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BEST PROXIMITY POINT THEOREMS FOR $(\alpha - \psi)$ RATIONAL PROXIMAL CONTRACTION MAPPINGS IN MULTIPLICATIVE METRIC SPACES

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Abstract

In this paper, we discuss $\alpha - \psi$ rational proximal contraction mappings in setting of multiplicative metric spaces. Our main results improve and extend several results in the literature. As an application we deduce the best proximity point and fixed point results in multiplicative metric spaces.

Keywords:

Best proximity points, fixed points, multiplicative metric spaces, $\alpha - \psi$ rational proximal contractive mappings.

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1. Introduction

Bashirov et al., [1] studied the multiplicative calculus defined a new distance so called multiplicative distance. By using this idea Ozavsar and Cevike [2] introduced the concept of multiplicative metric spaces by using the idea of multiplicative distance and gave some topological properties in such space. They also introduced the concept of multiplicative Banach's contraction mapping and proved fixed point results for such mapping in multiplicative metric spaces.

Chirasak Mongkolkeha [9] introduce the new class of proximal contraction , in the framework of multiplicative metric spaces which is more more general than classes of proximal contraction mappings in [4] and multiplicative Banach's contraction for nan-self mapping.The notation of best proximity point is introduced in [3].It turns out that many of the contractive conditions which are investigated for fixed points ensure the existence of best proximity points.

Precisely, we introduce the notions of $\alpha - \psi$ rational proximal contraction mappings , then we establish some corresponding best proximity theorems for such contraction and give some applications. Our main results generalize, extend and improve the corresponding results on the topics given in the literature.

2. PRELIMINARIES

In this section, we give some definitions and basic concept of multiplicative metric space for our consideration. Throughout this paper, we denote, $\mathbb{N}$, $\mathbb{R}^+$ and $\mathbb{R}$ the sets of positive integers, positive real numbers and real numbers, respectively.

Definition 2.1 ([1]). Let X be a nonempty set. A mapping d : X x X → R is said to be multiplicative metric if it is satisfying the following conditions:

i) d(x, y) ≥ 1 for all x, y ∈ X and d(x, y) = 1 if and only if x = y
ii) d(x, y) = d(y, x) for all x, y ∈ X
iii) d(x, z) ≤ d(x, y). d(y, z) for all x, y, z ∈ X (multiplicative triangle inequality)

Also, the ordered pair (X, d) is called multiplicative metric space.

Example 2.2 ([2]). Let d : $(\mathbb{R}^+)^n$ x $(\mathbb{R}^+)^n \to \mathbb{R}$ be defined as follows.

$$d(x, y) = \left|\frac{x_1}{y_1}\right| . \left|\frac{x_2}{y_2}\right| \cdots \left|\frac{x_n}{y_n}\right| ,$$

where $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in (\mathbb{R}^+)^n$ and $|\,.\,| : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as follows

$$| a | = \begin{cases} a \text{ if } a \geq 1 \\ \dfrac{1}{a} \text{ if } a < 1 \end{cases}$$

Then $((\mathbb{R}^+)^n, d)$ is a multiplicative metric space.

Definition 2.3 ([2]). Let (X, d) be a multiplicative metric space, x ∈ X and $\varepsilon > 1$.

Define the following set $B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$,which is called the multiplicative open ball of radius ε with center x.Similarly, one can describe the multiplicative closed ball as follows:

$$\overline{B}_\varepsilon(x) = \{y \in X : d(x, y) \leq \varepsilon\} .$$

Definition 2.4 ([2]). Let (x, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and x ∈ X. If, for any multiplicative open ball $B_\varepsilon(x)$, there exists a natural number N such that, for all n ≥ N, $x_n \in B_\varepsilon(x)$, then the sequence $\{x_n\}$ is said to be multiplicative convergent to the point x, which is denoted by $x_n \to x$ as $n \to \infty$.

Lemma 2.5 ([2]). Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and x ∈ X. Then $x_n \to x$ as $n \to \infty$ if and only if $d(x_n, x) \to 1$ as $x_n \to x$ as $n \to \infty$.

Lemma 2.6 ([2]). Let (X, d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. If the sequence $\{x_n\}$ is multiplicative convergent, then the multiplicative limit point is unique.

Definition 2.7 ([2]). Let (X, d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is called a multiplicative Cauchy sequence if, for all $\in > 1$, there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \in$ for all $m, n \geq N$.

Lemma 2.8 ([2]). Let (X, d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a multiplicative Cauchy sequence if and only if $d(x_m, x_n) \to 1$ as $m, n \to \infty$.

Theorem 2.9 ([2]). Let (X, d) be a multiplicative metric space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $x_n \to x$ and $y_n \to y \in X$ as $n \to \infty$. Then $d(x_n, y_n) \to (x, y)$ as $n \to \infty$.

Definition 2.10 ([2]). Let (X, d) be a multiplicative metric space and $A \subseteq X$. Then, we call $x \in A$, a multiplicative interior point of A if there exists an $\in > 1$ such that $B_\varepsilon(x) \subseteq A$. The collection of all interior points of A is called multiplicative interior of A and denoted by int(A).

Definition 2.11 ([2]). Let (X, d) be a multiplicative metric space and $A \subseteq X$. If every point A is a multiplicative interior point of A,i.e, A = int(A), then A is called a multiplicative open set.

Definition 2.12 ([2]). Let (X, d) be a multiplicative metric space A subset $S \subseteq X$ is called multiplicative closed in (X, d) if S contains all of its multiplicative limit points.

Theorem 2.13([2]). Let (X, d) be a multiplicative metric space. A subset $S \subseteq X$ is multiplicative closed if and only if X\S, the complement of S, is multiplicative open.

Theorem 2.14 ([2]). Let (X, d) be a multiplicative metric space and $S \subseteq X$. Then the set S is multiplicative closed if and only if every multiplicative convergent sequence in S has a multiplicative limit point that belongs to S.

Theorem 2.15 ([2]). Let (X, d) be a multiplicative metric space and $S \subseteq X$. Then (S, d) is complete if and only if S is multiplicative closed.

Theorem 2.16 ([2]). Let (X, d_x) and (Y, d_y) be two multiplicative metric spaces, $f : X \to Y$ be a mapping and $\{x_n\}$ be any sequence in X. Then f is multiplicative continuous at the point $x \in X$ if and only if $f(x_n) \to f(x)$ as $n \to \infty$ for every sequence $\{x_n\}$ with $x_n \to x$ as $n \to \infty$.

Next we give the notations A_0, B_0 and d(A,B) for nonempty subsets A and B of a multiplicative metric space (X, d) in the same sense in metric spaces.

Let A and B be non-empty subsets of a multiplicative metric space (X, d), we recall the following notations and notions that will be used in what follows.

$$d(A, B) = \inf\{d(x, y) : x \in A \text{ and } y \in B\}$$
$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$

$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}$

Definition 2.17([7]): Let A be non-empty subset of a multiplicative metric space (X, d).

A mapping g: A → A is said to be isometry if d(gx, gy) = d(x, y) for all

x, y ∈ A.

Definition 2.18 ([7]). Let A and B be non-empty subsets of a multiplicative metric space

(X, d). A point x ∈ A is called a best proximity point of a mapping T : A → B if it satisfies the condition that d(x, Tx) = d(A, B).

Definition 2.19([7]). A subset A of a multiplicative metric space (X, d) is said

to be approximately compact with respect to B, if every sequence $\{x_n\}$ in A satisfies the condition that $d(y, x_n) \to d(y, A)$ as $n \to \infty$ for some y ∈ B has a convergent subsequence.

Definition 2.20 ([10]): Let A, B be two non-empty subsets of a multiplicative metric space

(X, d) and α: A × A → [0,∞) be a given mapping. A mapping T: A→ B is said to be

an α- proximal admissible if it satisfies the following condition:

$$\left.\begin{array}{c} u,v,x,y \in A, \\ \alpha(x,y) \geq 1, \\ d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B) \end{array}\right\} \Rightarrow \alpha(u,v) \geq 1$$

Definition 2.21([10]): Let A, B be two non-empty subsets of a multiplicative metric space (X, d) and α: A × A → [0,∞) be a given mapping. A mapping T: A→ B is said to be an α-multiplicative proximal contraction if there exist $k \in \{0,1)$ it satisfies the following condition:

$$\left.\begin{array}{c} u,v,x,y \in A, \\ \\ d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B) \end{array}\right\} \Rightarrow \alpha(x,y)d(u,v) \geq d(x,y)^k$$

Remark ([10]): If α: A × A → [0,∞) is defined by $\alpha(x,y) = 1$ for all $x, y \in A$, then an α- multiplicative proximal contraction mapping reduces to a multiplicative proximal contraction mapping due to Mongkolkeha and sintunavarat[3]. Therefore the concept of α- multiplicative proximal contraction self mapping is also a generalization of Banach's contraction mapping in the setting of multiplicative metric spaces which was introduced by Ozavsar and Cevikel.

Let ψ_3 be the family of functions ψ:[0,+∞) → [0,+∞) satisfyinf the following properties:

(i) ψ is upper semi-continous and nondecreasing function ,

(ii) $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for each $t > 0$,where ψ^n is the n-th iterate of ψ,

(iii) $\psi(t) < t \; for \; every \; t > 0$.

Definition 2.22([12]): Let (X, d) be a metric and A,B be two non-empty subsets of X. Let T: A→ B and α, η: A × A → [0,∞) be functions. We say that T is generalized $\alpha - \psi$ proximal contraction with respect to η if , for all $x, y \in A$ and ψ in ψ_3 such that

$$\alpha(x,y) \geq \eta(x,y) \Rightarrow d(Tx,Ty) \leq \psi(M(x,y))$$

where

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2} - d(A,B), \frac{d(x,Ty)+d(y,Tx)}{2} - d(A,B)\right\}$$

Definition 2.23([12]): Let (X, d) be a metric and A,B be two non-empty subsets of X .

Let T: A→ B and α,η: A × A → [0,∞) be functions. We say that T is generalized $\alpha - \psi$ proximal contraction with respect to η if , for all $x, y, u, v \in A$ and ψ in ψ_3 such that

$$\begin{cases} \alpha(x,y) \geq \eta(x,y) \\ d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{cases} \Rightarrow d(u,v) \leq \psi(M(x,y))$$

3.Main Results

Now, we formulate the notion of $(\alpha - \psi)$ rational proximal contractive mappings in the context of a multiplicative metric space as follows.

Definition 3.1

Let (X , d) be a multiplicative metric space and A , B be two nonempty subsets of X. Let T: A→ B and $\alpha: A \times A \to [0,\infty)$ be functions. Then T is said to be $(\alpha - \psi)$ rational proximal contraction, if for all x,y u,v ϵ A and $\psi \in \psi_3$ such that

$$\left.\begin{matrix} d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{matrix}\right\} \Longrightarrow \alpha(x,y)d(u,v) \leq \psi(M(x,y)) \text{ .... (3.1)}$$

where,

$$M(x,y) = \max\{d(x,y), \frac{d(x,Tx).d(y,Ty)}{1+d(x,y)} - d(A,B), \frac{d(x,Ty).d(y,Tx)}{1+d(x,y)} - d(A,B), \frac{d(x,Ty).d(y,Tx)}{1+d(Tx,Ty)} - d(A,B)\}$$

Theorem:3.2

Let A and B be a two nonempty closed subsets of a complete multiplicative metric space (X,d) such that A_0 and B_0 are nonempty. Let the mappings $\alpha: A \times A \to [0,\infty)$, T:A →B and G: A →A satisfy the following conditions.

(i) T is $(\alpha - \psi)$ rational proximal contraction mapping and T is an $\alpha -$ proximal admissible mapping

(ii) T is continuous

(iii) g is an isometry

(iv) $A_0 \subseteq g(A_0)$

(v) $T(A_0) \subseteq B_0$.

(vi) There exists $x_0, x_1 \in A_0$ such that $d(gx_1,Tx_0) = d(A,B)$ and $\alpha(x_0, x_1) \geq 1$

Then T has a Unique best proximity point if for every y,z ∈ A such that $d(gy,Ty) = d(A,B) = d(gz,Tz)$ and $\alpha(gy,gz) \geq 1$.

Proof :

Choose $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists $x_1 \in A_0$ such that

$$d(gx_1,Tx_0) = d(A,B) \quad \text{and} \quad \alpha(x_1, x_0) \geq 1 \qquad \text{...... (3.2)}$$

Again choose a point $x_1 \in A_0$, there exists $x_2 \in A_0$

$$d(gx_2,Tx_1) = d(A,B) \quad \text{and } \alpha(x_1, x_2) \geq 1 \qquad \text{......(3.3)}$$

since T is an $(\alpha - \psi)$ rational proximal contraction mapping and g is an isometry .
By repeating this process, having chosen by induction $\{x_n\} \in A_0$ such that

$$\alpha(x_n, x_{n-1}) \geq 1 \qquad \text{.......... (3.4)}$$
$$d(gx_n,Tx_{n-1}) = d(A,B) \qquad \text{.......... (3.5)}$$
$$d(gx_{n+1},Tx_n) = d(A,B) \qquad \text{.......... (3.6)}$$

for all n∈ ℕ. .Since T is a $(\alpha - \psi)$ rational proximal contraction mapping ,we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq d(gx_{n+1}, gx_n) \\ &\leq \alpha(x_{n-1}, x_n) d(gx_{n+1}, gx_n) \\ &\leq \psi(M(x_n, x_{n-1})) \qquad \text{..............} \quad (3.7) \end{aligned}$$

for all n ∈ ℕ. where

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\Big\{d(x_n, x_{n-1}), \frac{d(x_n, Tx_n).d(x_{n-1}, Tx_{n-1})}{1+d(x_n, x_{n-1})} - d(A,B), \\ &\quad \frac{d(x_n, Tx_{n-1}).d(x_{n-1}, Tx_n)}{1+d(x_n, x_{n-1})} - d(A,B), \frac{d(Tx_n, x_{n-1}).d(Tx_{n-1}, x_n)}{1+d(Tx_n, Tx_{n-1})} - d(A,B)\Big\} \\ &\leq \max\Big\{d(x_n, x_{n-1}), \frac{d(x_n, x_{n+1}).d(x_{n-1}, x_n)}{1+d(x_n, x_{n-1})} - d(A,B), \\ &\quad \frac{d(x_n, x_n).d(x_{n-1}, x_{n+1})}{1+d(x_n, x_{n-1})} - d(A,B), \frac{d(x_{n+1}, x_{n-1}).d(x_n, x_n)}{1+d(x_{n+1}, x_n)} - d(A,B)\Big\} \\ &\leq \max\Big\{d(x_n, x_{n-1}), \frac{d(x_{n-1}, x_{n+1})}{1+d(x_n, x_{n-1})} - d(A,B), \\ &\quad \frac{d(x_{n-1}, x_{n+1})}{1+d(x_n, x_{n-1})} - d(A,B), \frac{d(x_{n+1}, x_{n-1})}{1+d(x_{n+1}, x_n)} - d(A,B)\Big\} \\ &\leq \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\} \end{aligned}$$

Since

$$\frac{d(x_n, x_{n+1})}{1+d(x_n, x_{n-1})} - d(A,B) \leq d(x_n, x_{n+1})$$
$$\frac{d(x_{n-1}, x_{n+1})}{1+d(x_n, x_{n-1})} - d(A,B) \leq d(x_n, x_{n+1})$$
$$\frac{d(x_{n+1}, x_{n-1})}{1+d(x_{n+1}, x_n)} - d(A,B) \leq d(x_{n-1}, x_n)$$

Case (i) : If for some n , we have $M(x_n, x_{n-1}) = d(x_n, x_{n+1})$, then

$$\begin{aligned} d(x_{n+1}, x_n) &\leq d(gx_{n+1}, gx_n) \\ &\leq \psi(M(x_n, x_{n-1})) \\ &= \psi(d(x_n, x_{n+1})) \\ &< d(x_n, x_{n+1}) \end{aligned}$$

Which is impossible .

Case (ii) :

If $M(x_n, x_{n-1}) = d(x_{n-1}, x_n)$ for all n ∈ ℕ.

$$d(x_{n+1}, x_n) \leq d(gx_{n+1}, gx_n)$$

$$\leq \psi\left(M(x_n, x_{n-1})\right)$$

$$= \psi\left(d(x_{n-1}, x_n)\right)$$
$$< d(x_{n-1}, x_n)$$

Since ψ is nondecreasing , we get by induction

$$d(x_{n+1}, x_n) \leq \psi^n(d(x_1, x_0) \;\forall\, n \in \mathbb{N}.$$

For (ii) condition , we have $\lim_{n\to\infty} d(x_{n+1}, x_n) = 1$.

Putting $x = x_{n+1}$ and $y = x_{n-1}$ in (3.1) & using (3.3) we have

$$\begin{aligned} d(x_{n+2}, x_n) &= d(gx_{n+1}, gx_{n-1}) \\ &\leq \alpha(x_{n+1}, x_{n-1}).\, d(gx_{n+2}, gx_n) \\ &\leq \psi(M(x_{n+1}, x_{n-1})) \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3.8)$$

where , $M(x_n, x_{n-1}) = \max\{d(x_n, x_{n-1}), \frac{d(x_n, Tx_n).d(x_{n-1}, Tx_{n-1})}{1+d(x_n, x_{n-1})} - d(A,B),$
$\frac{d(x_n, Tx_{n-1}).d(x_{n-1}, Tx_n)}{1+d(x_n, x_{n-1})} - d(A,B), \frac{d(x_n, Tx_{n-1}).d(x_{n-1}, Tx_n)}{1+d(x_n, x_{n-1})} - d(A,B)\}$
$\leq \max\{d(x_n, x_{n-1}), \frac{d(x_n, Tx_n).d(x_{n-1}, Tx_{n-1})}{1+d(x_n, x_{n-1})} - d(A,B),$
$\frac{d(x_n, Tx_{n-1}).d(x_{n-1}, Tx_n)}{1+d(x_n, x_{n-1})} - d(A,B), \frac{d(x_n, Tx_{n-1}).d(x_{n-1}, Tx_n)}{1+d(x_n, x_{n-1})} - d(A,B)$
}

Define $a_n = d(x_n, x_{n+2})$ and $b_n = d(x_n, x_{n+1})$ .Then

$$M(x_{n+1}, x_{n-1}) = \max\{a_{n-1}, \frac{b_{n+1}.b_{n-1}}{1+a_{n-1}} - d(A,B), \frac{b_{n-1}.b_n}{1+a_{n-1}} - d(A,B), \frac{b_{n-1}.b_n}{1+a_n} - d(A,B)\}$$
$$\leq \max\{a_{n-1}, \frac{b_{n+1}.b_{n-1}}{1+a_{n-1}}, \frac{b_{n-1}.b_n}{1+a_{n-1}}, \frac{b_{n-1}.b_n}{1+a_n}\}$$

If $M(x_{n+1}, x_{n-1}) = a_{n-1}$ (or) $\frac{b_{n+1}.b_{n-1}}{1+a_{n-1}}$ (or) $\frac{b_{n+1}.b_n}{1+a_{n-1}}$ (or) $\frac{b_{n-1}.b_n}{1+a_n}$ then (3.8)

implies $a_n \leq \psi(a_{n-1}) < a_{n-1}$

due to the property (iii) of ψ ,that is sequence $\{a_n\}$ is positive monotone decreasing , and hence it converges to some $t \geq 1$.

Assume that $t > 1$. Now by (3.8) , we get

$$t = \lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \sup \psi(a_{n-1}) = \psi(\lim_{n\to\infty} \sup(a_{n-1}) = \psi(t) < t\ .$$

Which is a contradiction.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} d(x_{n+2}, x_n) = 1$$

Let $x_n = x_m$ for some m , n ∈ ℕ with $m \neq n$.

without loss of generality , we assume that m > n+1 .Since $\alpha(x_m , x_{m-1}) \geq 1$,

we get , d($x_n, x_{n+1}) = d(gx_n , gx_{n+1}) = d(gx_m , gx_{m+1})$

$$\leq \psi(M(x_{m-1} , x_m))$$

where ,

$$M(x_{m-1} , x_m) = \max\{d(x_{m-1} , x_m) , \frac{d(Tx_{m-1} ,x_{m-1}).d(Tx_m ,x_m)}{1+d(x_{m-1} , x_m)} - d(A,B),$$

$$\frac{d(Tx_{m-1} ,x_m).d(Tx_m ,x_{m-1})}{1+d(x_{m-1} , x_m)} - d(A,B), \frac{d(Tx_{m-1} ,x_m).d(Tx_m ,x_{m-1})}{1+d(Tx_{m-1} ,Tx_m)} - d(A,B), \}$$

$$\leq \max\{d(x_{m-1} , x_m) , \frac{d(x_m ,x_{m-1}).d(x_{m+1} ,x_m)}{1+d(x_{m-1} , x_m)} - d(A,B),$$

$$\frac{d(x_m ,x_m).d(x_{m+1} ,x_{m-1})}{1+d(x_{m-1} , x_m)} - d(A,B), \frac{d(x_m ,x_m).d(x_{m+1} ,x_{m-1})}{1+d(x_m ,x_{m+1})} - d(A,B) \}$$

$$= \max\{d(x_m , x_{m-1}), d(x_{m+1} , x_m)\} \quad \ldots\ldots\ldots\ldots\ldots (3.9)$$

If $M(x_{m+1} , x_m) = d(x_m , x_{m-1})$ then (3.10) implies

$$d(x_n , x_{n-1}) \leq \psi(d(x_{m-1}, x_m))$$

$$\leq \psi^{m-n}(d(x_n , x_{n+1})) \text{ for all m ,n} \in \mathbb{N} \quad \ldots\ldots\ldots\ldots\ldots (3.10)$$

If on the other hand , If $M(x_{m-1} , x_m) = d(x_{m+1} , x_n)$ then from we have

$$d(x_n , x_{n+1}) \leq \psi(d(x_{m+1}, x_m))$$

$$\leq \psi^{m-n+1}(d(x_n , x_{n+1})) \text{ for all m ,n} \in \mathbb{N}$$

Using the property (iii) of ψ ,the two inequalities (3.9) and (3.10) imply

$$d(x_n , x_{n+1}) < d(x_n , x_{n+1})$$

Which is impossible.

Thus $x_n \neq x_m$ for all m,n ∈ ℕ with m≠ n.

Now we prove that is a Cauchy sequence, that is $\lim_{n\to\infty} d(x_n , x_{n+k}) = 1$ .for all $k \in \mathbb{N}$.we have already proved the cases for k=1 & k=2 in (3.7) & (3.8) respectively.

Take arbitrary k ≥ 3. we discuss two cases.

Case (i) :

Suppose that k=2m+1 , where m ≥ 1,using the multiplicative triangle inequality , we have

$$d(x_n,x_{n+k}) = d(gx_n,gx_{n+k}) = d(x_n , x_{n+2m+1}) = d(x_n , x_{n+2m+1})$$

$$\leq d(x_n , x_{n+1}).d(x_{n+1} , x_{n+2}) \ldots d(x_{n+2m} , x_{n+2m+1})$$

$$\leq \prod_{p=n}^{n+2m} \psi^p(d(x_0, x_1))$$

$$\leq \prod_{p=n}^{\infty} \psi^p(d(x_0, x_1)) \to 1 \quad \text{as } n \to \infty.$$

Case (ii) :

Suppose that k=2m , where $m \geq 2$,using the multiplicative triangle inequality , we have

$$d(x_n,x_{n+k}) = d(gx_n , gx_{n+k})$$

$$= d(x_n , x_{n+2m})$$

$$\leq d(x_n , x_{n+2}).d(x_{n+2} , x_{n+3}) \ldots d(x_{n+2m-1} , x_{n+2m})$$

$$\leq d(x_n , x_{n+2}). \prod_{p=n}^{n+2m-1} \psi^p(d(x_0, x_1))$$
$$\leq d(x_n , x_{n+2}). \prod_{p=n}^{\infty} \psi^p\big(d(x_0, x_1)\big) \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

By 2, we have $\lim_{n\to\infty} d(x_{n+2} , x_n) = 1.$

Therefore $\lim_{n\to\infty} d(x_{n+1} , x_n) = 1$ for all k ≥ 3 .Hence x_n is a Cauchy sequence .

Moreover by the continuity of g,we have $gx_n \rightarrow gx$. (i.e) $d(gx,gx_n) \rightarrow 1$ as $n \rightarrow \infty$.
By the completeness of x and A_0 is closed, we have x∈ A_0 such that $x_n \rightarrow x$.
For each n ∈ ℕ , we have

$$d(gx , Tx_n) \leq d(gx , gx_{n+1}). d(gx_{n+1} , Tx_n)$$
$$\leq d(gx , gx_{n+1}). d(A, B)$$
$$\leq d(gx , B) \qquad \text{(Since } (gx , gx_{n+1}) \rightarrow 1 \text{)}$$

On the other hand ,
$X_0 \in A_0$ and $T(A_0) \subseteq B_0$ there exists $z \in A_0$ such that d(z,Tx) =d(A,B) and $\alpha(x_n , x_{n+1}) \geq 1$ for all n ∈ ℕ and $x_n \rightarrow z$ and $n \rightarrow \infty$, we get,

$$\alpha(x_n , z) \geq 1$$
$$d(gx_{n+1}, z) \leq \psi\big(M(x_n, x)\big) < M(x_n, x)$$

Then , $d(z, gx_{n+1}) \rightarrow 1 \text{ and } z = gx \Longrightarrow d(gx, Tx) = d(A, B)$
This completes the proof.

Theorem:3.3

Let A and B be a two nonempty closed subsets of a complete multiplicative metric space (X,d) such that A_0 and B_0 are nonempty. Let the mappings $\alpha: A \times A \rightarrow [0 , \infty)$, T: A →B and g: A →A satisfy the following conditions.

(i) T is $(\alpha - \psi)$ rational proximal contraction mapping and T is an $\alpha -$ proximal admissible mapping

(ii) g is an isometry

(iii) $A_0 \subseteq g(A_0)$

(iv) $T(A_0) \subseteq B_0$.

(v) if {$\{x_n\}$ is a sequence in A_0 such that $\alpha(gx_n, x_{n+1}) \geq 1$ and $gx_n \rightarrow gx \in A$, then $\alpha(x_n, gx) \geq 1$ for all $n \in \mathbb{N}$.

(vi) there exists x_0, x_1∈ A_0 such that $d(gx_1,Tx_0)$ = d(A,B) and $\alpha(x_0 , x_1) \geq 1$

Then T has a Unique best proximity point if for every y,z ∈ A such that d(gy,Ty) =d(A,B) =d(gz ,Tz) and $\alpha(gy, gz) \geq 1$.

Proof:

Following the proof of the theorem 3.2 , We check $\{x_n\}$ be a sequence in B and $\{Tx_{nk}\}$ is a convergent subsequence of $\{Tx_n\}$ such that converging to some element u∈ B.Further for k ∈ ℕ.we have

$$\begin{aligned} d(A,B) &\le d(gx,u) \\ &\le d(gx, gx_{n_{k+1}}).d(gx_{n_{k+1}}, Tx_{n_k}).d(Tx_{n_k}, u) \\ &\le d(gx, gx_{n_{k+1}}).d(A,B).d(Tx_{n_k}, u) \end{aligned} \quad ....(3.11)$$

Letting $k \to \infty$ in equation (3.11) d(gx,u) = d(A,B)

Therefore , we have that,

d(gz,Tx) = d(A,B) for some z ∈A_0.

Here Tx ∈ B_0 , g is an isometry .From assumption (v) there exists a subsequence of such that $\alpha(gx_{n_k}, gx) \ge 1$ for k∈ ℕ.Since T is $(\alpha - \psi)$ rational proximal contraction mapping and g is an isometry,

we have $d(x_{n_{k+1}}, z) \le d(gx_{n_{k+1}}, gz) \le \psi\left(M(x_{n_k}, z)\right)$

for all k ∈ ℕ .On the other hand

$$\begin{aligned} M(x_{n_k}, z) &= \max\left\{d(x_{n_k}, z), \frac{d(x_{n_k}, Tx_{n_k}).d(z,Tz)}{1+d(x_{n_k}, z)} - d(A,B), \right. \\ &\left. \frac{d(x_{n_k}, Tz).d(z, Tx_{n_k})}{1+d(x_{n_k}, z)} - d(A,B), \frac{d(x_{n_k}, Tz).d(z, Tx_{n_k})}{1+d(x_{n_k}, z)} - d(A,B)\right\} \\ &= \max\{d(x_{n_k}, z), d(x_{n_k}, z)\} \end{aligned}$$

In the above inequality, we have

$$M(x_{n_k}, z) = d(x_{n_k}, z)$$

Further $d(x_{n_{k+1}}, z) \le M(x_{n_k}, z)$

Letting $k \to \infty$, $d(x,z) \le d(x,z)$

Which is a contradiction and this $\Longrightarrow d(x,z) \longrightarrow 1.$ (i.e) z = x.

From this we have , d(gx,Tx) = d(A,B) .

Since $A_0 \subseteq g(A_0)$ and g is an isometry.
Hence x is a best proximity point of T. Next we prove the Uniqueness, suppose that there exists $x^*, y^* \in A$ with $x^* \ne y^*$ and $d(gx^*, Ty^*) = d(A,B)$
From hypothesis, we obtain $\alpha(x, x^*) \ge 1$. Since g is an isometry and T is an $(\alpha - \psi)$ rational proximal contraction mapping , we get

$$\begin{aligned} d(x,x^*) &= d(gx, gx^*) \\ &\le \alpha(x,x^*)d(gx,gx^*) \\ &\le \psi(M(x,x^*)) \end{aligned}$$

where ,

$$M(x,x^*) = \max\left\{d(x,x^*), \frac{d(x,Tx).d(x^*,Tx^*)}{1+d(x,Tx^*)} - d(A,B), \frac{d(x,Tx^*).d(x^*,Tx)}{1+d(x,Tx^*)} - d(A,B), \frac{d(x,Tx^*).d(x^*,Tx)}{1+d(Tx,Tx^*)} - d(A,B)\right\}$$

$$= d(x,x^*)$$

Hence we get , $d(x,x^*) \leq \psi\big(d(x,x^*)\big) < d(x,x^*)$

which is impossible. That is $x = x^*$. This completes the proof .

Taking $\alpha(x,y) \geq \eta(x,y)$ in Theorem 3.2 and theorem 3.3, we obtain the following results.

Corollary :3.4

Let A and B be a two nonempty closed subsets of a complete multiplicative metric space (X,d) such that A_0 and B_0 are nonempty. Let the mappings : $A \times A \rightarrow [0,\infty)$, T:A →B and g: A →A satisfy the following conditions.

(i) T is $(\alpha - \psi)$ rational proximal contraction mapping with respect to η.
(ii) T is continuous
(iii) g is an isometry
(iv) $A_0 \subseteq g(A_0)$
(v) $T(A_0) \subseteq B_0$.
(vi) there exists $x_0, x_1 \in A_0$ such that $d(gx_1,Tx_0) = d(A,B)$ and $\alpha(x_0,x_1) \geq \eta((x_0,x_1)$

Then T has a Unique best proximity point if for every y, z ∈ A such that $d(gy,Ty) = d(A,B) = d(gz,Tz)$ and $\alpha(gy,gz) \geq \eta(gy,gz)$.

Corollary :3.5

Let A and B be a two nonempty closed subsets of a complete multiplicative metric space (X, d) such that A_0 and B_0 are nonempty. Let the mappings $\alpha: A \times A \rightarrow [0,\infty)$, T: A →B and g: A →A satisfy the following conditions.

(i) T is $(\alpha - \psi)$ rational proximal contraction mapping with respect to η.
(ii) g is an isometry
(iii) $A_0 \subseteq g(A_0)$
(iv) $T(A_0) \subseteq B_0$.
(v) if $\{x_n\}$ is a sequence in A_0 such that $\alpha(gx_n, x_{n+1}) \geq \eta(gx_n, x_{n+1})$ and $gx_n \rightarrow gx \in A$, then $\alpha(x_n, gx\ ) \geq \eta(x_n, gx\ )$ for all $n \in \mathbb{N}$.
(vi) there exists $x_0, x_1 \in A_0$ such that $d(gx_1,Tx_0) = d(A,B)$ and $\alpha(x_0,x_1) \geq \eta((x_0,x_1)$.

Then T has a Unique best proximity point if for every y,z ∈ A such that $d(gy,Ty) = d(A,B) = d(gz,Tz)$ and $\alpha(gy,gz) \geq \eta(gy,gz)$.

Taking $\eta(x,y) \leq 1$ in Theorem 3.2 and theorem 3.3, we obtain the following results.

Corollary :3.6

Let A and B be a two nonempty closed subsets of a complete multiplicative metric space (X,d) such that A_0 and B_0 are nonempty. Let the mappings $\alpha: A \times A \rightarrow [0, \infty)$, T:A →B and g: A →A satisfy the following conditions.

(i) T is ψ rational proximal contraction .

(ii) T is continuous

(iii) g is an isometry

(iv) $A_0 \subseteq g(A_0)$

(v) $T(A_0) \subseteq B_0$ and T is η - subadmissible.

(vi) there exists $x_0, x_1 \in A_0$ such that $d(gx_1, Tx_0) = d(A,B)$ and $\eta((x_0, x_1) \leq 1$

Then T has a Unique best proximity point if for every y, z ∈ A such that $d(gy,Ty) = d(A,B) = d(gz, Tz)$ and $\eta(gy, gz) \leq 1$.

Corollary :3.7

Let A and B be a two nonempty closed subsets of a complete multiplicative metric space (X, d) such that A_0 and B_0 are nonempty. Let the mappings $\alpha: A \times A \rightarrow [0, \infty)$, T: A →B and g: A →A satisfy the following conditions.

(i) T is ψ rational proximal contraction .

(ii) g is an isometry

(iii) $A_0 \subseteq g(A_0)$

(iv) $T(A_0) \subseteq B_0$ and T is η - subadmissible.

(v) if $\{x_n\}$ is a sequence in A_0 such that $\eta(gx_n, x_{n+1}) \leq 1$ and $gx_n \rightarrow gx \in A$, then $\eta(x_n, gx) \leq 1$ for all $n \in \mathbb{N}$.

(vi) there exists $x_0, x_1 \in A_0$ such that $d(gx_1, Tx_0) = d(A,B)$ and $\eta((x_0, x_1) \leq 1$.

Then T has a Unique best proximity point if for every y,z ∈ A such that $d(gy,Ty) = d(A,B) = d(gz, Tz)$ and $\eta(gy, gz) \leq 1$.

If g is the identity mapping and $\alpha(x,y) \geq \eta(x,y)$ in theorem 3.2 and theorem 3.3 ,then we obtain the following.

Corollary :3.8

Let A and B be a two nonempty closed subsets of a complete multiplicative metric space (X,d) such that A_0 and B_0 are nonempty. Let the mappings $\alpha: A \times A \rightarrow [0, \infty)$, T:A →B and g: A →A satisfy the following conditions.

(i) T is $(\alpha - \psi)$ rational proximal contraction mapping with respect to η.

(ii) T is continuous

(iii) $T(A_0) \subseteq B_0$.

(iv) there exists $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A,B)$ and $\alpha(x_0, x_1) \geq \eta((x_0, x_1)$

Then T has a Unique best proximity point if for every y, z ∈ A such that $d(y,Ty) = d(A,B) = d(z, Tz)$ and $\alpha(y,z) \geq \eta(y,z)$.

Corollary :3.9

Let A and B be a two nonempty closed subsets of a complete multiplicative metric space (X, d) such that A_0 and B_0 are nonempty. Let the mappings $\alpha: A \times A \rightarrow [0, \infty)$, T: A →B and g: A →A satisfy the following conditions.

(i) T is $(\alpha - \psi)$ rational proximal contraction mapping with respect to η.

(ii) $T(A_0) \subseteq B_0$.

(iii) if $\{x_n\}$ is a sequence in A_0 such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ and $gx_n \rightarrow gx \in A$, then $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N}$.

(iv) there exists $x_0, x_1 \in A_0$ such that $d(x_1,Tx_0) = d(A,B)$ and $\alpha(x_0, x_1) \geq \eta((x_0, x_1)$.

Then T has a Unique best proximity point if for every y,z ∈ A such that d(y,Ty) =d(A,B) =d(z ,Tz) and $\alpha(y, z) \geq \eta(y, z)$.

4.Applications

An application of our results, we give some new fixed point theorems which can be deduced from our results.In theorem 3.2 and 3.3 ,If we take A=B then we deduce the following result.

Theorem :4.1

Let T be a complete multiplicative metric space (X , d) into itself and $\alpha: A \times A \rightarrow [0, \infty)$ be a given function satisfying the following conditions.

(i) T is $(\alpha - \psi)$ rational contraction and α-admissible mapping.

(ii) T is continuous

(iii) g is an isometry and A ⊆ g(A) such that $\alpha(x, y) \geq 1$ and $d(Tx, Ty) \leq \psi(M(x, y))$,

where,

$$M(x, y) = \max\{d(x, y), \frac{d(x,Tx).d(y,Ty)}{1+d(x,y)}, \frac{d(x,Ty).d(y,Tx)}{1+d(x,y)}, \frac{d(x,Ty).d(y,Tx)}{1+d(Tx,Ty)}\}$$

Then T has a Unique fixed point .

Theorem :4.2

Let T be a complete multiplicative metric space (X , d) into itself and $\alpha: A \times A \rightarrow [0, \infty)$ be a given function satisfying the following conditions.

(i) T is $(\alpha - \psi)$ rational proximal contraction and α-admissible mapping.

(ii) T is continuous

(iii) g is an isometry and A ⊆ g(A) such that $\alpha(x, y) \geq 1$ and $(Tx, Ty) \leq k(M(x, y))$,

where,

$$M(x, y) = \max\{d(x, y), \frac{d(x,Tx).d(y,Ty)}{1+d(x,y)}, \frac{d(x,Ty).d(y,Tx)}{1+d(x,y)}, \frac{d(x,Ty).d(y,Tx)}{1+d(Tx,Ty)}\}$$

Then T has a Unique fixed point .

Theorem :4.3

Let T be a complete multiplicative m7etric space (X , d) into itself and $\alpha: A \times A \rightarrow [0 , \infty)$ be a given function satisfying the following conditions.

(i) T is $(\alpha - \psi)$ rational proximal contraction and α-admissible mapping.

(ii) T is continuous

(iii) g is an isometry and A ⊆ g(A) such that $\alpha(x, y) \geq 1$ and $(Tx, Ty) \leq \psi\big(d(x, y)\big)$,

Then T has a Unique fixed point .

Theorem :4.4

Let T be a complete multiplicative metric space (X , d) into itself and $\alpha: A \times A \rightarrow [0 , \infty)$ be a given function satisfying the following conditions.

(i) T is $(\alpha - \psi)$ rational proximal contraction and α-admissible mapping.

(ii) T is continuous

(iii) g is an isometry and A ⊆ g(A) such that $\alpha(x, y) \geq 1$ and $(Tx, Ty) \leq kd(x, y)$, Then T has a Unique fixed point.

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