $T^*(M)$, with projection to the base $p: T^*(M) \to M$, let X be tangent vector to $T^*(M)$ at the point $\zeta \in T_a^* M$ then $D_p(X) \in T^*(M)$ so that $\varphi(X) = \zeta_a(D_p(x))$ defines a conical a conical 1-form φ on $T^*(M)$ in coordinates $(x, y) \to \sum_i y_i \, dy$ the projection p is $p(x, y) = x$ so if $X = \sum a_i \frac{\partial}{\partial x_i} + \sum b_i \frac{\partial}{\partial y_i}$ so if given take the exterior derivative $w = -d\varphi = \sum dx_i \wedge dy_i$ which is the canonical 2-from on the cotangent bundle it is non-degenerate, so that the map $X \to (i \times w)$ from the tangent bundle of $T^*(M)$ to its contingent bundle is isomorphism. Now suppose f is smooth function an $T^*(M)$ its derivative is a 1-form df .Because of the isomorphism a above there is a unique vector field X on $T^*(M)$ such that $df = (i \times w)$ from the g another function with vector field Y , then .

(46) $$Y(t) = df\,(Y) = i_Y \cdot i^i X^w = -i\, X^i Y^w = -(X)_g$$

On a Riemannian manifold we shall see next there is natural function on $T^*(M)$. In fact a metric defines an inner on T^* as well as on T for the map $X \to g(X, -)$ defines an isomorphism form T to T^* then $g\left(\sum_j g_{ij} dx_j \times \sum_k g_{kl}\, dx_l\right) = g_{ik}$ which means that $g^*(dx_j, dx_k) = g^{ik}$ where g^{ik} denotes the matrix to g_{ik} we consider the function $T^*(M)$ defined by $H(\zeta_a) = g^*(\zeta_a, \zeta_a)$.

Definition 3.4.6 Geodesist Metric

The vector field X on $T^*(M)$ given by $I_i w = dH$ is called the geodesist flow of the metric g .

Definition 3.4.7 Geodesic Curve

If $\gamma: (a, b) \to T^*(M)$ Is an integral curve of the geodesic flow. Then the curve $P(\gamma)$ in (M) is called a geodesic . In locally coordinates, if the geodesic flow .

(47) $$X = \left(a_i \frac{\partial}{\partial x_i}\right) + \left(b_j \frac{\partial}{\partial y_j}\right)$$

Proposition 3.4.7 Projects Riemannian Manifold

The function f a above is If $f(\zeta_x) = \zeta_x(X_x)$

Proof :

Write in coordinates If $X = \sum a_i \left(\frac{\partial}{\partial x_i}\right) b_j \left(\frac{\partial}{\partial y_j}\right)$ where If $\phi = \sum_i y_i dx_i$ since $\tilde{X}$ projects on M then $X = \sum a_i \frac{\partial}{\partial x_i}$ by the definition of ϕ.Now let M be a Riemannian manifold and H , the function on

$T^*(M)$ defined by the metric as a above , if φ_t is an one parameter group of isometrics , then the induced diffeomorphisms of $T^*(M)$ will preserve the function H so the vector field $\tilde{Y}$ will satisfy $\tilde{Y}(H)=0$. that $X(f)=0$ where X is the geodesic flow a long the geodesic flow, and is therefore a constant of integration of the geodesic equations

3.5 Harmonic forms

We mentioned a above that a metric g , defines an inner product not just on T_a but also an inner product g^* on T_a^* , with this we can define an inner product on pth exterior power $T_a^*(\wedge^p)$:

(48) $$(\alpha_1\Lambda\alpha_2\Lambda........\Lambda\alpha_P, \beta_1\Lambda\beta_2\Lambda.........\Lambda\beta_P) = Det\ g^*(\alpha_i, \beta_i)$$

Thus if $dx_1 \Lambda\, dx_2\, \Lambda, ..., \Lambda\, dx_n$ defines the orientation $w=\sqrt{\det g_{ij}}\, dx_1\,\Lambda, ..., \Lambda\, dx_n$ on a compact manifold we can integrate this to obtain total volume – so a metric defines not only length but also volumes, Now take $\alpha \in \Lambda^P(T_a^*), \beta \in \Lambda^{n-P}(T_a^*)$ and define $f_\beta : \wedge^p T_a^* \to R, by\ \ f_\beta(\alpha)w = \beta \wedge \alpha$ .But we have an inner product , so any liner map on $\Lambda^P T_a^*$ is of the form $\alpha \to (\alpha, \gamma)$ for some $\gamma \in \Lambda^P(T_a^*)$ so we have a well –defined liner map $\beta \to \gamma\beta$ form $\Lambda^{n-P}(T_a^*)$ to $\Lambda^P(T_a^*)$ satisfying $(\gamma_\beta, \alpha)w = \beta\Lambda\alpha$.

Definition 3.5.1 Hodge Star Operator

The Hodge star operator is the linear map $*: \Omega^P(M) \to \Omega^{n-P}(M)$ with the property that at each point.

(49) $$(\alpha, \beta)w = \alpha\Lambda * \beta$$

Proposition 3.5.2 Compact Support M manifold

Let M be an oriented Riemannian manifold with volume for w , and let $\alpha \in \Omega^P(M)$, $\beta \in \Omega^{P-1}(M)$ be forms of compact support then . $\int_M (d^*\alpha, \beta)w = \int_M (\alpha, d\beta)w$

Definition 3.5.3 Deferential Laplacian on p-forms

Let M be an oriented Riemannian manifold , then the Laplacian on p-forms is the deferential operator $\Delta : \Omega^P(M) \to \Omega^P(M)$ defined by $\Delta : dd^* + d^*d$

Definition 3.5.4 Starting Point

A differential form $\alpha \in \Omega^P(M)$ is harmonic if $\Delta\alpha = 0$,on a compact manifold harmonic ply a important role, which there is no time to explore in this course
here is the starting point.

Definition 3.5.5 Harmonic and de Rham Manifold

Let M be a compact oriented Riemannian manifold then :

(i) a p-form is harmonic if and only if $d\alpha = 0$ and $d^*\alpha = 0$

(ii) In each de Rham cohomology class there is at most one harmonic from.

Theorem 3.5.6 The Fundamental Theorem of Riemannian Geometry

Suppose M is An m-dimensional smooth manifold , and G is a symmetric covariant tensor field of rank 2 on M if (U_i, u^i) is a local coordinate system on M then the tensor field G can be expressed as.

(50) $$G = g_{ij}du^i \otimes du^j$$

On U, where $g_{ij} = g_{ji}$ is a smooth function on U . U provides a bilinear function on $T_p(M)$ at every point $p \in M$ .Suppose $X = X^i \frac{\partial}{\partial u^i}, Y = Y^i \frac{\partial}{\partial u^i}$ then $G(X,Y) = g_{ij}X^iY^J$ .We say that the tensor G is no negated at the point if , whenever $X \in T_p(M)$ and $G(X,Y) = 0$. For all $Y \in T_p(M)$ it must be true that $X = 0$ this implies that G is no degenerate at p if and only if the system of linear equations $g_{ij}(p)X^i = 0 \quad 1 \le j \le m$ has zero as its only solution (i,e) det $(g_{ij}(p)) \neq 0$ if for all $X \in T_p(M)$ we have $G(X,Y) \ge 0$ And the equality holds only if $X = 0$ then we say G is positive definite at p . From liner algebra a necessary and sufficient condition for G to be positive definite that matrix (g_{ij}) is positive definite . Thus a positive definite tensor G is necessarily non degenerate .

Definition 3.5.7 Generalized Tensor is Riemannian

If an m-dimensional smooth manifold M is given a smooth every no degenerate symmetric covariant tensor field of rank-2 , G then M is called a generalized tensor or metric tensor or metric of M . If G is positive definite then M is called Riemannian manifold.

(Not) : for a generalized Riemannian manifold M ,$G = g_{ij} du^i \otimes du^j$ specifies an inner product on the tangent space $T_p(M)$ at every point $p \in M$ for any $X,Y \in T_p(M)$. let $X.Y = G(X.Y) = g_{ij}(p)X^iY^j$ When G is positive definite, it is meaningful to define the length of a tangent vector and the angle between two tangent vectors at the some point $|X| = \sqrt{g_{ij}X^iX^J}$. Thus a Riemannian manifold is a differentiable manifold which has a positive definite inner product on the tangent space at every point . The inner product is required to smooth X,Y are smooth tangent vector fields then X,Y is a smooth on M

Definition 3.5.8 Smooth Parametrzel Curve

$dS^2 = g_{ij} du^i du^j$ is independent of the choice of the local coordinate system u^i and usually called the metric form or Riemannian metric (dS) is precisely the length of an infinitesimal tangent vector and is called the element of are length . Suppose a $C = u^i = u^i(t)$ and $t_0 \le t \le t_1$ is a continuous and piecewise smooth parameterized curve on M ,then the are length of C is defined to be .

(51) $$S = \int_{t_0}^{t_1} \sqrt{g_{ij} \frac{du^i}{dt} \frac{du^j}{dt}}\, dt$$

Remark 3.5.9 Exist a smooth is nonzero everywhere

The existence of a Riemannian metric on a smooth manifold is an extraordinary result. In general there may not exist a non-positive . In the context of fiber bundles , the existence of a Riemannian metric on M implies the existence of a positive definite smooth of bundle of symmetric covariant tensor of order 2-on M, However for arbitrary vector bundles there may not exist a smooth which is
nonzero everywhere.

Theorem 3.5.10 Fundamental of Riemannian Geometry

Suppose M is an m-dimensional generalized Riemannian manifold then there exists a unique tensor – Free and metric compatible connection on M , called the (Levi-civita connectin of M) o (Riemannian connection of M)

Proof :

Suppose D is a torsion-free and metric – compatible connection on M , denote the connection matrix of D under the local coordinates U^i by $W = (W_i^j)$ where $W_i^j = \Gamma_{ik}^j du^k$ .Then we have $dg_{ij} = W_i^j g_{kj} + W_j^k g_{ki}$, and $\Gamma_{ik}^j = \Gamma_{kj}^j$ Denote that $\Gamma_{ik}^j = g_{ij}\Gamma_{kj}^j$, $W_{ik} = g_{lk}W_i^l$. Then its follows from (49) and (50) we get that .

(52) $$\frac{\partial g_{ij}}{\partial u^k} = \Gamma_{jik} + \Gamma_{ikj}^j$$

$\Gamma_{jik} = \Gamma^{j}_{ikj}$ is cycling the indices in (51) we get $\frac{\partial g_{ik}}{\partial u^{j}} = \Gamma_{jik} + \Gamma^{j}_{ikj}$ and $\frac{\partial g_{ik}}{\partial u^{i}} = \Gamma_{jik} + \Gamma^{j}_{ikj}$ .And calculating (50)and (51) we then obtain .

(53) $$\Gamma_{ij} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u_{j}} + \frac{\partial g_{jk}}{\partial u_{i}} - \frac{\partial g_{jk}}{\partial u_{k}}\right) \qquad \Gamma^{k}{}_{ji} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{il}}{\partial u^{j}} + \frac{\partial g_{jl}}{\partial u^{i}} - \frac{\partial g_{ij}}{\partial u^{l}}\right)$$

The equation is (Levi-civita connectin of M) or (Riemannian connection of M)

3.6 The Spectral Geometry of operators of Dirac and Laplace Type

We have also given in each a few additional references to relevant . The constraints of space have of necessity forced us to omit many more important references that it was possible to include and we a apologize in a dance for that . We a the following notational conventions , let (M, g) (be compact Riemannian manifold of dim. M with boundary ∂M .Let Greek indices γ, μ range from $1\, to\, m$, and index a local system of coordinates $x = (x^1, \ldots\ldots\ldots, x^m)$ on the interior of M expand the metric in the form $dS^2 = g_{\mu\gamma} dx^{\mu} dx^{\gamma}$ were $g_{\mu\gamma} = g(\partial_{x_\mu}, \partial_{x_\gamma})$ and where we adopt the Einstein convention of summing over repeated indices we let $g^{\mu\gamma}$ be the inverse matrix the Riemannian measure is given by $dx = (dx^1, \ldots\ldots\ldots, dx^m)$ for $g = \sqrt{\det(g_{\mu\gamma})}$ let ∇ be the" levi-Civita" connection. We expand $\nabla_{\partial x,\gamma} . \partial_{x\mu} = \Gamma_{\gamma\mu} \sigma \partial x_{\sigma}$. Where $\Gamma_{\gamma\mu}\sigma$ are the m, R are may then be given by. $R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ and given by .

(54) $$R(X,Y,Z,W) = g(R(X,Y),Z,W)$$

We shall let Latin indices i, j range from 1 to m and index a local orthonormal frame $(e_1, \ldots\ldots\ldots, e_m)$ for the components of the curvature tensor scalar curvature τ . Are then given by setting $P_{ij} = R_{ijkk}$ *and* $\tau = P_{ij} = R_{ikki}$. We shall often have an auxiliary vector bundle set V and an auxiliary given on V , we use this connection and the " Levi-Civita" connection to covariant differentiation , let dy be the measure of the induced metric on boundary ∂M , we choose a local orthonormal from near the boundary M , so that (e_m) is the inward unit normal . We let indices a,b range from 1 to m-1 and index the induced local frame $(e_1, \ldots\ldots\ldots, e_{m-1})$ for the tangent bundle f the boundary , let $L_{a,b} = {}_g(\nabla_{e_a} eb, e_m)$ denote the second fundamental form . we some over indices with the implicit range indicated . Thus the geodesic curvature K_g is given by $K_g = L_{aa}$. We shall let denote multiple tangential covariant differentiation with respect to the " Levi-Civita" connection the boundary the difference between and being of course measured by the fundamental form.

3.7 The Geometric of Operators of Laplace and Dirac Type

In this section we shall establish basic definitions discuss operator of Laplace and of Dirac type introduce the De-Rham complex and discuss the Bochner Laplacian and the weitzenboch formula .

Let D be a second of smooth sections $C^{\infty}(v)$ of a vector bundle v over space M , expand $D = -(g^{\mu\nu} \partial x_{\mu} \partial x_{\nu} + a^{\sigma} \partial x_{\sigma} + b)$ where coefficient $(g^{\mu\nu}, a^{\mu}, b)$ are smooth endomorphism's of v, we suppress the fiber indices . We say that D is an operator of Laplace type if A^2 ,on $C^{\infty}(v)$ is said to be an operator of Dirac type if A^2 is an operator of laplace operator of Dirac type if and only if the endomorphism's γ^{ν} satisfy the Clifford commutation relations $\gamma^{\nu}\gamma^{\mu} + \gamma^{\mu}\gamma^{\nu} = -2g^{\nu\mu}(id)$. Let A be an operator of Dirac type and let $\zeta = \zeta_{\nu} dx^{\nu}$ be a smooth 1-form on M we let $\gamma(\zeta) = \zeta_{\nu} v^{\nu}$ define a Clifford module structure on V . This is independent of the particular coordinate system chosen .

We can always choose a fiber metric on so that γ is skew adjoin . We can then construct a unitary connection ∇ on V so that $\nabla\gamma=0$ such that a connection is called compatible the endomorphism if ∇ is compatible we expand $A=\gamma\nabla^{v}_{\partial x_v}+\psi_A$, ψ_A is tonsorial and does not depend on the particular coordinate system chosen it does of course depend on the particular compatible connection chosen.

Definition3.7.1 The De-Rham Complex

The prototypical example is given by the exterior algebra , let $C^{\infty}(\Lambda^p M)$ be the space of smooth p forms. Let $d:C^{\infty}(\Lambda^p M)\to C^{\infty}(\Lambda^{p-1}M)$ be exterior differentiation if ζ is cotangent vector , Let $ext(\zeta):w\to\zeta\Lambda w$ denote exterior multiplication and let $\operatorname{int}(\zeta)$ be the Dual , Interior multiplication, $v(\zeta)=ext(\zeta)-\operatorname{int}(\zeta)$ define module on exterior algebra $\Lambda(M)$. Since $d+\delta=v(dx^v)\nabla_{\partial x_v}$. $d+\delta$ is an operator of " Diract type" the a associated laplacian $\Delta_m=(d+\delta)^2=\Delta^0_m\oplus......\oplus\Delta^p_m\oplus.....\oplus\Delta^m_M$ decomposes as the "Direct sum" of operators of laplace type Δ^p_m on the space of smooth p forms $C^{\infty}(\Lambda^p M)$ on has $\Delta^0_M=-g^{-1}\partial x_\mu g\,g^{\mu v}(\partial x_v)$ it is possible to write the p-form valued Laplacian in an invariant form . Extend the " Levi-Civita" comection to act on tensors of all types .Let $\tilde{\Delta}_{M^w}=-g^{\mu v}w$, μv define Bochner or reduced Laplacian , let R given the associated action of curvature tensor . The "Weitzenbock" formula terms of the " Bochner Laplacian" in the form

$$\Delta_M=\tilde{\Delta}_M+\frac{1}{2}\gamma(dx^\mu)\gamma(dx^\gamma)R_{\mu\gamma} \tag{55}$$

This formalism can be applied more generally.

Lemma 3.7.2 Spinner Bundle

Let D be an operator of Laplace type on a Riemannian manifold, there exists a unique connection ∇ on V and there exists a unique endomorphism E of V , so that $D\phi=-\phi_{;ii}-E\phi$ if we express D locally in the form $D=-(g^{\mu\gamma}\partial x_\gamma\partial x_\mu+a^\mu\partial x_\mu+b)$ then the connection 1-form w of ∇ and the endomorphism E are given by .

$$w_\gamma=\frac{1}{2}(g_{\gamma\mu}a^\mu+g^{\sigma E}\Gamma_{\sigma E_\gamma}id)\,and\;E=b-g^{\gamma\mu}(\partial x_\gamma w_\mu+w_\gamma w_\mu-w_\sigma\Gamma^\sigma{}_{\gamma\mu}) \tag{56}$$

Let V be equipped with an auxiliary fiber metric, then D is self-adjoin if and only if ∇ is unitary and E is self-adjoin we note if D is the Spinner bundle and the "Lichnerowicz formula" with our sign convention that $E=-\frac{1}{4}J(id)$ where J is the scalar curvature.

Definition 3.7.3 Heat Trace Asymptotic for closed manifold

Throughout this section we shall assume that D is an operator of Laplace type on a closed Riemannian manifold (M,g). We shall discuss the L^2 - spectral resolution if D is self adjoin , define the heat equation introduce the heat trace and the heat trace asymptotic present the leading terms in the heat trace

Asymptotic references for the material of this section and other references will be cited as needed , we suppose that D is self-adjoin there is then a complete spectral resolution of D on $L^2(V)$. This means that we can find a complete orthonormal basis $\{\phi_n\}$ for $L^2(V)$ where the ϕ_n are a smooth sections to V which satisfy the equation $D\phi_n=\lambda_n\phi_n$.

3.8 Inverse Spectral Problems in Riemannian Geometry

In al-arguably one the simplest inverse problem in pure mathematics " can on hear the shape of drum " mathematically the question is formulated as follows , let Ω be a simply connected plane domain (The drumhead bounded by a smooth curve γ) , and consider the wave equation on Ω with . Dirichlet boundary condition on γ - (the drumhead is clamped at boundary) .

(57) $$\Delta u(x,t) = \frac{1}{C^2} U_{tt}(x,t) \ \ in\ \Omega \quad , \quad U(x,t) = 0 \quad in\ \gamma$$

The function $U(x,t)$ is the displacement of drumhead as vibrates at position x at time t , looking for solutions of the form $U(x,t) = \operatorname{Re} e^{iwt} v(x)$ (normal modes) leads to an eigenvalue problem for the Dirichlet Laplacian on Ω $\Delta v(x) + \lambda v(x) = 0 \ \ in\ \Omega \ , \ v(x) = 0 \quad on \gamma$

Where $\lambda = w^2 / c^2$, we write the infinite sequins of Dirichlet eigenvalues for this problem as $\{\lambda_n(\Omega)\}_{n=1}^{\infty}$ or simply $\{\lambda_n\}_{n=1}^{\infty}$, if the choice of domain Ω is clear in context , Kans question means the following is it possible to distinguish " drums " Ω_1 and Ω_2 with distinct (modulo isometrics) bounding curves γ_1 and γ_2 simply by (hearing) all of the eigenvalues of Dirichlet Laplacian some surprising and interesting results are obtained by considering the heat equation on Ω with Dirichlet boundary conditions, which given rise to the same boundary value problem as before the heat equation is .

(58) $$\begin{aligned} &\Delta U(x,t) = U_t(x,t) && in\ \Omega \\ &U(x,t) = 0 && on\ \gamma \\ &U(x,0) = f(x) && \end{aligned}$$

Where $U(x,t)$ is the temperature at point x and time t , and f(x) is the initial temperature distribution. This evolution equation is formal solution. $U(x,t) = (e^{t\Delta} f)(x)$. Where the operator $e^{t\Delta}$ can be calculated using the spectral resolution of Δ . Indeed if $\phi_j(x)$ is the the normalized eigenfunction of the boundary value problem with eigenvalue λ_j the operator $e^{t\Delta}$ has integral Kernel $k(t,x,y)$ the heat Kernel given by .

(59) $$k(t,x,y) = \sum_{j=1}^{\infty} e^{-t\lambda_j} \phi(x) \phi(y)$$

The trace of $k(t,x,y)$ is actually a spectral in variant by (59) we can compute .

(60) $$k(t,x,y) = \sum_{j=1}^{\infty} e^{-t\lambda_j}$$

(Not) that the function (60) determines the spectrum $\{\lambda_n\}_{n=1}^{\infty}$, to analyze the geometric content of spectrum, one calculates the by completely different method one constructs the heat kernel by perturbation from the explicit heat kernel for the plane, and then on computes the trace explicitly . It turns out that the trace has a small-t asymptotic expansion.

(61) $$\int_{\Omega} k(t,x,t)\,dx \approx \frac{1}{4\pi t}\left(a_0 + a_{1/2} + a_1 t + \ldots\ldots\ldots\right)$$

Where $a_0 = area(\Omega)$, $a_1 = length(\gamma)$, Al though a strict derivation is a bit involved which shows why a_0 and a_1 should given the area of Ω and length of γ the heat kernel in the plan is . $k_0(x,y,t) = \frac{1}{4\pi t} \exp\left(|x-y|^2 / 4\pi t\right)$, we expect particle that , for small times $K(t,x,t) \approx k_0(t,x,t)$ (a Brownian particle starting out the interior doesn't the boundary for a time of order $\sqrt{t}$).

(62) $$\int_{\Omega} K(t,x,t)\,dx \approx \int_{\Omega} k_0(t,x,t)\,dx = \frac{1}{4\pi t}\ are\ (\Omega)$$

For times of order $\sqrt{t}$, boundary effects become important we can approximate the heat kenrnel near the boundary locally by (method images) locally the boundary looks the line $x_1 = 0$ *in the* $x_1 - x_2$ plane , letting $x \to x^*$ be , $K_2(x,x,t) = k_0(x,x,t) - k(x,y,t^*)$ vanishes $x_1 = 0$ hence $K_\Omega(x,x,t) \approx \frac{1-e^{-2\delta^{2/t}}}{4\pi t}$ where δ. Is the distance from x to the boundary , writing the volume integral for the additional term as an integral over the boundary curve and distance from the boundary $\int_\gamma \int_0^\infty \frac{1}{4\pi t} e^{-2\delta/t} d\delta\, ds$ we have.

(63) $$\int_\Omega K(x,x,t)\,dx \approx \frac{area\,(\Omega)}{4\pi t} - \frac{length\,(\gamma)}{4} \frac{1}{\sqrt{2\pi t}} + \left(\frac{1}{\sqrt{t}}\right)$$

it follows that the is spectral set of a given (drum) Ω contains only drums with the same area and perimeter here we will briefly discuss the generalization of kais problem and some of the known results. A Riemannian manifold of dimension n is a smooth n-dimensional manifold M . equipped with a Riemannian metric g which defines the length of tangent vectors and determines distances and angles on the manifold . The metric also determines the Riemann curvature tensor of M . In two dimensions , the Riemannian curvature tensor is in turn determined by the scalar curvature, and in three dimensions it is completely determined by the Ricci curvature tensor. If M is compact the associated Laplacian has infinite set of discrete eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ what is the geometric content of the spectrum for a compact Riemannian manifold. Constructs a pair 16-dimensional torii with with the same spectrum . The torii T_1^n *and* T_2^n are quotients of R^n by lattices Γ_1 *and* Γ_2 of translations of R^n. Since the tow torii are isometric of and only if their lattices are congruent, it suffices to construct a pair of non-congruent 16-dimensional lattices whose a associated torii have the same spectrum .To understand the analysis involved in Milnor's construction consider the following simple " trace formula" for a torus $T^n = R^n/\Gamma$ which computes the trace of the heat kernel on a torus in terms of lengths of the lattice vectors to the heat kernel on the torus is given by the formula.

(64) $$K_\Gamma(x,x,t) = \sum_{w\in\Gamma} k_0(x+w,y,t) \quad \text{Where} \quad K_0(x,x,t)\,dx = \frac{vol(T^n)}{4\pi t} \sum_{w\in\Gamma} e^{-|w|^2/4t}$$

Milnor noted that there exist non-congruent lattices in 16-dim. With the same set of " length " $\{|w| : w \in \Gamma\}$ first discovered by the trace of the heat kernel determines the spectrum and the heat trace is in turn determined by the lengths, it follows that the corresponding non-isometric torii have the same spectrum.

Example 3.8.1 Riemannian Manifold with Same Spectrum

Riemannian manifold with the same spectrum letter constructed continuous families of is spectral manifold in sufficiently high dimension $n \geq 5$ two major questions remained:

(i) can one show that the is spectral set of given manifold at finite in low dimension .

(ii) can on find counterexamples for Kicks original problem , can one construct is spectral , non-congruent planar .

Definition 3.8.2 Some Positive Results

In proved one of the first major positive results on is spectral sets of surfaces and planar domains informally. A sequence of planar domains Ω_j converges in C^∞ since to a limiting non-degenerate set compact surfaces S_j converges in C^∞ sense to limiting non-degenerate surface S , converge in C^∞ sense to a positive definite metric on S .

Theorem 3.8.3

(i) Let Ω_j be a sequence of is spectral planar domains there is a subsequence which converges in C^∞ sense to no degenerate limiting surface.

(ii) Let S_j be a sequence of is spectral compact surfaces there is a subsequence of the S_j converging in C^∞ sense to a non-degenerate surface S .

3-5-5 Theorem : [B]

Suppose M_j is a sequence if is spectral manifold such that either .

(i) All of the M_j - have negative sectional curvatures .

(ii) All of the M_j -have Ricci curvatures bounded below .

Then there are finitely many diffeomorphism types and there is a subsequence which convergent in C^∞ to a no degenerate limiting manifold .

(65) $$\frac{d\gamma}{dt}(t)=\left(\frac{d\gamma^1}{dt}t_0,\ldots.,\frac{d\gamma^n}{dt}t_0\right)$$

we many k bout smooth curves that is curves with all continuous higher derivatives cons the level surface $f(x^1,x^2,\ldots,x^n)=c$ of a differentiable function f where x^i to $(i-th)$ coordinate the gradient vector of f at point $P=x^1(P),x^2(P),\ldots.,x^n(P)$ is $\nabla f=\left(\frac{\partial f}{\partial x^1},\ldots\ldots,\frac{\partial f}{\partial x^n}\right)$ is given a vector $u=(u^1,\ldots,u^n)$ the direction derivative $D_u f=\nabla f\cdot\overline{u}=\frac{\partial f}{\partial x^1}u^1+\ldots+\frac{\partial f}{\partial x^n}u^n$, the point P on level surface $f(x^1,x^2,\ldots,x^n)$ the tangent is given by equation.

(66) $$\frac{\partial f}{\partial x^1}(P)(x^1-x^1)(P)+\ldots.+\frac{\partial f}{\partial x^n}(P)(x^n-x^n)(P)=0$$

For the geometric views the tangent space shout consist of all tangent to smooth curves the point P , assume that is curve through $t=t_0$ is the level surface.

(63) $$f(x^1,x^2,\ldots,x^n)=c\ ,\ f(\gamma^1(t),\gamma^2(t),\ldots.,\gamma^n(t))=c$$ by

taking derivatives on both $\frac{\partial f}{\partial x^1}(P)\gamma'(t_0)+\ldots.+\frac{\partial f}{\partial x^n}(P)\gamma^n(t))=0$ and so the tangent line of γ is really normal orthogonal to ∇f , where γ runs over all possible curves on the level surface through the point P. The surface M be a C^∞ manifold of dimension n with $k\geq 1$ the most intuitive to define tangent vectors is to use curves , $p\in M$ be any point on M and let $\gamma:]-\varepsilon,\varepsilon[\to M$ be a C^1 curve passing through p that is with $\gamma(M)=p$ unfortunately if M is not embedded in any R^N the derivative $\gamma'(M)$ does not make sense ,however for any chart (U,φ) at p the map $(\varphi\circ\gamma)$ at a C^1 curve in R^n and tangent vector $v=(\varphi\circ v)'(M)$ is will defined the trouble is that different curves the same v given a smooth mapping $f:N\to M$ we can define how tangent vectors in T_pN are mapped to tangent vectors in T_qM with (U,φ) choose charts $q=f(p)$ for $p\in N$ and (V,ψ) for $q\in M$ we define the tangent map or flash-forward of f as a given tangent vector. $X_p=[\gamma]\in T_pN$ and $d\,f_*:T_pM$, $f_*([\gamma])=[f\circ\gamma]$

A tangent vector at a point p in a manifold M is a derivation at p, just as for R^n the tangent at point p form a vector space $T_p(M)$ called the tangent space of M at p, we also write $T_p(M)$ a differential of map $f:N\to M$ be a C^∞ map between two manifolds at each point $p\in N$ the map F induce a linear map of tangent space called its differential p, $F_*:T_pN \to T_{F(p)}N$ as follows it $X_p \in T_pN$ then $F_*(X_p)$ is the tangent vector in $T_{F(p)}M$ defined

(64) $$\left(F_*(X_p)\right)f = X_p\left(f \circ F\right) \in R \quad , \quad f \in C^\infty(M)$$

The tangent vectors given any C^∞ - manifold M of dimension n with $k \geq 1$ for any $p \in M$,tangent vector to M at p is any equivalence class of C^1 - curves through p on M modulo the equivalence relation defined in the set of all tangent vectors at p is denoted by T_pM we will show that T_pM is a vector space of dimension n of M . The tangent space T_pM is defined as the vector space spanned by the tangents at p to all curves passing through point p in the manifold M , and the cotangent T_p^*M of a manifold at $p \in M$ is defined as the dual vector space to the tangent space T_pM , we take the basis vectors $E_i = \left(\frac{\partial}{\partial x^i}\right)$ for T_pM and we write the basis vectors T_p^*M as the differential line elements $e^i = dx^i$ thus the inner product is given by.

(65) $$\left\langle \partial/\partial x, dx^i \right\rangle = \delta_i^j \ .$$

3.1 Definition

Let M_1 and M_2 be differentiable manifolds a mapping $\varphi: M_1 \to M_2$ is a differentiable if it is differentiable , objective and its inverse φ^{-1} is diffeomorphism if it is differentiable φ is said to be a local diffeomorphism at $p \in M$ if there exist neighborhoods U of p and V of $\varphi(p)$ such that $\varphi:U \to V$ is a diffeomorphism , the notion of diffeomorphism is the natural idea of equivalence between differentiable manifolds , its an immediate consequence of the chain rule that if $\varphi: M_1 \to M_2$ is a diffeomorphism then.

(66) $$d\varphi:T_pM_1 \to T_{\varphi(p)}M_2$$

Is an isomorphism for all $\varphi: M_1 \to M_2$ in particular , the dimensions of M_1 and M_2 are equal a local converse to this fact is the following $d\varphi: T_pM_1 \to T_{\varphi(p)}M_2$ is an isomorphism then φ is a local diffeomorphism at p from an immediate application of inverse function in R^n, for example be given a manifold structure again A mapping $f^{-1}: M \to N$ in this case the manifolds N and M are said to be homeomorphism , using charts (U,φ) and (V,ψ) for N and M respectively we can give a coordinate expression $\tilde{f}: M \to N$

3.2 Definition

Let M_1^{-1} and M_2^{-1} be differentiable manifolds and let $\varphi: M_1 \to M_2$ be differentiable mapping for every $p \in M_1$ and for each $v \in T_pM_1$ choose a differentiable curve $\alpha:(-\varepsilon,\varepsilon) \to M_1$ with $\alpha(M) = p$ and $\alpha'(0) = v$ take $\alpha \circ \beta = \beta$ the mapping $d\varphi_p: T_\varphi(p)M_2$ by given by $d\varphi(v) = \beta'(M)$ is line of α and $\varphi: M_1^{-1} \to M_2^{-1}$ be a differentiable mapping and at $p \in M_1$ be such $d\varphi: T_pM_1 \to T_\varphi M_2$ is an isomorphism then φ is a local homeomorphism .

3.3 Proposition

Let M_1^n and M_1^m be differentiable manifolds and let $\varphi: M_1 \to M_2$ be a differentiable mapping , for every $p \in M_1$ and for each $v \in T_pM_1$ choose a differentiable curve $\alpha:(-\varepsilon,\varepsilon) \to M_1$ with

$\alpha(o) = p$, $\alpha'(o) = v$ take $\beta = \varphi \circ \alpha$ the mapping $d\varphi : T_p M_1 \to T_{\varphi(p)} M_2$ given by $d\varphi_p(v) = \beta'(o)$ is a linear mapping that dose not depend on the choice of α .

3.5Theorem

The tangent bundle TM has a canonical differentiable structure making it into a smooth 2n-dimensional manifold , where N=dim. The charts identify any $U_p \in U(T_p M) \subseteq (TM)$ for an coordinate neighborhood $U \subseteq M$, with $U \times R^n$ that is hausdorff and second countable is called (The manifold of tangent vectors)

Definition 4.6

A smooth vectors fields on manifolds M is map $X : M \to TM$ such that

(i) $X(P) \in T_p M$ for every G .

(ii) in every chart X is expressed as $a_i\,(\partial / \partial x_i)$ with coefficients $a_i(x)$ smooth functions of the local coordinates x_i .

4.7 Theorem

Suppose that on a smooth manifold M of dimension n there exist n vector fields $\{x^{(1)}, x^{(2)}, \ldots., x^{(n)}\}$ for a basis of $T_p M$ at every point p of M , then $T_p M$ is isomorphic to $M \times R^n$ m here isomorphic means that TM and $M \times R^n$ are homeomorphism as smooth manifolds and for every $p \in M$, the homeomorphism restricts to between the tangent space $T_p M$ and vector space $\{a_i\} R^n$.

Proof:

define $\pi : \vec{a} \in T_p M \subset TM$ on other hand , for any $M \times R^n$ for some $a_i \in R$ now define $\Phi : \vec{a} \in TM \to \{\pi(s) : a_1, \ldots., a_n \in M \times R^n\}$ is it clear from the construction and the hypotheses of theorem that Φ and Φ^{-1} are smooth using an arbitrary chart $\varphi : U \subseteq M \to R^n$ and corresponding chart

(67) $$\varphi T : \pi^{-1}(U) \subseteq TM \to R^n \times R^m$$

IV. GET PEER REVIEWED

I The basic notions on differential geometry knowledge of calculus , including the geometric formulation of the notion of the differential and the inverse function theorem.

II A certain familiarity with the elements of the differential Geometry of surfaces with the basic definition of differentiable manifolds , starting with properties of covering spaces and of the fundamental group and its relation to covering spaces .

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