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A THIRD ORDER FOUR POINT MIXED BOUNDARY VALUE PROBLEM

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Abstract: In this article a more generalized four point boundary value problem associated with a third order differential equation involving mixed boundary conditions is proposed. First we constructed the Green's function for the associated third order three point boundary value problem through undetermined parametric method and then the idea is extended to the third order four point boundary value problem with mixed boundary conditions. Using this idea and an Iteration method is proposed to solve the corresponding nonlinear problem.

Key Words: Green's function, Schauder fixed point theorem, Vitali's convergence theorem.

AMS MSC: 34B18, 34B99, 35J05

1. INTRODUCTION:

Mixed boundary value problems arise in many areas of applied problems like modeling of nonlinear diffusion via nonlinear sources, chemical concentration in biological problems, thermal conduction of heat, electromagnetic conduction of thermal power etc. Non local boundary value problems raise much attention because of its ability to accommodate more boundary points than their corresponding order of differential equations [2], [6], [12]-[15]. Considerable studies were made by Bai and Fag [2], Gupta [4] and Web [9]. This research article is concerned with solving the nonlocal third order four point boundary value problem with mixed type boundary conditions

$$u'''(t)+a(t)f(t,u(t))=0 \qquad (1.1)$$

$$u(a)=0 \ , \quad u(a)+u'(a)=k_1 u(\eta_1) \ , \quad u(b)+u'(b)=k_2 u(\eta_2) \qquad (1.2)$$

where $f: [a, b] \times \mathbb{R} \to \mathbb{R}$ is a given function, $a < \eta_1 < \eta_2 < b$ and $k_1, k_2 \in R$

The Green's function plays an important role in solving boundary value problems of differential equations through integral equations. The exact expressions of the solutions for some linear ODEs boundary value problems can be expressed by the corresponding Green's functions of the problems. The Green's function method will be used to obtain an initial estimate for shooting method. The Greens function method for solving the boundary value problem is an effect tools in numerical experiments. Some BVPs for nonlinear integral equations the kernels of which are the Green's functions of corresponding linear differential equations. The undetermined parametric method we use in this paper is a universal method, the Green's functions of many boundary value problems for ODEs can be obtained by similar method.

In (2008), Zhao discussed the solutions and Green's functions for non local linear second-order Three-point boundary value problems.

$u'' + f(t) = 0, \quad t \in [a, b]$

subject to one of the following boundary value conditions:

i. $u(a) = ku(\eta), \quad u(b) = 0$ ii. $u(a) = 0, \quad u(b) = ku(\eta)$ iii. $u(a) = ku'(\eta), \quad u(b) = 0$

iv. $u(a) = 0, \ u(b) = ku'(\eta)$ where k was the given number and $\eta \in (a, b)$ is a given point.

In (2013), Mohamed investigate the positive solutions to a singular second order boundary value problem with more generalized boundary conditions. He consider the Sturm-Liouville boundary value problem

$u'' + \lambda g(t) f(t) = 0, \quad t \in [0,1]$ with the boundary conditions

$\propto u(0) - \beta u'(0) = 0, \ \gamma u(1) + \delta u'(1) = 0$

where $\propto > 0, \ \beta > 0, \ \gamma > 0 \ and \ \delta > 0$ are all constants, λ is a positive parameter and $f(.)$ is singular at $u = 0$.

Also the existence of positive solutions of singular boundary value problems of ordinary differential equations has been studied by many researchers such as **Agarwal and Stanek** established the existence criteria for positive solutions singular boundary value problems for nonlinear second order ordinary and delay differential equations using the Vitali's convergence theorem. **Gatical et al** proved the existence of positive solution of the problem

$u'' + f(t) = 0, \quad t \in [0,1]$ with the boundary conditions

$\propto u(0) - \beta u'(0) = 0 \ , \ \gamma u(1) + \delta u'(1) = 0$

using the iterative technique and fixed point theorem for cone for decreasing mappings.

Recently **Goteti V.R.L. Sarma** et al., studied the solvability of a second order four point boundary value problem with ordinary boundary conditions $u'' + f(t) = 0, \quad t \in [a,b]$ satisfying the boundary conditions

$u(a) = k_1 u(\eta_1), \ u(b) = k_2 u(\eta_2);$ where $a < \eta_1 < \eta_2 < b \ and \ k_1 \ and \ k_2$ are real constants.

This article generalizes the existing results and paves the way to study associated multi point boundary value problems and periodic boundary value problems also.

This article is organized as follows: In section 2 we considered associated third order three point mixed boundary condition and a method to its Green's function is proposed and solved. In section 3 we considered the main non local third order four point mixed boundary value problem and it is solved with the help of suitable Green's function which was constructed under the idea in section 2. We illustrated our results by constructing a suitable example.

Section 2. A Third Order Three Point Mixed Boundary Value Problem:

In this section a third order three point boundary value problem

$$u'''(t) + a(t) f(t,u) = 0, \quad t \in [a,b], \tag{2.1}$$

$$u(a) = 0, \qquad u(a) + u'(a) = k_1 u(\eta), \qquad u(b) + u'(b) = k_2 u(\eta) \tag{2.2}$$

where $f: [a, b] \times \mathbb{R} \to \mathbb{R}$ is a given function $a < \eta < b$ and $k_1, k_2 \in R$

is solved by constructing its green's function through the method of undetermined coefficients. This method is verified through a suitable example.

Theorem 2.1 The Green's function for (2.1), (2.2) is given by

$$G(t,s) = K(t,s) + \frac{\left[(k_1)(b-a+1)-k_2\right]t^2+\left[(k_1)\left(a^2-2b-b^2\right)+2ak_2\right]t+(k_1)\left(-a^2b-a^2+2ab+ab^2\right)-a^2k_2}{\left(2a-k_1(\eta^2-a^2)\right)\left(b-a+1-k_2(\eta-a)\right)-\left(1-k_1(\eta-a)\right)\left(b^2-a^2+2b-k_2(\eta^2-a^2)\right)}K(\eta,s) \tag{2.3}$$

Where
$$K(t,s)=\begin{cases}\dfrac{(b-s)(b-s+2)(t-a)^2}{2(b-a)(b-a+2)}, & a\le t\le s\le b\\[2ex] \dfrac{(b-s)(b-s+2)(t-a)^2}{2(b-a)(b-a+2)}-\dfrac{1}{2}(t-s)^2, & a\le s\le t\le b\end{cases} \tag{2.4}$$

Proof: First consider the corresponding homogeneous equation with homogeneous boundary conditions as

$$u'''(t)=0, \quad , \quad u(a)=0, \quad u(a)+u'(a)=0, \quad u(b)+u'(b)=0 \tag{2.5}$$

Let us take the Green's function in the form
$$K(t,s)=\begin{cases}A_1(s)t^2+B_1(s)t+C_1(s), & a\le t\le s\le b\\ A_2(s)t^2+B_2(s)t+C_2(s), & a\le s\le t\le b\end{cases}$$

From the properties of Green's function

$$A_2 - A_1 = \frac{-1}{2}$$

$$2\left(A_2-A_1\right)s+\left(B_2-B_1\right)=0 \quad \Rightarrow \quad B_2=B_1+s$$

$$\left(A_2-A_1\right)s^2+\left(B_2-B_1\right)s+\left(C_2-C_1\right)=0 \quad \Rightarrow \quad C_2=C_1-\frac{1}{2}s^2$$

Hence

$$K(t,s)=\begin{cases}A_1(s)t^2+B_1(s)t+C_1(s), & a\le t\le s\le b\\ A_1(s)t^2+B_1(s)t+C_1(s)-\dfrac{1}{2}(t-s)^2, & a\le s\le t\le b\end{cases}$$

$$u(a)=0 \quad \Rightarrow \quad C_1=a^2A_1-aB_1$$

$$u(a)+u'(a)=0 \quad \Rightarrow \quad B_1=-2aA_1 \ , \quad C_1=a^2A_1$$

$$u(b)+u'(b)=0 \quad \Rightarrow \quad A_1(s)=\frac{(b-s)(b-s+2)}{2(b-a)(b-a+2)}$$

$$K(t,s)=\begin{cases}\dfrac{(b-s)(b-s+2)}{2(b-a)(b-a+2)}(t-a)^2, & a\le t\le s\le b\\[2ex] \dfrac{(b-s)(b-s+2)}{2(b-a)(b-a+2)}(t-a)^2-\dfrac{1}{2}(t-s)^2, & a\le s\le t\le b\end{cases}$$

And the solution of $\begin{array}{l} u'''(t)+a(t)f(t,u(t))=0,\\ u(a)=0, \quad u(a)+u'(a)=0, \quad u(b)+u'(b)=0\end{array}$ is given by

$$w(t)=\int_a^b K(t,s)a(s)f(s,u(s))ds, \tag{2.6}$$

And clearly

$$w(a)=0, \quad w(a)+w'(a)=0, \quad w(b)+w'(b)=0, \quad w(\eta)=\int_a^b K(\eta,s)a(s)f(s,u(s))ds.$$

(2.7)

Now let us assume the solution of (2.1), (2.2) can be expressed by

$$u(t)=w(t)+\left(At^2+Bt+C\right)w(\eta)$$

(2.8)

where A,B and C are constants that will be determined.

From (2.7), (2.8) we know that

$$u(a)=\left(Aa^2+Ba+C\right)w(\eta)$$

$$u(a)+u'(a)=\left(Aa^2+Ba+C\right)w(\eta)+\left(2Aa+B\right)w(\eta)$$

$$u(b)+u'(b)=\left(Ab^2+Bb+C\right)w(\eta)+\left(2Ab+B\right)w(\eta)$$

$$u(\eta)=w(\eta)+\left(A\eta^2+B\eta+C\right)w(\eta)$$

Putting this into (2.2) yields

$$\begin{cases}\left[2a-k_1(\eta^2-a^2)\right]w(\eta)A+\left[1-k_1(\eta-a)\right]w(\eta)B=k_1w(\eta)\\ \left[b^2-a^2+2b-k_2(\eta^2-a^2)\right]w(\eta)A+\left[b-a+1-k_2(\eta-a)\right]w(\eta)B=k_2w(\eta)\end{cases}$$

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Solving the system of linear equations on the unknown numbers A and B, using Cramer's rule we obtain

$$\begin{cases} A=\dfrac{(k_1)(b-a+1)-k_2}{\left(2a-k_1(\eta^2-a^2)\right)\left(b-a+1-k_2(\eta-a)\right)-\left(1-k_1(\eta-a)\right)\left(b^2-a^2+2b-k_2(\eta^2-a^2)\right)} \\ B=\dfrac{(k_1)\left(a^2-2b-b^2\right)+2ak_2}{\left(2a-k_1(\eta^2-a^2)\right)\left(b-a+1-k_2(\eta-a)\right)-\left(1-k_1(\eta-a)\right)\left(b^2-a^2+2b-k_2(\eta^2-a^2)\right)} \end{cases}$$

$$C=-a^2A-aB=\frac{(k_1)\left(-a^2b-a^2+2ab+ab^2\right)-a^2k_2}{\left(2a-k_1(\eta^2-a^2)\right)\left(b-a+1-k_2(\eta-a)\right)-\left(1-k_1(\eta-a)\right)\left(b^2-a^2+2b-k_2(\eta^2-a^2)\right)}$$

Hence, the solution of (2.1) with the boundary condition (2.2) is

$$u(t)=w(t)+(At^2+Bt+C)w(\eta)$$

$$=w(t)+\frac{\left[(k_1)(b-a+1)-k_2\right]t^2+\left[(k_1)\left(a^2-2b-b^2\right)+2ak_2\right]t+(k_1)\left(-a^2b-a^2+2ab+ab^2\right)-a^2k_2}{\left(2a-k_1(\eta^2-a^2)\right)\left(b-a+1-k_2(\eta-a)\right)-\left(1-k_1(\eta-a)\right)\left(b^2-a^2+2b-k_2(\eta^2-a^2)\right)}w(\eta)$$

This together with (2.6) implies that

$$u(t)=\int_a^b K(t,s)a(s)f(s,u(s))ds+\frac{\left[(k_1)(b-a+1)-k_2\right]t^2+\left[(k_1)\left(a^2-2b-b^2\right)+2ak_2\right]t+(k_1)\left(-a^2b-a^2+2ab+ab^2\right)-a^2k_2}{\left(2a-k_1(\eta^2-a^2)\right)\left(b-a+1-k_2(\eta-a)\right)-\left(1-k_1(\eta-a)\right)\left(b^2-a^2+2b-k_2(\eta^2-a^2)\right)}\int_a^b K(\eta,s)a(s)f(s,u(s))ds$$

Consequently, the Green's function G(t, s) for the boundary value problem (2.1), (2.2) is as described in (2.3).

Corollary 2.2: The Greens function of $u'''(t)+a(t)f(t,u)=0,\quad t\in[a,b]$,

$u(0)=0,\qquad u(0)+u'(0)=k_1u(\eta),\qquad u(1)+u'(1)=k_2u(\eta)$ is

$$G(t,s)=K(t,s)+\frac{\left[2k_1-k_2\right]t^2-3k_1t}{\left(-k_1\eta^2\right)\left(2-k_2\eta\right)-\left(1-k_1\eta\right)\left(3-k_2\eta^2\right)}K(\eta,s)$$, where

$$K(t,s)=\frac{1}{6}\begin{cases}(s^2-4s+3)t^2,\ a\le t\le s\le b\\(s^2-4s+3)t^2-3(t-3)^2,\ a\le s\le t\le b\end{cases}$$ and

$$K(\eta,s)=\frac{1}{6}\begin{cases}(s^2-4s+3)\eta^2,\ a\le \eta\le s\le b\\(s^2-4s+3)\eta^2-3(\eta-3)^2,\ a\le s\le \eta\le b\end{cases}$$

Example: Consider the third-order three-point boundary value problem:

$$\begin{cases}u'''(t)+\cos t=0, \quad t\in[0,1],\\u(0)=0, \quad u(0)+u'(0)=\frac{1}{6}u\left(\frac{1}{2}\right), \qquad u(1)+u'(1)=\frac{1}{5}u'\left(\frac{1}{2}\right)\end{cases}$$

The general solution is given by $u(t)=\int_0^1 G(t,s)\cos(s)ds$,

where $G(t,s)=K(t,s)+\dfrac{[2k_1-k_2]t^2-3k_1t}{(-k_1\eta^2)(2-k_2\eta)-(1-k_1\eta)(3-k_2\eta^2)}K(\eta,s)$

$$K(t,s)=\frac{1}{6}\begin{cases}(s^2-4s+3)t^2,\ a\le t\le s\le b\\(s^2-4s+3)t^2-3(t-3)^2,\ a\le s\le t\le b\end{cases}$$ and

$$K(\eta,s)=\frac{1}{6}\begin{cases}(s^2-4s+3)\eta^2,\ a\le \eta\le s\le b\\(s^2-4s+3)\eta^2-3(\eta-3)^2,\ a\le s\le \eta\le b\end{cases}$$

Upon simplifying we will get

$$u(t)=\sin(t)-\frac{110t^2+5t}{2(167)}(\sin 1+\cos 1)+\frac{30t-8t^2}{167}\sin\left(\frac{1}{2}\right)+\frac{114t^2-177t}{167}$$

Section 3: A Third Order Four Point Mixed Boundary Value Problem

In this section we will generalize the ideas used in section 2 to solve the third order four point mixed boundary value problem

$$u'''(t)+a(t)f(t,u(t))=0 \tag{3.1}$$

$$u(a)=0\ ,\quad u(a)+u'(a)=k_1u(\eta_1)\ ,\quad u(b)+u'(b)=k_2u(\eta_2) \tag{3.2}$$

Theorem 3.1 Assume that

$$(2a+k_1(a^2-\eta^2{}_1))(b-a+1+k_2(a-\eta_2))-(1+k_1(a-\eta_1))(b^2-a^2+2b+k_2(a^2-\eta^2{}_2))\neq 0$$

Then the Solution to the differential equation mixed boundary value problem (3.1) satisfying (3.2)

Is given by

$$u(t)=w(t)+\left[\frac{(k_1w(\eta_1)\left(b-a+1+k_2(a-\eta_2)\right)-k_2w(\eta_2)(1+k_1(a-\eta_1)))t^2}{(2a+k_1(a^2-\eta^2{}_1))(b-a+1+k_2(a-\eta_2))-(1+k_1(a-\eta_1))(b^2-a^2+2b+k_2(a^2-\eta^2{}_2))}\right]$$

+

$$\left[\frac{((k_2w(\eta_2)(2a+k_1(a^2-\eta^2{}_1))-(k_1w(\eta_1)(b^2-a^2+2b+k_2(a^2-\eta^2{}_2)))t}{(2a+k_1(a^2-\eta^2{}_1))(b-a+1+k_2(a-\eta_2))-(1+k_1(a-\eta_1))(b^2-a^2+2b+k_2(a^2-\eta^2{}_2))}\right]+$$

$$\left[\frac{k_2w(\eta_2)((a^2(1+k_1(a-\eta_1))-a(2a+k_1(a^2-\eta^2{}_1)))+k_1w(\eta_1)(a(b^2-a^2+2b+k_2(a^2-\eta^2{}_2))-a^2(b-a+1+k_2(a-\eta_2)))}{(2a+k_1(a^2-\eta^2{}_1))(b-a+1+k_2(a-\eta_2))-(1+k_1(a-\eta_1))(b^2-a^2+2b+k_2(a^2-\eta^2{}_2))}\right]$$

(3.3)

where $w(t)=\int_a^b K(t,s)a(s)f(s,u(s))ds$, $w(\eta)=\int_a^b K(\eta,s)a(s)f(s,u(s))ds$ and

$$K(t,s)=\begin{cases}\dfrac{(b-s)(b-s+2)}{2(b-a)(b-a+2)}(t-a)^2, & a\le t\le s\le b\\[2mm] \dfrac{(b-s)(b-s+2)}{2(b-a)(b-a+2)}(t-a)^2-\dfrac{1}{2}(t-s)^2, & a\le s\le t\le b\end{cases}$$

Proof: Assume the solution set of (3.1) with the boundary condition (3.2) is given by

$u(t)=w(t)+(At^2+Bt+C)(w(\eta_1)+w(\eta_2)$,then using the boundary conditions we have

$$u(a)=w(a)+(Aa^2+Ba+C)(w(\eta_1)+w(\eta_2)=0 \qquad (3.4)$$

$$u(b)=w(b)+(Ab^2+Bb+C)(w(\eta_1)+w(\eta_2)) \qquad (3.5)$$

$$u(\eta_1)=w(\eta_1)+(A\eta_1{}^2+B\eta_1+C)(w(\eta_1)+w(\eta_2)) \qquad (3.6)$$

$$u(\eta_2)=w(\eta_2)+(A\eta_2{}^2+B\eta_2+C)(w(\eta_1)+w(\eta_2)) \qquad (3.7)$$

$$u'(a) = w'(a) + (2aA + B)(w(\eta_1) + w(\eta_2)) \qquad (3.8)$$

$$u'(b) = w'(b) + (2bA + B)(w(\eta_1) + w(\eta_2)) \qquad (3.9)$$

From the equation (3.3) we have $a^2A + aB + C = 0 \rightarrow C = -a^2A - aB$ and using equations (3.4) . . . (3.8) and the boundary conditions of (3.2) we get

$$A = \left[\frac{(k_1 w(\eta_1)\left(b - a + 1 + k_2(a - \eta_2)\right) - k_2 w(\eta_2)(1 + k_1(a - \eta_1)))}{(2a + k_1(a^2 - \eta^2{}_1))(b - a + 1 + k_2(a - \eta_2)) - (1 + k_1(a - \eta_1))(b^2 - a^2 + 2b + k_2(a^2 - \eta^2{}_2))\left(w(\eta_1) + w(\eta_2)\right)}\right]$$

$$B = \left[\frac{(k_2 w(\eta_2)(2a + k_1(a^2 - \eta^2{}_1)) - (k_1 w(\eta_1)(b^2 - a^2 + 2b + k_2(a^2 - \eta^2{}_2))}{(2a + k_1(a^2 - \eta^2{}_1))(b - a + 1 + k_2(a - \eta_2)) - (1 + k_1(a - \eta_1))(b^2 - a^2 + 2b + k_2(a^2 - \eta^2{}_2))\left(w(\eta_1) + w(\eta_2)\right)}\right]$$

$$C = -a^2A - aB$$

$$C = \left[\frac{k_2 w(\eta_2)((a^2(1 + k_1(a - \eta_1)) - a(2a + k_1(a^2 - \eta^2{}_1))) + k_1 w(\eta_1)(a(b^2 - a^2 + 2b + k_2(a^2 - \eta^2{}_2)) - a^2(b - a + 1 + k}{(2a + k_1(a^2 - \eta^2{}_1))(b - a + 1 + k_2(a - \eta_2)) - (1 + k_1(a - \eta_1))(b^2 - a^2 + 2b + k_2(a^2 - \eta^2{}_2))\left(w(\eta_1) + w(\eta\right.}\right.$$

Putting the values of A , B , C in the equation $u(t) = w(t) + (At^2 + Bt + C)(w(\eta_1) + w(\eta_2)$ we get (3.3)

Corollary 3.2 If $(2a + k_1(a^2 - \eta^2{}_1))(b - a + 1 + k_2(a - \eta_2)) - (1 + k_1(a - \eta_1))(b^2 - a^2 + 2b + k_2(a^2 - \eta^2{}_2)) \neq 0$, then

the solution to the boundary value problem :

$$u'''(t) + a(t) f(t, u(t)) = 0$$

$$u(a) = 0 \ , \quad u(a) + u'(a) = k_1 u(\eta_1) \ , \quad u(b) + u'(b) = 0$$ is given by

$$u(t) = w(t) + \left[\frac{(k_1 w(\eta_1)\left(b - a + 1\right)t^2 - (k_1 w(\eta_1)(b^2 - a^2 + 2b)t + k_1 w(\eta_1)(a(b^2 + 2b) - a^2(b + 1)}{(2a + k_1(a^2 - \eta^2{}_1))(b - a + 1 + k_2(a - \eta_2)) - (1 + k_1(a - \eta_1))(b^2 - a^2 + 2b + k_2(a^2 - \eta^2{}_2))}\right]$$

Proof : Direct substitution of equation (3.3) .

Corollary 3.3 If $k_2 - 2k_1 \neq 0$, $\eta k_1 \neq 1$,then the boundary value problem

$$u'''(t) + a(t) f(t, u(t)) = 0$$

$$u(0) = 0 \ , \quad u(0) + u'(0) = k_1 u(\eta) \ , \quad u(1) + u'(1) = k_2 u(\eta)$$

has a solution same as the solution of **Corollary 2.3**

Example 3.1: Let us consider the following example

$$\begin{cases} u'''(t) + \cos(t) = 0, \\ u(0) = 0, \ \ u(0) + u'(0) = \frac{1}{2} u\left(\frac{1}{3}\right), \quad u(1) + u'(1) = \frac{1}{4} u\left(\frac{1}{5}\right) \end{cases}$$

Solution: Substituting the values in (3.3) and simplifying we get the following solution

$$u(t) = t^2 \left[\frac{-305}{936} \sin(1) - \frac{3}{8} \sin(\frac{1}{3}) + \frac{25}{52} \sin(\frac{1}{5}) - \frac{305}{936} \cos(1) + \frac{49}{72} \right] +$$
$$t \left[\frac{23}{40} \sin(\frac{1}{3}) + \frac{5}{156} \sin(\frac{1}{5}) - \frac{61}{2808} \sin(1) - \frac{61}{2808} \cos(1) - \frac{1247}{1080} \right] + \sin t$$

4. Application To Nonlinear Problem:

In this section, we study the iterative solutions which will converge to the solution of the following nonlinear three-point boundary value problem

$$\begin{cases} u''' + f(t,u) = 0, \quad t \in (0,1), \\ u(0) = 0, \qquad u(0) + u'(0) = k_1 u(\eta), \qquad u(1) + u'(1) = k_2 u(\eta) \end{cases}$$

(4.1)

with $\eta \in (0,1), \ k_1 > 0, \ 2k_1 - k_2 < 0$

Let $J = (0,1), \quad I = [0,1], \quad \mathbb{R}^+ = [0, \infty),$

$$D = \left\{ x \in C(I) \middle| \exists M_x \geq m_x > 0, \text{such that } m_x (1-t) \leq x(t) \leq M_x (1-t), t \in I \right\}.$$

Concerning the function f we impose the following hypotheses:

$$\begin{cases} f(t,u) \text{ is nonnegative continuous on } J \times \mathbb{R}^+, \\ f(t,u) \text{ is monotone increasing on u, for fixed t} \in \text{J}, \\ \exists q \in (0,1) \text{ such that } f(t,ru) \geq r^q f(t,u), \forall 0 < r < 1, (t,u) \in J \times \mathbb{R}^+. \end{cases}$$

(4.2)

Obviously, from (4.2) we obtain

$$f(t,\lambda u)\geq \lambda^{q} f(t,u), \quad \forall \lambda>1, \quad (t,u)\in J\times \mathbb{R}^{+}.$$

We can see that if $0<\alpha_i<1,\ a_i(t)$ are nonnegative continuous on J, for i = 0,1,2,…,m, then

$f(t,u)=\sum_{i=1}^{m} a_i(t)u^{\alpha_i}$ satisfy the condition (4.2).

Concerning the boundary value problem (4.1), we have following conclusions.

Theorem 4.1. Suppose the function f(t, u) satisfy the condition (4.2), it may be singular at t=0 and/or t=1, and

$$0<\int_0^1 f(t,1-t)dt<\infty.$$

Then nonlinear singular boundary value problem (4.1) has a unique solution w(t) in $C(I)\cap C^2(J)$. Constructing successively the sequence of functions

$$h_n(t)=\int_0^1 G(t,s)f(s,h_{n-1}(s))ds, \qquad n=1,2,\ldots.$$

for any initial function $h_0(t)\geq 0(\not\equiv 0), t\in I$ then $\{h_n(t)\}$ must converge to w(t) uniformly on I and the rate of convergence is

$$\max_{t\in I}\left|h_n(t)-w(t)\right|=O\left(1-N^{q^n}\right),$$

where 0< N <1 , which depends on the initial function $h_0(t), G(t,s)$ as in (2.20).

Proof. Let

$$P=\left\{x(t)\big|x(t)\in C\left(I\right), x(t)\geq 0\right\},$$

$$Fx(t)=\int_0^1 G(t,s)f(s,x(s))ds, \quad \forall x(t)\in D.$$

It is easy that the operator $F:D\to P$ is increasing; From Corollary 2.3 we know that if $u\in D$ satisfies

$$u(t)=Fu(t), \quad t\in I,$$

then $u\in C^1\left(I\right)\cap C^2\left(J\right)$ is a solution of (4.1).

For any $x \in D$, there exist positive numbers $0 < m_x < 1 < M_x$ such that

$$m_x(1-s) \le x(s) \le M_x(1-s), \quad s \in I,$$

$$(m_x)^q f(s,1-s) \le f(s,x(s)) \le (M_x)^q f(s,1-s), \quad s \in J.$$

From corollary (2.1) we have $G(t,s) = K(t,s) + \dfrac{[2k_1 - k_2]t^2 - 3k_1 t}{(\eta^2)(k_2 - 2k_1) + 3(k_1\eta - 1)} K(\eta, s)$, where

$$K(t,s) = \frac{1}{6}\begin{cases} (s^2 - 4s + 3)t^2, \; a \le t \le s \le b \\ (s^2 - 4s + 3)t^2 - 3(t-3)^2, \; a \le s \le t \le b \end{cases}$$

Now clearly $G(t,s) \ge t\left(\dfrac{-[k_1 + k_2]K(\eta,s)}{(\eta^2)(k_2 - 2k_1) + 3(k_1\eta - 1)}\right)$

$$\textit{and} \quad G(t,s) \le 3t^2 + \frac{\{(2k_1 - k_2)t^2 - 3k_1 t^2\} K(\eta,s)}{(\eta^2)(k_2 - 2k_1) + 3(k_1\eta - 1)}$$

$$\Rightarrow \quad G(t,s) \le t\left[3 - \frac{(k_1 + k_2)K(\eta,s)}{(\eta^2)(k_2 - 2k_1) + 3(k_1\eta - 1)}\right]$$

Using above inequalities and the conditions (4.2), we obtain

$$Fx(t) = \int_0^1 G(t,s) f(s,x(s))ds$$

$$\ge \int_0^1 t\left(\frac{-[k_1 + k_2]K(\eta,s)}{(\eta^2)(k_2 - 2k_1) + 3(k_1\eta - 1)}\right)\left((m_x)^q f(s,1-s)\right)ds$$

$$\ge t(m_x)^q \frac{-[k_1 + k_2]}{(\eta^2)(k_2 - 2k_1) + 3(k_1\eta - 1)}\left[\int_0^1 K(\eta,s)\left(f(s,1-s)\right)ds\right], t \in I$$

$$Fx(t) = \int_0^1 G(t,s) f(s, x(s)) ds$$

$$\leq \int_0^1 t \left[3 - \frac{(k_1 + k_2) K(\eta, s)}{(\eta^2)(k_2 - 2k_1) + 3(k_1\eta - 1)} \right] \left((M_x)^q f(s, 1-s) \right) ds$$

$$\leq t (M_x)^q \int_0^1 \left(3 - \frac{(k_1 + k_2) K(\eta, s)}{(\eta^2)(k_2 - 2k_1) + 3(k_1\eta - 1)} \right) (f(s, 1-s)) ds, \quad t \in I$$

Hence we obtain $F : D \to D.$

For any $h_o \in D$, we let

$$l_{h_o} = \sup \left\{ l > 0 \middle| l h_o(t) \leq (F h_o)(t), t \in I \right\},$$

$$L_{h_o} = \inf \left\{ L > 0 \middle| L h_o(t) \geq (F h_o)(t), t \in I \right\},$$

$$m = \min \left\{ 1, (l_{h_o})^{\frac{1}{1-q}} \right\}, \quad M = \max \left\{ 1, (L_{h_o})^{\frac{1}{1-q}} \right\}$$

$$u_0(t) = m h_0(t), \qquad u_n(t) = F u_{n-1}(t)$$

$$v_0(t) = M h_0(t), \qquad v_n(t) = F v_{n-1}(t), \qquad n = 0, 1, 2, \ldots$$

Since the operator F is increasing

$$u_0(t) \leq u_1(t) \leq \cdots \leq u_n(t) \cdots \leq v_n(t) \leq \cdots \leq v_1(t) \leq v_0(t), \qquad t \in I.$$

For $t_0 = \dfrac{m}{M}$, from (4.2) it can obtained by induction that

$$u_n(t) \geq (t_0)^{q^n} v_n(t), \qquad t \in I, n = 0, 1, 2, \cdots$$

Hence

$$0 \leq u_{n+p}(t) - u_n(t) \leq v_n(t) - u_n(t) \leq \left(1 - (t_0)^{q^n} M h_0(t) \right), \qquad \forall n, p$$

so that there exist function $w(t) \in D$ such that

$$u_n(t) \to w(t), \qquad v_n(t) \to w(t), \quad \text{(uniformly on I)},$$

and

$$u_n(t) \le w(t) \le v_n(t), \quad t \in I, n = 0,1,2,\cdots$$

From the operator F is increasing we have

$$u_{n+1}(t) = Fu_n(t) \le Fw(t) \le Fv_n(t) = v_{n+1}(t), \quad n = 0,1,2,\cdots$$

Hence $w(t) \in C^1(I) \cap C^2(J)$ is a solution of (4.1).

Since the operator F is increasing, we obtain $u_n(t) \le h_n(t) \le v_n(t), \quad t \in I, n = 0,1,2,\cdots$

Now clearly

$$\begin{aligned} |h_n(t) - w(t)| &\le |h_n(t) - u_n(t)| + |u_n(t) - w(t)| \\ &\le 2|v_n(t) - u_n(t)| \le \left(1 - (t_0)^{q^n}\right) M\,|h_0(t)|, \end{aligned}$$

so that $\max\limits_{t \in I} |h_n(t) - w(t)| \le \left(1 - (t_0)^{q^n}\right) M \max\limits_{t \in I} |h_0(t)|.$

Remark: If $f(t,u)$ is continuous on $I \times \mathbb{R}^+$, then it is quite evident that the condition (4.2) holds. Hence $w(t) \in C^2(J)$ is the unique solution.

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