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- FRAMES OVER HIBERT C ALEBRAS

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Abstract

Star Frame are explained with different examples. The operators on * frames are defined in Hilbert C* modules and Hilbert C* algebras.The relationship between frames and * frames are in Hilbert C* algebras are explained.

1 Introduction

Frames are first introduced by in 1952 by Duffin and Schaeffer. They abstracted the fundamental notion of Gabor to study signal processing and non -harmonic Fourier series. Alijani and Dehgan introduced * frames as a generalization of frames in Hilbert C* modules. They studied operators associated to given * frames for Hilbert C* modules over commutative unitary C* algebras. Problems about frames and * frames for Hilbert C* modules are more complicated than those for Hilbert spaces. This makes the study of * frames for Hilbert C* modules important and interesting.

2. Basic Definition

Definiton 2.1:- A C* algebra is a Banach algebra equipped with an involution $a \to a^*$ satisfying the condition $\|aa^*\| = \|a\|^2$

Defintion 2.2:- The standard Hilbert A- Module ℓ_2(A) defined by

$$\ell_2(A) = \left\{\{a_j\}_{j\in J} \subseteq A, \sum_{j\in J} a_j a^*_j converges\ in\ A\right\}$$

Defintion 2.3:- Let A be a C * algebra and H be a A-module. Suppose that the linear structures given on A and H are compatible. i.e $\lambda(ax) = (\lambda a)x$ for every $\lambda \in \mathbb{C}, a \in A$ and x∈ H.

If there exists a mapping $\langle .,. \rangle$: H× H → Awith the properties

(i) $\langle x,x\rangle \geq 0$ for every $x \in H$
(ii) $\langle x,x\rangle = 0$ if and only if x=0
(iii) $\langle x,y\rangle = \langle y,x\rangle^*$ for every x,y ∈ H
(iv) $\langle ax,y\rangle = a\langle x,y\rangle$ for every a ∈ A and x,y ∈ H
(v) $\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$ for every x,y,z∈ H

Then the pair $\{H, \langle .,. \rangle\}$ is called a pre – Hilbert A- module. The map $\langle .,. \rangle$ is said to be an A-valued inner product. If the pre-Hilbert module $\{H, \langle .,. \rangle\}$ is complete with respect to the norm $\|x\| = \|\langle x,x\rangle\|^{1/2}$ then it is called a Hilbert A- Module.

Definition 2.4:- A sequence $\{x_j\}_{j\in J}$ of vectors in a Hilbert space H is called a frame if there exists constants A,B such that 0<A≤ $B < \infty$ such that

$$A\|x\|^2 \leq \sum_{j\in J} |\langle x, x_j\rangle|^2 \leq B\|x\|^2 \text{ for all } x \in H.$$

Definition 2.5:- A synthesis operator T: ℓ_2 → H is defined as Te_j = x_j where $\{e_j\}_{j\in J}$ is an orthonormal basis for ℓ_2.

Definition 2.6:- Let $\{x_j\}_{j\in J}$ be a frame for H and $\{e_j\}_{j\in J}$ be an orthonormal basis for ℓ_2, then the analysis operator T*: H → ℓ_2 is the adjoint of the systhesis operator T and is defined as T*x= $\sum_{j\in J}\langle x, x_j\rangle$ for all x ∈ H.

Definition 2.7:- Let $\{x_j\}_{j\in J}$ be a frame for the Hilbert space H. A frame operator S=TT*:H→H is defined as Sx= $\sum_{j\in J}\langle x, x_j\rangle\, x_j$ for all x ∈ H.

3.Definion on * Frames

Definition 3.1:- Let H be a Hilbert A-Module . A family $\{x_j\}_{j\in J}$of elements of H is a frame for H, If there exists constants 0< $A \leq B < \infty$, such that for all x∈H

A$\langle x, x\rangle \leq \sum_{j\in J}\langle x, x_j\rangle\langle x_j, x\rangle \leq B\langle x, x\rangle$

The numbers A and B are called lower and upper frame bounds respectively.

If A=B Then $\{x_j\}_{j\in J}$is a tight frame

If A=B=1 then $\{x_j\}_{j\in J}$is a normalized tight frame or a parseval frame

If A=B=λ then $\{x_j\}_{j\in J}$ is called a λ –tight frame

Definiton 3.2:- Let A be a C* algebra and J be a finite or countable index set. A sequence $\{x_j\}_{j\in J}$of elements in a Hilbert A-module H is said to be a *-frame for H if there exists strictly non- zero elements A and B of H such that

$A\langle x, x\rangle A^* \leq \sum_{j\in J}\langle x, x_j\rangle\langle x_{j,}x\rangle \leq$B$\langle x, x\rangle$B* , x∈H

Where the sum in the middle of the inequality is convergent in norm.

Then the elements A and B are called lower and upper *- frame bounds respectively.

We note that every frame for a Hilbert module is a *-frame .

If A=B then the *-frame $\{x_j\}_{j\in J}$is a tight *-frame .

If A=B=1 then the *-frame is called a normalized *-frame or parseval *-frame.

Definition 3.3:- Let $\{x_j\}_{j\in J}$be a *-frame for H. The pre *-frame operator T:H→ ℓ_2(A) defined as T(x)= $\{\langle x, x_j\rangle\}_{j\in J}$is an injective and closed range adjointable A-module map.

Definition 3.4:-Let $\{x_j\}_{j\in J}$*be a* *- frame for H . The Adjoint operator T is T*:ℓ_2(A)→H Which is surjective and defined as T*$(e_j) = x_j$ for j∈ J where $\{e_j\}_{j\in J}$is the standard basis for ℓ_2(A).

Definition 3.5:- Let $\{x_j\}_{j\in J}$be a *-frame for H. The *-frame operator S:H→ $H$is defined as Sx=T*Tx =$\sum_{j\in J}\langle x, x_j\rangle x_j$.

4.Examples of *-frames:-

4.1 EXAMPLE:- Let ℓ^∞be the unitaray C* algebra of all bounded complex valued sequences with the following operations.

uv= $\{u_i v_i\}_{i\in\mathbb{N}}$ u^*= $\{\bar{u}_i\}_{\in\mathbb{N}}$ $\|u\|$ = $sup_{i\in\mathbb{N}}|u_i|$ ∀ u = $\{u_i\}_{i\in\mathbb{N}}$ v = = $\{v_i\}_{i\in\mathbb{N}}$ in ℓ^∞

Let C_0 be the set of all sequences converging to zero. Then C_0 is a Hilbert ℓ^∞ module with ℓ^∞ valued inner product $\langle u, v\rangle$ = $\{u_i\bar{v}_i\}_{i\in\mathbb{N}}$ for u,v ∈ C_0 .Let J= ℕ and define f_j ∈ C_0 by

$f_j = \{f_i{}^j\}_{i\in\mathbb{N}}$ such that $f_i{}^j = \begin{cases} \frac{1}{2}+\frac{1}{i} & i \neq j \\ 0 & i = j \end{cases}$ for all j∈ ℕ

We observe that

$$\sum_{j\in J}\langle u, f_i\rangle\langle f_i, u\rangle = \left\{|u_i|^2\left(\frac{1}{2}+\frac{1}{i}\right)^2\right\}_{i\in\mathbb{N}} = \left\{\frac{1}{2}+\frac{1}{i}\right\}_{i\in\mathbb{N}} \langle\{u_i\}_{i\in\mathbb{N}}, \{u_i\}_{i\in\mathbb{N}}\rangle \left\{\frac{1}{2}+\frac{1}{i}\right\}_{i\in\mathbb{N}}$$

For u = $\{u_i\}_{i\in\mathbb{N}} \in C_0$. The sequence $\{f_j\}_{j\in\mathbb{N}}$is a $\left\{\frac{1}{2}+\frac{1}{i}\right\}_{i\in\mathbb{N}}$tight *- frame but it is not tight frame for Hilbert ℓ^∞ module C_0. Note that $\{f_j\}_{j\in\mathbb{N}}$ is a frame for Hilbert ℓ^∞ module C_0 with optimal lower and upper real bounds $\frac{1}{2}$ and $\frac{3}{2}$ respectively.

4.2 EXAMPLE:- Let A be a C* algebra of the set of all diagonal matrices in $M_{2X2}(\mathbb{C})$ and suppose A is a Hilbert A module over itself. Consider

$A_i = \begin{bmatrix} \frac{1}{2^i} & 0 \\ 0 & \frac{1}{3^i} \end{bmatrix}$ for all i ∈ ℕ .For A= $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ ∈ A , we have

$$\sum_{i\in\mathbb{N}}\langle A, A_i\rangle\langle A_i, A\rangle = \begin{bmatrix} \frac{|a|^2}{3} & 0 \\ 0 & \frac{|b|^2}{8} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{8}} \end{bmatrix}\langle A, A\rangle \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{8}} \end{bmatrix}$$

Then $\{A_i\}_{i\in\mathbb{N}}$ is a $\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{8}} \end{bmatrix}$ tight *- frame for Hilbert A module A but this frame for A with optimal lower and upper real bounds $\frac{1}{\sqrt{8}}, \frac{1}{\sqrt{3}}$ respectively.

4.3 Example:- Let C_0 be the Hilbert ℓ^∞ the same as above example 5.2.1. For j∈ J consider

$f_j = \{f_i{}^j\}_{i\in\mathbb{N}}$ such that $f_i{}^j = \begin{cases} \frac{1}{i} & i = j \\ 0 & i \neq j \end{cases}$

If u = $\{u_i\}_{i\in\mathbb{N}}$ is a sequences in c_0 then we have

$$\sum_{j\in J}\langle u, f_i\rangle\langle f_i, u\rangle = \left\{\frac{|u_i|^2}{i}\right\}_{i\in\mathbb{N}} \left\{\frac{1}{i}\right\}_{i\in\mathbb{N}} \langle\{u_i\}_{i\in\mathbb{N},}\{u_i\}_{i\in\mathbb{N},}\rangle \left\{\frac{1}{i}\right\}_{i\in\mathbb{N}}.$$

Since $\left\{\frac{1}{i}\right\}_{i\in\mathbb{N}}$is not strictly non zero in ℓ^∞, the sequence $\{f_j\}_{j\in J}$ has not lower bound condition in ℓ^∞ and then it is not a * frame for c_o but $\{f_j\}_{j\in J} \in \ell^\infty$.

5.RESULTS ON *-FRAMES

Theorem 5.1:- Let $\{x_j\}_{j\in J}$ be a *- frame for H with *- frame operator S.

$Then\ \|A^{-1}\|^{-2} \leq \|S\|\ \|B\|^2$.

Proof:- Given that $\{x_j\}_{j\in J}$ be a *- frame for H by definition

A$\langle x,x\rangle\ A^* \leq\ \sum_{j\in J}\langle x,x_j\rangle\langle x_j,x\rangle \leq$ B $\langle x,x\rangle\ B^*$ for all x∈ H

$\Longrightarrow$ A$\langle x,x\rangle\ A^* \leq \langle Sx,x\rangle \leq$ B $\langle x,x\rangle\ B^*$

$\Longrightarrow$ A$\langle x,x\rangle\ A^* \leq \langle Sx,x\rangle$ and $\langle Sx,x\rangle \leq$ B $\langle x,x\rangle\ B^*$

$\Longrightarrow \langle x,x\rangle \leq A^{-1}\langle Sx,x\rangle\ A^{*^{-1}}$ and $\langle Sx,x\rangle \leq$ B $\langle x,x\rangle\ B^*$

$\Longrightarrow\ \ \|A^{-1}\|^{-2}\ \|\langle x,x\rangle\| \leq \|\langle Sx,x\rangle\|$ and $\|\langle Sx,x\rangle\|\ \leq \|B\|\ \|\langle x,x\rangle\|\|B^*\|$

$\Longrightarrow\ \ \|A^{-1}\|^{-2}\ \|\langle x,x\rangle\| \leq \|\langle Sx,x\rangle\|$ and $\|\langle Sx,x\rangle\|\ \leq \|B\|^2\|\langle x,x\rangle\|$

$\Longrightarrow\ \ \|A^{-1}\|^{-2}\ \|\langle x,x\rangle\| \leq \|\langle Sx,x\rangle\| \leq \|B\|^2\|\langle x,x\rangle\|$

By taking supremum over all x∈ H with $\|x\| \leq 1$ we get

$\|A^{-1}\|^{-2} \leq \|S\|\ \|B\|^2$

THEOREM 5.3.2:- Let $\{x_j\}_{j\in J}$ be a *- frame for H with pre *- frame operator T. Then $\{x_j\}_{j\in J}$ be a frame for H.

PREPOSITION 5.3.3:- Let A be a C* module over itself every *- frame $\{x_j\}_{j\in J}$ be a tight *- frame for A.

Proof:- Suppose $\{x_j\}_{j\in J}$ be a *- frame for A with *-frame operator S.

Consider I_A = S$S^{-1}I_A$ = $\sum_{j\in J}\langle S^{-1}I_A,x_j\rangle\ x_j$ = $S^{-1}I_A\ \ \sum_{j\in J}\left|x_j\right|^2$

The above inequality shows that $\sum_{j\in J}\left|x_j\right|^2$ is an invertible element in A and

$\sum_{j\in J}\left|x_j\right|^2$ is a strictly positive element of A

So $\sum_{j\in J}\langle x,x_j\rangle\langle x_j,x\rangle$ = $\sum_{j\in J}\left|x_j\right|^2\ \langle x,x\rangle\ \forall\ x\in A$

Then $\{x_j\}_{j\in J}$is tight * frame for A.

THEOREM 5.3.4:- Let $\{f_j\in H : j\in J\ \}$ be a *- frame for H with lower and upper * frame bounds A,B respectively. The * frame transform or pre * frame operator T: H→ ℓ_2(A) defined by T(f) = $\{\langle f,f_j\rangle\}_{j\in J}$ is an injective and closed range adjointable A module map and $\|T\| \leq \|B\|$. The adjoint operator T^* is surjective and it is given by $T^*(e_i)$ = f_j for j ∈ J where $\{e_j : j\ \in J\}$ is the standard basis for ℓ_2(A) .

THEOREM 5.3.5:- Let $\{f_j \in H: j \in J\}$ be a *- frame for H with pre * frame operator T and lower and upper * frame bounds A and B respectively. Then $\{f_j\}_{j\in J}$ is a frame for H with lower and upper frame bounds $\|(T^*T)^{-1}\|^{-1}$ and $\|T\|^2$ respectively.

Proof: -By theorem 5.3.4 T is injective and has closed range and

$\|(T^*T)^{-1}\|^{-1}\langle f,f\rangle \le \sum_{j\in J}\langle f,f_j\rangle\langle f_j,f\rangle \le \|T\|^2\langle f,f\rangle \ \forall f \in$ H

Hence $\{f_j\}_{j\in J}$ is a frame for H with lower and upper frame bounds $\|(T^*T)^{-1}\|^{-1}$ and $\|T\|^2$ respectively.

Reference:

[1] K. Amir and BehroozKhosravai, Frame bases in tensor product of Hilbert spaces and Hilbert C* modules . Pro Indian Acadamy science Vol.117, No.1 Feb 2007 pp1-12

[2]A.Alljanil,M.A Dehgan *-frames in Hilbert C* modules ,P.B Sci,Bull.Series A,Vol-73,J114-2011.

[3]P.G.Casuzza ,The art of frame theory, Taiwanese Journal of Maths, 4(2) 2000, 192-202

[4] Bahram Dastourian, Mohammad Janfada *-frames for operators on Hilbert modulus,Wavelets and Linear algebra .3(2016) 27-43.

[5] D.Han and D.R.Larson , " Frames Bases and Group representations" Memories Ams Nov7 (2000) ,Providence RI.

[6] M.Frank and D.R.Larson .A module frame concepts for Hilbert C* modules functional and harmonic analysis of wavelets, contemp,math 247(200),247-223

[7] L.Gavrutta, Frames for operators ,App .Comput.Harmon,Anal,32(2012),139-144.