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UNSTEADY MHD FLOW OF TWO IMMISCIBLE OLDROYD FLUIDS UNDER VARIOUS PRESSURE GRADIENTS

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ABSTRACT

The flow of two immiscible electrically conducting Oldroyd fluids through a straight rectangular tube has been studied. The flow has been considered in presence of transverse uniform magnetic field and under the influence of time –varying pressure gradient. Using integral transform technique, the exact solutions for the velocities of two immiscible fluids have been obtained. Finally a few particular cases of pressure gradient have been discussed.

1.INTRODUCTION

There are fluids, which exhibit the elasticity property of solids and viscosity property of fluids, which are adequate in nature, and relevant fields of fluid dynamics. These types of fluids are called non-Newtonian fluids or visco-elastic fluids. The present authors have consulted freely some of the books[1-7] in this reference. The flow of visco-elastic fluid between two parallel plates under uniform, exponential or periodic pressure gradient has been investigated by Das[8] and Pal and Sengupta[9]. Drake[10] studied the flow of an incompressible viscous fluid along a rectangular channel due to a periodic pressure gradient. Panja and Sengupta[11] investigated the unsteady hydrodynamic flow of two immiscible visco-elastic fluids between two inclined parallel plates. Sengupta and Chakraborty[12] studied the MHD flow of two immiscible visco-elastic Rivlin-Ericksen fluids through a non-conducting channel. The problem of unsteady flow of two immiscible visco-elastic fluids under a certain pressure gradient between two fixed plates was studied by Kapur and Shukla[13]. Sengupta and Raymahapatra[14] investigated the flow of two immiscible visco-elastic Maxwell fluids with transient pressure gradient through a rectangular tube. Chakraborty and Sengupta[15] studied the hydromagnetic flow of two immiscible visco-elastic Walter conducting liquids between two inclined parallel plates. In this paper the authors have investigated the unsteady MHD flow of two immiscible Oldroyd fluids through a straight rectangular tube under various pressure gradients.

2. MATHEMATICAL FORMULATION

For the slow motion the equation of state relating to the stress tensor τ_{ik} and the rate of strain tensor e_{ik} for visco-elastic Oldroyd type are of the form:

$$\tau_{ik} = -pg_{ik} + \tau'_{ik}$$

$$(1+\lambda_1 \frac{\partial}{\partial t})\tau'_{ik} = 2\mu(1+\mu_1 \frac{\partial}{\partial t})e_{ik}$$

$$e_{ik} = \frac{1}{2}(\mathrm{v}_{i,k} + \mathrm{v}_{k,i})$$

where τ'_{ik} is the part of the stress tensor associated with the change of shape of the material element, p is the isotropic pressure of arbitrary type , g_{ik} the metric tensor, μ(>0) the co-efficient of viscosity and v_i the velocity vector, λ_1 and μ_1 (λ_1, μ_1>0) are the stress relaxation time parameter and rate of strain retardation time, respectively. The metric tensor g_{ik} in Cartesian co-ordinates become $g_{ik}=\delta_{ik}$. Now from above:

$$\tau'_{ik} = \left(\frac{1+\mu_1 \frac{\partial}{\partial t}}{1+\lambda_1 \frac{\partial}{\partial t}}\right) 2\mu e_{ik} = 2\mu^* e_{ik}(say)\ and\ \nu^* = \frac{\mu^*}{\rho} = \nu\left(\frac{1+\mu_1 \frac{\partial}{\partial t}}{1+\lambda_1 \frac{\partial}{\partial t}}\right)$$

where $\nu = \frac{\mu}{\rho}$ is the kinematical co-efficient of viscosity.

Fundamental Navier Stokes equation of motion is:

$$\frac{\partial \vec{q}}{\partial t} = -\frac{1}{\rho}\vec{\nabla}p + \nu^* \nabla^2 \vec{q} + \vec{F}$$

$$i.e. \left(1+\lambda_1 \frac{\partial}{\partial t}\right)\frac{\partial \vec{q}}{\partial t} = -\left(1+\lambda_1 \frac{\partial}{\partial t}\right)\vec{\nabla}p + \nu\left(1+\mu_1 \frac{\partial}{\partial t}\right)\nabla^2 \vec{q} + \left(1+\lambda_1 \frac{\partial}{\partial t}\right)\vec{F} \qquad ...(1)$$

where $\vec{q}$ is the velocity vector. This is the Navier Stokes equation of motion in case of Oldroyd fluid.

With reference to rectangular Cartesian co-ordinate system we consider the boundary of the walls of the channel as $x = \pm a$ and $\mathrm{y} = \pm \mathrm{b}$. The z-axis is chosen on the surface of the fluids and towards the direction of motion of both fluids, the x-axis perpendicular to the interface drawn into the upper fluid and the y-axis in the plane of the interface.

Let $\rho_i, \lambda_i, \mu_i, \bar{\mu}_i, \sigma_i, \nu_i (i=1,2)$ be the density, relaxation time, retardation time, co-efficient of viscosity, electrical conductivity and kinematical co-efficient of viscosity of the upper and lower fluids respectively each occupying height 'a'. We also suppose that two media have approximately the same permeability μ_e throughout and thus the same magnetic field H_0 is interacting to both the conducting fluids, the velocities of the lower and upper fluids are respectively $w_i(x,y,t)$ [i=1,2], in the z-direction.

The equations of motion of Oldroyd fluids in the presence of a transverse magnetic field in view of the above assumptions become:

$$\left(1+\lambda_i \frac{\partial}{\partial t}\right)\frac{\partial w_i}{\partial t} = -\frac{1}{\rho_i}\left(1+\lambda_i \frac{\partial}{\partial t}\right)\frac{\partial p}{\partial z} + \nu_i\left(1+\mu_i \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 w_i}{\partial x^2}+\frac{\partial^2 w_i}{\partial y^2}\right) - \frac{\sigma_i B_0^{\ 2}}{\rho_i}\left(1+\lambda_i \frac{\partial}{\partial t}\right) w_i \quad ..(2)$$

where $\nu_i = \dfrac{\bar{\mu}_i}{\rho_i}$ are the kinematical co-efficients of viscosity of the upper and lower fluids and $B_0=\mu_e H_0$ is the magnetic induction vector. [i=1,2]

3. SOLUTION OF THE PROBLEM

We introduce firstly the non-dimensional quantities:

$$x' = \frac{x}{a}, y' = \frac{y}{a}, z' = \frac{z}{a}, t' = \frac{t\nu_i}{a^2}, p' = \frac{pa^2}{\nu_i^{\ 2}\rho_i}, w_i' = \frac{w_i a}{\nu_i}, \lambda_i' = \frac{\lambda_i \nu_i}{a^2}, \mu_i' = \frac{\mu_i \nu_i}{a^2} \quad \text{[i=1,2]}$$

Now dropping the primes from equation (2) we have:

$$\lambda_i \frac{\partial^2 w_i}{\partial t^2} + \left(1+M_i^{\ 2}\lambda_i\right)\frac{\partial w_i}{\partial t} + M_i^{\ 2} w_i = -\left(1+\lambda_i \frac{\partial}{\partial t}\right)\frac{\partial p}{\partial z} + \left(1+\mu_i \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 w_i}{\partial x^2}+\frac{\partial^2 w_i}{\partial y^2}\right) \quad \ldots (3)$$

where $M_i = \sqrt{\dfrac{\sigma_i B_0^{\ 2} a^2}{\bar{\mu}_i}}$ [i=1,2] is the Hartmann numbers corresponding to the upper and lower fluids, respectively.

Initially the fluids are at rest and the flow takes place under the time-varying pressure gradient. The initial and boundary conditions of the upper fluid are:

$$w_1(x,y,0) = 0 \quad \ldots(4)$$

$$w_1(1,y,t) = 0; \; -l \le y \le l, t \rangle 0 \quad \ldots(5.1)$$

$$\frac{\partial w_1}{\partial x} = 0; \; x = 0 \quad \ldots(5.2)$$

$$w_1(x,\pm l,t); \; -1 \le x \le 1, \; t \rangle 0 \quad \ldots(6.1)$$

$$\frac{\partial w_1}{\partial y} = 0; \; y = 0 \quad \ldots(6.2)$$

The initial and boundary conditions of the lower fluid are:

$$w_2(x,y,0)=0 \qquad ...(7)$$

$$w_2(-1,y,t)=0;\ -l\le y\le l, t\rangle 0 \qquad ...(8.1)$$

$$\frac{\partial w_2}{\partial x}=0;\ x=0 \qquad ...(8.2)$$

$$w_2(x,\pm l,t);\ -1\le x\le 1,\ t\rangle 0 \qquad ...(9.1)$$

$$\frac{\partial w_2}{\partial y}=0;\ y=0 \qquad ...(9.2)$$

where $l=\frac{b}{a}$

The method of integral transform is used to find out the solution and we define the finite Fourier cosine transform as:

$$w_c(n,y,t)=\int_0^1 w_1(x,y,t)\cos(p_n x)dx \qquad ...(10)$$

$$w_c^*(n,m,t)=\int_0^l w_c(n,y,t)\cos(p_m y)dy \qquad ...(11)$$

where $p_n=(2n+1)(\pi/2),\ p_m=(2m+1)(\pi/2l)$

The inverse finite cosine transform defined in equations (10) and (11) can be obtained as:

$$w_1(x,y,t)=2\sum_{n=0}^{\infty} w_c(n,y,t)\cos(p_n x) \qquad ...(12)$$

$$w_c(n,y,t)=2\sum_{m=0}^{\infty} w_c^*(n,m,t)\cos(p_m y) \qquad ...(13)$$

Taking finite Fourier cosine transform to boundary conditions (6.1) and (6.2) we obtain:

$$w_c(n,l,t)=0 \qquad ...(14.1)$$

$$\frac{\partial w_c(n,y,t)}{\partial y}=0 \qquad ...(14.2)$$

Applying transforms (10) and (11) to initial condition (4) we have:

$$w_c^*(n,m,0)=0 \qquad ...(15)$$

Using equations (10) and (11) to the equation of motion (3) and using (4), (5.1), (5.2), (6.1), (6.2), (14.1) and (14.2) we get:

$$\frac{\partial^2 w_c^*}{\partial t^2}+\xi_1\frac{\partial w_c^*}{\partial t}+\eta_1 w_c^*=\frac{(-1)^{m+n}F(t)}{p_m p_n \lambda_1} \qquad ...(16)$$

where $\xi_1=\frac{1}{\lambda_1}\left[1+M_1^2\lambda_1+\mu_1(p_m^2+p_n^2)\right], \eta_1=\frac{(M_1^2+p_m^2+p_n^2)}{\lambda_1}, \frac{\partial p}{\partial z}=-F(t)$

where F(t) is an arbitrary function of time.

Now we use the Laplace transform defined by :

$$W_1(s)=\int_0^\infty w_c^{\,*}e^{-st}dt,\ \ \overline{F}(s)=\int_0^\infty F(t)e^{-st}dt$$

From equation (16) and by (15) we get:

$$s^2W_1(s)+\xi_1 sW_1(s)+\eta_1W_1(s)=\frac{(-1)^{m+n}\overline{F}(s)}{p_m p_n\lambda_1} \qquad ...(17)$$

We solve this equation by Laplace inversion and we use convolution integral to get:

$$w_c^{\,*}=\frac{2(-1)^{m+n}}{p_m p_n\lambda_1\sqrt{4\eta_1-\xi_1^{\,2}}}\int_0^t\left[F(t-u)e^{-\frac{\xi_1 u}{2}}\sin\left(\sqrt{4\eta_1-\xi_1^{\,2}}\,\frac{u}{2}\right)\right]du \qquad ...(18)$$

Now by (12) and (13) from (18) we have the velocity of the upper fluid as:

$$w_1(x,y,t)=\frac{8}{l}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+n}}{p_m p_n\lambda_1\sqrt{4\eta_1-\xi_1^{\,2}}}\int_0^t\left[F(t-u)e^{-\frac{\xi_1 u}{2}}\sin\left(\sqrt{4\eta_1-\xi_1^{\,2}}\,\frac{u}{2}\right)\right]du\cos(p_n x)\cos(p_m y)$$

for $0\le x\le 1$ ...(19)

Similarly under the boundary conditions (8.1), (8.2) and (9.1), (9.2) using the above transforms we obtain the velocity of the lower fluid as:

$$w_2(x,y,t)=\frac{8}{l}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+n}}{p_m p_n\lambda_2\sqrt{4\eta_2-\xi_2^{\,2}}}\int_0^t\left[F(t-u)e^{-\frac{\xi_2 u}{2}}\sin\left(\sqrt{4\eta_2-\xi_2^{\,2}}\,\frac{u}{2}\right)\right]du\cos(p_n x)\cos(p_m y)$$

for $-1\le x\le 0$ ...(20)

where $\xi_2=\frac{1}{\lambda_2}\left[1+M_2^{\,2}\lambda_2+\mu_2(p_m^{\,2}+p_n^{\,2})\right],\ \eta_2=\frac{(M_2^{\,2}+p_m^{\,2}+p_n^{\,2})}{\lambda_2}$

4.FLOW UNDER VARIOUS PRESSURE GRADIENT

CASE I: ***Flow under constant pressure gradient***

Let F(t)=F_0 (a constant)

The velocities of the upper and lower fluids from (19 and (20) reduce as:

$$w_1=\frac{4}{l}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+n}F_0}{p_m p_n\lambda_1\eta_1}\left[1-e^{-\frac{\xi_1 t}{2}}\left\{\frac{\xi_1}{\sqrt{4\eta_1-\xi_1^{\,2}}}\times\sin\sqrt{4\eta_1-\xi_1^{\,2}}\,\frac{t}{2}\right\}+\cos\left\{\sqrt{4\eta_1-\xi_1^{\,2}}\,\frac{t}{2}\right\}\right]\cos(p_n x)\cos(p_m y)$$

for $0\le x\le 1$ ...(21.1)

$$w_2=\frac{4}{l}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+n}F_0}{p_m p_n\lambda_2\eta_2}\left[1-e^{-\frac{\xi_2 t}{2}}\left\{\frac{\xi_2}{\sqrt{4\eta_2-\xi_2^{\,2}}}\times\sin\sqrt{4\eta_2-\xi_2^{\,2}}\,\frac{t}{2}\right\}+\cos\left\{\sqrt{4\eta_2-\xi_2^{\,2}}\,\frac{t}{2}\right\}\right]\cos(p_n x)\cos(p_m y)$$

for $-1\le x\le 0$ ...(21.2)

CASE II: ***Flow under impulsive pressure gradient***

Let $F(t)=F_1.\delta(t)$ where F_1 is a constant and $\delta(t)$ is the unit impulse function defined as:

$\delta(t) = 0, t \neq 0$ and $\delta(t) = 1, t = 0$

The velocities of the upper and lower fluids from (19 and (20) reduce as:

$$w_1 = \frac{8}{l}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+n}F_1}{p_m p_n \lambda_1 \sqrt{4\eta_1 - \xi_1^2}}\left[e^{-\frac{\xi_1 t}{2}}\left\{\sin\sqrt{4\eta_1 - \xi_1^2}\,\frac{t}{2}\right\}\right]\cos(p_n x)\cos(p_m y)$$

for $0 \le x \le 1$ ...(22.1)

$$w_2 = \frac{8}{l}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+n}F_1}{p_m p_n \lambda_2 \sqrt{4\eta_2 - \xi_2^2}}\left[e^{-\frac{\xi_2 t}{2}}\left\{\sin\sqrt{4\eta_2 - \xi_2^2}\,\frac{t}{2}\right\}\right]\cos(p_n x)\cos(p_m y)$$

for $-1 \le x \le 0$ ...(22.2)

CASE III: ***Flow under transient pressure gradient***

Let $F(t)=F_2 e^{-\omega t}$ $(\omega > 0)$ where F_2 is a constant

The velocities of the upper and lower fluids from (19 and (20) reduce as:

$$w_1 = \frac{16}{l}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+n}F_2 e^{-\omega t}}{p_m p_n \lambda_1 (\omega^2 - 2\omega\xi_1 + 4\eta_1)}\begin{bmatrix} 1 - e^{-(\frac{\xi_1}{2}-\omega)t}\left\{\dfrac{\xi_1 - 2\omega}{\sqrt{4\eta_1 - \xi_1^2}} \times \sin\sqrt{4\eta_1 - \xi_1^2}\,\dfrac{t}{2}\right\} \\ + \cos\left\{\sqrt{4\eta_1 - \xi_1^2}\,\dfrac{t}{2}\right\}\end{bmatrix}\cos(p_n x)\cos(p_m y)$$

for $0 \le x \le 1$ ...(23.1)

$$w_2 = \frac{16}{l}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+n}F_2 e^{-\omega t}}{p_m p_n \lambda_2 (\omega^2 - 2\omega\xi_2 + 4\eta_2)}\begin{bmatrix} 1 - e^{-(\frac{\xi_2}{2}-\omega)t}\left\{\dfrac{\xi_2 - 2\omega}{\sqrt{4\eta_2 - \xi_2^2}} \times \sin\sqrt{4\eta_2 - \xi_2^2}\,\dfrac{t}{2}\right\} \\ + \cos\left\{\sqrt{4\eta_2 - \xi_2^2}\,\dfrac{t}{2}\right\}\end{bmatrix}\cos(p_n x)\cos(p_m y)$$

for $-1 \le x \le 0$ ...(23.2)

CASE IV: ***Flow under periodic pressure gradient***

Let F(t)= Re ($F_3 e^{int}$) where F_3 is a constant

The velocities of the upper and lower fluids from (19 and (20) reduce as:

$$w_1 = \frac{4}{l}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+n}F_3}{p_m p_n \lambda_1\sqrt{4\eta_1-\xi_1{}^2}}\left[\begin{array}{l}\dfrac{(N^2+\eta_1)(\xi_1\sin Nt-\cos Nt)+2N^2\sqrt{4\eta_1-\xi_1{}^2}}{R_1S_1}\\ -\dfrac{e^{-\frac{\xi_1 t}{2}}}{2R_1}\{P_1\cos(N+\frac{P_1}{2})t+\xi_1\sin(N+\frac{P_1}{2})t\}\\ -\dfrac{e^{-\frac{\xi_1 t}{2}}}{2S_1}\{Q_1\cos(N-\frac{Q_1}{2})t-\xi_1\sin(N-\frac{Q_1}{2})t\}\end{array}\right]\cos(p_n x)\cos(p_m y)$$

for $0\le x\le 1$ ...(24.1)

$$P_1 = 2N-\sqrt{4\eta_1-\xi_1{}^2} \quad\text{and}\quad Q_1 = 2N+\sqrt{4\eta_1-\xi_1{}^2}$$
$$R_1 = N^2+\eta_1-N\sqrt{4\eta_1-\xi_1{}^2} \quad\text{and}\quad S_1 = N^2+\eta_1+N\sqrt{4\eta_1-\xi_1{}^2}$$

$$w_2 = \frac{4}{l}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+n}F_3}{p_m p_n \lambda_2\sqrt{4\eta_2-\xi_2{}^2}}\left[\begin{array}{l}\dfrac{(N^2+\eta_2)(\xi_2\sin Nt-\cos Nt)+2N^2\sqrt{4\eta_2-\xi_2{}^2}}{R_2S_2}\\ -\dfrac{e^{-\frac{\xi_2 t}{2}}}{2R_2}\{P_2\cos(N+\frac{P_2}{2})t+\xi_2\sin(N+\frac{P_2}{2})t\}\\ -\dfrac{e^{-\frac{\xi_2 t}{2}}}{2S_2}\{Q_2\cos(N-\frac{Q_2}{2})t-\xi_2\sin(N-\frac{Q_2}{2})t\}\end{array}\right]\cos(p_n x)\cos(p_m y)$$

for $-1\le x\le 0$ ...(24.2)

$$P_2 = 2N-\sqrt{4\eta_2-\xi_2{}^2} \quad\text{and}\quad Q_2 = 2N+\sqrt{4\eta_2-\xi_2{}^2}$$
$$R_2 = N^2+\eta_2-N\sqrt{4\eta_2-\xi_2{}^2} \quad\text{and}\quad S_2 = N^2+\eta_2+N\sqrt{4\eta_2-\xi_2{}^2}$$

5. DISCUSSIONS

From the velocities of the upper and lower fluids obtained in equations (19) and (20) we can easily find out the corresponding velocities in case of two immiscible **Maxwell fluids** just making both μ_1 and μ_2 zero, respectively. Then only ξ_1 and ξ_2 will change in equations (19) and (20). Similarly, making λ_1 and λ_2 both zero in equations (19) and (20) we can obtain the corresponding velocities of two immiscible visco-elastic **Rivlin-Ericksen fluids**. Hence, just by means of the above changes we can also determine the velocities of two immiscible Maxwell fluids and Rivlin-Ericksen fluids under constant, impulsive, transient and periodic pressure gradient.

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