International Journal of Engineering, Science and Mathematics
Vol. 6 Issue 8, December 2017 (Special Issue)
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijesm.co.in, Email: ijesmj@gmail.com Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

LAGUERRE WAVELET BASED GALERKIN METHOD FOR THE NUMERICAL SOLUTION OF ELLIPTIC PROBLEMS

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Abstract

In this paper, we proposed Laguerre wavelet based Galerkin method for the numerical solution of elliptic problems. Here, we obtain numerical solutions of elliptic problems using modified Laguerre wavelets with respect to the given boundary conditions by Galerkin method. The numerical results obtained by this method are compared with the exact solution and solutions of other existing method. Some of the test problems are considered to demonstrate the applicability and validity of the proposed method.

Keywords:

Galerkin method;
Laguerre wavelet;
Elliptic problems.

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1. Introduction

Differential equations play an important role in modelling of physical problems in science and engineering. These equations describe a wide range of natural phenomena, such as sound, heat, electrostatics, electrodynamics, fluid flow, elasticity and quantum mechanics. These seemingly distinct physical phenomena can be formulized similarly in terms of differential equations. While, analytical methods can be used to solve some differential equations, many, if not most differential equations, can't be solved analytically. In general, it is not always possible to obtain exact solution of an arbitrary differential equation. This necessitates either discretization of differential equations leading to numerical solutions, or their qualitative study which is concerned with deduction of important properties of the solutions without actually solving them.
Recently, some of the numerical methods are used for the numerical solutions of differential equations. For example, Haar wavelet collocation method [4, 5], Legendre wavelet collocation method [6], New wavelet Galerkin method [7] etc.

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Wavelet theory is relatively new and an emerging area in mathematical research. In recent years, wavelets have been applied in different fields of science, engineering, and other areas needing numerical approximations. Different types of wavelets and approximating functions have been used in numerical solutions of differential equations. The concepts for understanding wavelets were provided by Meyer, Mallat, Daubechies, and many others. Since then, the number of applications where wavelets have been having exploded.

A promising idea to use wavelets in numerical solution of differential equations is to combine orthonormal wavelet bases with variational type methods. The Galerkin method is one of the best known methods for finding numerical solutions of differential equations which is invented by Russian mathematician Boris Grigoryevich Galerkin [1].

The one dimensional elliptic problem is of the form,

$$\frac{\partial^2 u}{\partial x^2} + \alpha u = f(x), \quad 0 \le x \le 1 \tag{1.1}$$

With boundary conditions $u(0) = a, \quad u(1) = b$ (1.2)

where α, a, b are constants and $f(x)$ is function of x

In this paper, we developed Laguerre wavelet-Galerkin method for the numerical solution of elliptic problems. This method is based on expanding the solution by Laguerre wavelets with unknown coefficients. The properties of Laguerre wavelets together with the Galerkin method are utilized to evaluate the unknown coefficients and then a numerical solution to eq. (1.1) is obtained.

The organization of the paper is as follows. In section 2, Preliminaries of Laguerre wavelets are given. Method of solution is discussed in section 3. Numerical Results and Analysis are given in section 4. Finally, conclusions of the proposed work are discussed in section 5.

2. Preliminaries of Laguerre wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets [2]:

$$\psi_{a,b}(x) = |a|^{\frac{-1}{2}} \psi\left(\frac{x-b}{a}\right), \quad a,b \in R \ \& \ a \ne 0$$

If we restrict the parameters a & b to discrete values as

$$a = a_0^{-k}, \quad b = n b_0 a_0^{-k}, \quad a_0 > 1, \ b_0 > 0$$

we have the following family of discrete wavelets

$$\psi_{k,n}(x) = |a_0|^{\frac{1}{2}} \psi\left(a_0^k x - n b_0\right), \quad k,n \in Z$$

Where $\Psi_{k,n}$ form a wavelet basis for a,b. In particular, when $a_0 = 2$ & $b_0 = 1$, then $\psi_{k,n}(x)$ forms an orthonormal basis. The Laguerre wavelets $\psi_{k,n}(x) = \psi(k, n, m, x)$ involve four arguments, k is assumed any positive integer, m is the degree of the Laguerre polynomials and it is the Normalized time. They are defined on the interval $[0,\infty)$ as

$$\psi_{k,n}(x) = \begin{cases} 2^{\frac{1}{2}} \bar{L}_m\left(2^k x - 2n + 1\right), & \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}} \\ 0, & Otherwise \end{cases} \tag{2.1}$$

Where $$\bar{L}(x) = \frac{1}{m!} L_m(x) \tag{2.2}$$

$m = 0, 1, 2, 3.......M-1$. In eq. (2.2) the coefficients are used for orthonormality. Here $L_m(x)$ are the Laguerre polynomials of degree m with respect to the weight function $W(x) = 1$ on the interval $[0,\infty)$ and satisfy the following recursive formula $L_0(x) = 1, \ L_1(x) = 1 - x,$

$$L_{m+2}(x) = \frac{(2m+3-x)L_{m+1}(x) - (m+1)L_m(x)}{m+2}, \quad m = 0, 1, 2, 3.....$$

For $k = 1$ & $n = 1$ in (2.1) and (2.2), then the Laguerre wavelets are given by

$$\psi_{1,0}(x) = \sqrt{2}, \quad \psi_{1,1}(x) = 2\sqrt{2}\, x(1-x), \quad \psi_{1,2}(x) = \frac{\sqrt{2}}{4}(4x^2 - 12x + 7),$$

$$\psi_{1,3}(x) = \frac{\sqrt{2}}{18}(-4x^3 + 24x^2 - 39x + 17),$$

$$\psi_{1,4}(x) = \frac{\sqrt{2}}{24}\left(\frac{2}{3}x^4 - \frac{20}{3}x^3 + 21x^2 - \frac{73}{3}x + \frac{209}{24}\right)$$ and so on.

3. Method of solution

Consider the trail solution generated by the Laguerre wavelet to the differential equation with satisfying the given boundary conditions which is involving unknown parameter. Accuracy in the solution is increased by choosing higher degree Laguerre wavelet polynomials.
Write the eq. (1.1) as,

$$R(x) = \frac{\partial^2 u}{\partial x^2} + \alpha u - f(x) \tag{3.1}$$

where $R(x)$ is the residual of the eq. (1.1). When $R(x) = 0$ for the exact solution $u(x)$ only which will satisfy the boundary conditions.
Consider the trail solution $u(x)$ for eq. (1.1) defined over $[0,1)$ can be expanded as a modified Laguerre wavelet [8] series with satisfying given boundary conditions as follows:

$$u(x) = \sum_{i=1}^{2^{k-1}} \sum_{j=1}^{M} c_{i,j}\psi_{i,j}(x) \tag{3.2}$$

where $c_{i,j}$'s are unknown coefficients to be determined. Differentiating eq. (3.2) twice w.r.t. x and substitute in eq. (3.1). To find $c_{i,j}$'s we choose weight functions as assumed basis elements and integrate on boundary values together with the residual to zero [1].

i.e. $\int_0^1 \psi_{1,j}(x) R(x)\, dx = 0, \; j = 1, 2, ...n$

then we obtain system of linear equations, on solving this system we get unknown parameters. Then substitute these unknowns in the trail solution, numerical solution of eq. (1.1) is obtained.

Method of implementation

Here, we applying the above procedure for eq. (1.1) with $\alpha = -1$, $a = 0$, $b = 0$ the eq.(1.1) can be written as,

$$\frac{\partial^2 u}{\partial x^2} - u = f(x) \tag{3.3}$$

with boundary conditions $u(0) = 0, \quad u(1) = 0$ (3.4)

Choose the trail solution of (3.3) for $k = 1$ & $m = 3$ is given by

$$u(x) = c_{1,1}\psi_{1,1}(x) + c_{1,2}\psi_{1,2}(x) + c_{1,3}\psi_{1,3}(x) \tag{3.5}$$

Here the modified Laguerre wavelet which satisfies the given boundary conditions (3.4) is considered as follows:

$$\psi_{1,1}(x) = 2\sqrt{2}\, x(1-x) = 2\sqrt{2}(x - x^2)$$

$$\psi_{1,2}(x) = \frac{\sqrt{2}}{4} x(1-x)(4x^2 - 12x + 7) = \frac{\sqrt{2}}{4}(-4x^4 + 16x^3 - 19x^2 + 7x)$$

$$\psi_{1,3}(x) = \frac{\sqrt{2}}{18} x(1-x)(-4x^3 + 24x^2 - 39x + 17) = \frac{\sqrt{2}}{18}(4x^5 - 28x^4 + 63x^3 - 56x^2 + 17x)$$

Then the eq. (3.5) becomes

$$u(x) = c_{1,1}\, 2\sqrt{2}\,(x - x^2) + c_{1,2}\frac{\sqrt{2}}{4}(-4x^4 + 16x^3 - 19x^2 + 7x) + c_{1,3}\frac{\sqrt{2}}{18}(4x^5 - 28x^4 + 63x^3 - 56x^2 + 17x) \quad (3.6)$$

Differentiating eq. (3.6) twice w.r.t. x we get,

$$\frac{\partial^2 u}{\partial x^2} = c_{1,1}\, 2\sqrt{2}\,(-2) + c_{1,2}\frac{\sqrt{2}}{4}(-48x^2 + 96x - 38) + c_{1,3}\frac{\sqrt{2}}{18}(80x^3 - 336x^2 + 378x - 112) \quad (3.7)$$

Using eq. (3.6) and (3.7), then eq. (3.3) can be rewritten as

$$c_{1,1}\, 2\sqrt{2}\,(-2) + c_{1,2}\frac{\sqrt{2}}{4}(-48x^2 + 96x - 38) + c_{1,3}\frac{\sqrt{2}}{18}(80x^3 - 336x^2 + 378x - 112) - \begin{pmatrix} c_{1,1}\, 2\sqrt{2}\,(x - x^2) + c_{1,2}\frac{\sqrt{2}}{4}(-4x^4 + 16x^3 - 19x^2 + 7x) + \\ c_{1,3}\frac{\sqrt{2}}{18}(4x^5 - 28x^4 + 63x^3 - 56x^2 + 17x) \end{pmatrix} = f(x)$$

$$\Rightarrow \begin{array}{l} c_{1,1}\, 2\sqrt{2}\,(x^2 - x - 2) + c_{1,2}\frac{\sqrt{2}}{4}(4x^4 - 16x^3 - 29x^2 + 89x - 38) + \\ c_{1,3}\frac{\sqrt{2}}{18}(-4x^5 + 28x^4 + 17x^3 - 280x^2 + 361x - 112) = f(x) \end{array}$$

$$\Rightarrow \begin{array}{l} c_{1,1}\, 2\sqrt{2}\,(x^2 - x - 2) + c_{1,2}\frac{\sqrt{2}}{4}(4x^4 - 16x^3 - 29x^2 + 89x - 38) + \\ c_{1,3}\frac{\sqrt{2}}{18}(-4x^5 + 28x^4 + 17x^3 - 280x^2 + 361x - 112) - f(x) = 0 \end{array}$$

$$\Rightarrow R(x) = 0$$

Where

$$R(x) = c_{1,1}\, 2\sqrt{2}\,(x^2 - x - 2) + c_{1,2}\frac{\sqrt{2}}{4}(4x^4 - 16x^3 - 29x^2 + 89x - 38) + c_{1,3}\frac{\sqrt{2}}{18}(-4x^5 + 28x^4 + 17x^3 - 280x^2 + 361x - 112) - f(x) \quad (3.8)$$

This is the residual.

The "weight functions" are the same as the basis functions. Then by the weighted Galerkin method to get system of equations, we consider the following:

$$\int_0^1 \psi_{1,j}(x) R(x)\, dx = 0, \quad j = 1, 2, 3 \quad (3.9)$$

After evaluating eq. (3.9), we have three equations with three unknown coefficients i.e. $c_{1,1}$, $c_{1,2}$ and $c_{1,3}$. By solving this by Gauss Elimination method, we obtain the values of $c_{1,1}, c_{1,2}, c_{1,3}$. Substituting these values in eq. (3.6), we get the numerical solution of eq. (3.3) by wavelet-Galerkin method using Laguerre wavelets.

4. Numerical Experiment

In this section, we applied Laguerre wavelet based Galerkin method (LWGM) for the numerical solution of elliptic problems and also to demonstrate the applicability of the proposed method.

Test Problem 4.1 Consider [3] $\frac{\partial^2 u}{\partial x^2} - u = x - 1, \quad 0 \le x \le 1$ (4.1)

With boundary conditions: $u(0)=0, \quad u(1) = 0$ (4.2)

Which has the exact solution $u(x) = -\frac{1}{1-e^2}e^x + \frac{e^2}{1-e^2}e^{-x} - x + 1$.

By applying the method explained in the section 3, we obtain the constants $c_{1,1} = 0.0617$, $c_{1,2} = 0.1028$ and $c_{1,3} = -0.0884$. Substituting these values in eq. (3.6) we get the numerical solution. Obtained numerical solutions are compared with exact and other existing method solution is presented in table 1 and figure 1.

Table 1. Comparison of numerical solutions and exact solution for test problem 4.1.

x	Numerical solution		Exact solution
	LWGM	Kostadinova et al. Ref[3]	
0.1	0.0264712	0.0276352	0.0265183
0.2	0.0443444	0.0453501	0.0442945
0.3	0.0546184	0.0545619	0.0545074
0.4	0.0583436	0.0566876	0.0582599
0.5	0.0565875	0.0531447	0.0565906
0.6	0.0504023	0.0453501	0.0504834
0.7	0.0407912	0.0347212	0.0408782
0.8	0.0286751	0.0226751	0.0286795
0.9	0.0148592	0.0106289	0.0147663

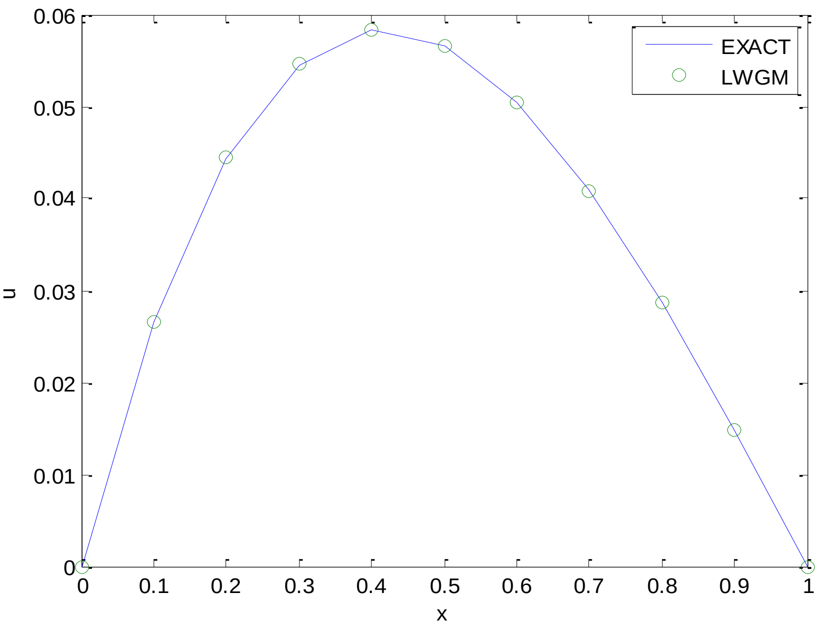

Fig. 1. Comparison of numerical and exact solutions for test problem 4.1.

Test Problem 4.2 Next, consider [9]

$$\frac{\partial^2 u}{\partial x^2} + \frac{16}{9}\pi^2 u = \frac{7}{9}\pi^2 \sin(\pi x), \quad 0 \le x \le 1 \tag{4.3}$$

With boundary conditions: $u(0)=0, \quad u(1) = 0$ (4.4) and which has the exact solution $u(x) = \sin(\pi x)$.

By applying the method explained in the section 3, we obtain the constants $c_{1,1} = 1.1947$, $c_{1,2} = 2.4825$ and $c_{1,3} = -4.8432$. Substituting these values in eq. (3.6), we get the numerical solutions. Obtained numerical solutions are compared with exact solution are presented in table 2 and figure 2.

Table 2. Comparison of exact and approximate solutions for test problem 4.2.

x	Numerical solution(LWGM)	Exact solution
0.1	0.3087468	0.3090169
0.2	0.5925196	0.5877852
0.3	0.8151813	0.8090169
0.4	0.9540854	0.9510565
0.5	0.9982500	1.0000000
0.6	0.9465312	0.9510565
0.7	0.7952968	0.8090169
0.8	0.5811001	0.5877852
0.9	0.3093530	0.3090169

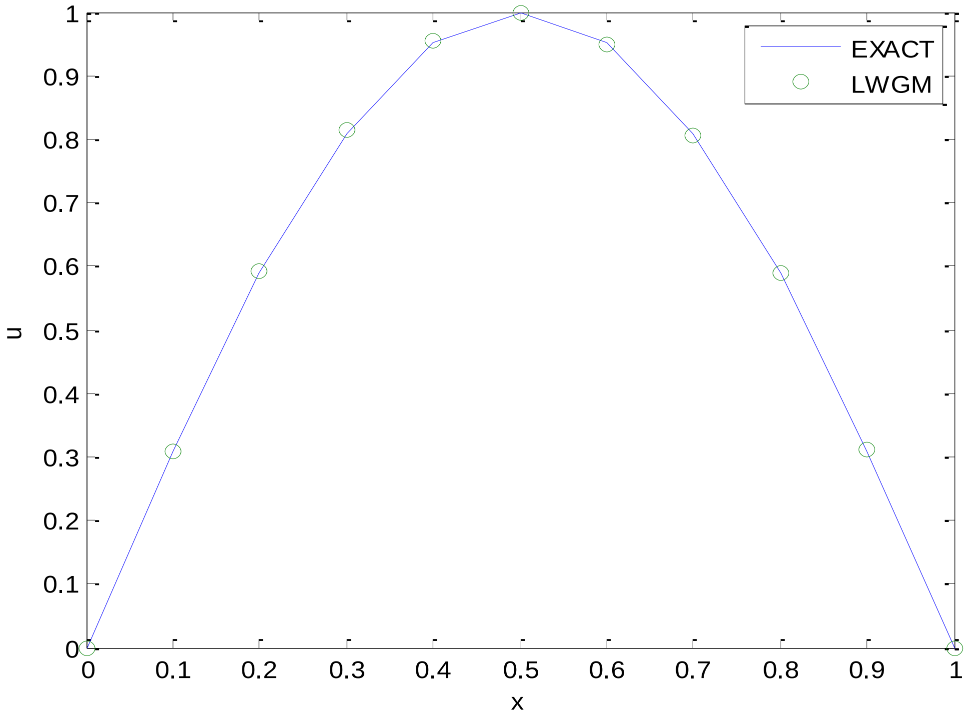

Fig. 2. Comparison of numerical and exact solutions for test problem 4.2.

5. Conclusion

In this paper, we applied the Laguerre wavelet based Galerkin method for the numerical solution of one dimensional elliptic problems. The tables and figures shows that the numerical solutions obtained by proposed method agrees with the exact solution. Hence the Laguerre wavelet based Galerkin method is effective for solving differential equations.

Acknowledgement

Authors acknowledge the support received from the University Grants Commission (UGC), Govt. of India for grant under UGC-SAP DRS-III for 2016-2021: F.510/3/DRS-III/2016(SAP-I) Dated: 29th Feb. 2016.

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