International Journal of Engineering, Science and Mathematics
Vol. 6 Issue 8, December 2017 (Special Issue)
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijesm.co.in, Email: ijesmj@gmail.com Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

Biorthogonal Spline Wavelet-Transform Method for the Numerical Solution of Integral and Integro-Differential Equations

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Abstract

Biorthogonal spline wavelet transform method is proposed for the numerical solution of integral and integro-differential equations. It uses biorthogonal spline wavelet filter coefficients matrix as a prolongation and restriction operators. The performance of the proposed method is better than the existing ones in terms of super convergence with low computational time. Some of the illustrative examples are demonstrate through error analysis.

Keywords:

Biorthogonal spline wavelets; Filter coefficients; Multigrid; Biorthogonal spline wavelet transform; integral and integro-differential equations.

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1. Introduction

Integral and integro-differential equations arise naturally in many applications in various fields of science and engineering and also have been studied extensively both at the theoretical and practical level. Specific applications of integral and integro-differential equations can be found in the mathematical modelling of spatiotemporal developments, epidemic modelling [1] and various biological and physical problems. Analytical solutions of integral and integro-differential equations, however, either do not exist or it is often hard to find. It is precisely due to this fact that several numerical methods have been developed for finding approximate solutions of integral and integro-differential equations [2-4].

Multigrid method is well known among the fastest solution method. Particularly, for elliptic problems, they have been proved to be highly accurate. Vectors from fine grids are transferred to coarser grids with Restriction operator, while vectors are transferred from coarse grids to the finer grids with a Prolongation operator. An introduction of multigrid method is found in Wesseling [5]. Multigrid Tutorial (Briggs [6] and Trottenberg et al. [7]), is helpful to get the basic ideas of multigrid techniques. The multigrid method is largely applicable in increasing the efficiency of iterative methods used to solve large system of algebraic equations resulted from discretization of the differential equations and integral equations are applicable to solve numerically. Multigrid method

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has been applied for the numerical solution of different types of integral equations. Hackbusch [8, 9] given the multigrid techniques and the integral equations from both theoretical and computational points of views. Schippers [10, 11] used multigrid methods for boundary integral equations. Gáspár [12] has given a new approach a fast multigrid solution of boundary integral equations. Lee [13] has solved multigrid method for nonlinear integral equations. Paul [14], applied the multigrid algorithm for solving integral equations.

Wavelet analysis is a new branch of mathematics and widely applied in signal analysis, image processing and numerical analysis etc. The wavelet methods have proved to be very effective and efficient tool for solving problems of mathematical calculus. In recent years, these methods have attracted the interest of researchers of structural mechanics and many papers in this field are published. In most papers the Daubechies wavelets are applied. These wavelets are orthogonal, sufficiently smooth and have a compact support. Wavelet methods solve system of equations with faster convergence and less computation cost. Recently, many authors (De Leon [15], Bujurke et al. [16-18], Avudainayagam & vani [19]) have worked on wavelet multigrid methods for the solution of differential equations. Shilralashetti et al. [20] has proposed the wavelet based decoupled method for the investigation of surface roughness effects in elastohydrodynamic lubrication problems using couple stress fluid. Also, Shilralashetti et al. [21] have introduced a new wavelet based full-approximation scheme for the numerical solution of nonlinear elliptic partial differential equations. Wang et al. [22] have applied a fast wavelet multigrid algorithm for the solution of electromagnetic integral equations.

Biorthogonal wavelet basis were introduced by Cohen-Daubechies-Feauveau in order to obtain wavelet pairs that are symmetric, regular and compactly supported. Unfortunately, this is incompatible with the orthogonality requirement that has to be dropped altogether. Biorthogonal wavelets build with splines are especially attractive because of their short support and regularity. So it is called a "Biorthogonal Spline Wavelets" [23]. In the biorthogonal case, rather than having one scaling and wavelet function, there are two scaling functions ϕ , $\tilde{\phi}$, that may generate different multiresolution analysis, and accordingly two different wavelet functions ψ , $\tilde{\psi}$. But biorthogonal wavelet based multigrid schemes are found to be effective [24]. Biorthogonal wavelet based multigrid schemes provide some remedy in such challenging cases. Sweldens [25] highlighted effectively the construction of biorthogonal wavelet filters for the solution of large class of ill-conditioned system. The DBSWT matrix designed and implemented by Ruch and Fleet [26] for decomposition and reconstruction of the given signals and images. Using these decomposition and reconstruction matrices we introduced restriction and prolongation operators respectively in the implementation of biorthogonal spline wavelet transform method (BSWTM). In this paper, we applied biorthogonal spline wavelet transform method (BSWTM) for the numerical solution of integral and integro-differential equations. Thus, the proposed method can be applied to a wide range of science and engineering problems.

The organization of this paper is as follows. In section 2, properties of biorthogonal spline wavelets are discussed. In Section 3, method of solution is discussed. In section 4, method of implementation of numerical experiments and results. Finally, conclusion of the proposed work is given in section 5.

2. Properties of Biorthogonal Wavelets

Discrete Biorthogonal Spline wavelet transform (DBSWT) matrix:

Let us consider the (5, 3) biorthogonal spline wavelet filter pair,

We have

$$\tilde{c} = (\tilde{c}_{-1}, \tilde{c}_0, \tilde{c}_1) = \left(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4} \right)$$

and

$$c = (c_{-2}, c_{-1}, c_0, c_1, c_2) = \left(\frac{-\sqrt{2}}{8}, \frac{\sqrt{2}}{4}, \frac{3\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{-\sqrt{2}}{8} \right)$$

To form the highpass filters, We have

$$d_k = (-1)^k \tilde{c}_{1-k} \text{ and } \tilde{d}_k = (-1)^k c_{1-k}$$

The highpass filter pair d and $\tilde{d}$ for the (5, 3) biorthogonal spline filter pair.

$$d_0 = \frac{\sqrt{2}}{4}, d_1 = \frac{-\sqrt{2}}{2}, d_2 = \frac{\sqrt{2}}{4} \text{ and } \tilde{d}_{-1} = \frac{\sqrt{2}}{8}, \tilde{d}_0 = \frac{\sqrt{2}}{4}, \tilde{d}_1 = \frac{-3\sqrt{2}}{4}, \tilde{d}_2 = \frac{\sqrt{2}}{4}, \tilde{d}_3 = \frac{\sqrt{2}}{8}$$

In this paper, we use the filter coefficients which are,

Low pass filter coefficients: c_{-2}, c_{-1}, c_0, c_1, c_2 and High pass filter coefficients: d_0, d_1, d_2 for decomposition matrix.

Low pass filter coefficients: $\tilde{c}_{-1} = d_2$, $\tilde{c}_0 = -d_1$, $\tilde{c}_1 = d_0$ and High pass filter coefficients: $\tilde{d}_{-1} = -c_2, \tilde{d}_0 = c_1, \tilde{d}_1 = -c_0, \tilde{d}_2 = c_{-1}, \tilde{d}_3 = -c_{-2}$ for reconstruction matrix.

The matrix formulation of the discrete biorthogonal spline wavelet transforms (DBSWT) plays an important role in both biorthogonal spline wavelet transforms method (BSWTM) and biorthogonal Spline wavelet full-approximation transform method (BSWFATM) for the numerical computations. As we already know about the DBSWT matrix and its applications in the wavelet method and is given in [26] as,

Decomposition matrix:

$$D_W = \begin{pmatrix} c_{-1} & c_0 & c_1 & c_2 & 0 & 0 & \dots & 0 & 0 & c_{-2} \\ d_1 & d_2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & d_0 \\ 0 & c_{-2} & c_{-1} & c_0 & c_1 & c_2 & \dots & 0 & 0 & 0 \\ 0 & d_0 & d_1 & d_2 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & & \ddots & & \dots & & \dots & & 0 & 0 \\ c_1 & c_2 & 0 & 0 & \dots & \dots & 0 & c_{-2} & c_{-1} & c_0 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & d_0 & d_1 & d_2 \end{pmatrix}_{N\times N}$$

Reconstruction matrix:

$$R_W = \begin{pmatrix} \tilde{c}_0 & \tilde{c}_1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \tilde{c}_{-1} \\ \tilde{d}_0 & \tilde{d}_1 & \tilde{d}_2 & \tilde{d}_3 & 0 & 0 & \dots & 0 & 0 & \tilde{d}_{-1} \\ 0 & \tilde{c}_{-1} & \tilde{c}_0 & \tilde{c}_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \tilde{d}_{-1} & \tilde{d}_0 & \tilde{d}_1 & \tilde{d}_2 & \tilde{d}_3 & \dots & 0 & 0 & 0 \\ \vdots & & \ddots & & \dots & & \dots & & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & & \dots & 0 & \tilde{c}_{-1} \tilde{c}_0 & \tilde{c}_1 \\ \tilde{d}_2 & \tilde{d}_3 & 0 & 0 & \dots & & \dots & 0 \, \tilde{d}_{-1} & \tilde{d}_0 & \tilde{d}_1 \end{pmatrix}_{N\times N}$$

Biorthogonal Spline Wavelet operators:

Using the above matrices, we introduced biorthogonal spline wavelet restriction and biorthogonal spline wavelet prolongation operators respectively. i.e.,

Biorthogonal spline wavelet restriction operator:

$$BSWT_R = \begin{pmatrix} c_{-1} & c_0 & c_1 & c_2 & 0 & 0 & \dots & 0 & 0 & c_{-2} \\ d_1 & d_2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & d_0 \\ 0 & c_{-2} & c_{-1} & c_0 & c_1 & c_2 & \dots & 0 & 0 & 0 \\ 0 & d_0 & d_1 & d_2 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & & \ddots & & \dots & & \dots & & 0 & 0 \\ 0 & 0 & \dots & 0 & d_0 & d_1 & d_2 & 0 & \dots & 0 \; 0 \end{pmatrix}_{\frac{N}{2}\times N}$$

Biorthogonal spline wavelet prolongation operator:

$$BSWT_P = \left(\begin{array}{cccccccccc} \tilde{c}_0 & \tilde{c}_1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \tilde{c}_{-1} \\ \tilde{d}_0 & \tilde{d}_1 & \tilde{d}_2 & \tilde{d}_3 & 0 & 0 & 0 & \dots & 0 & \tilde{d}_{-1} \\ 0 & \tilde{c}_{-1} & \tilde{c}_0 & \tilde{c}_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \tilde{d}_{-1} & \tilde{d}_0 & \tilde{d}_1 & \tilde{d}_2 & \tilde{d}_3 & 0 & \dots & 0 & 0 \\ \vdots & & \ddots & & & \ddots & & \ddots & & \vdots \\ 0 & \dots & 0 & \tilde{c}_{-1} & \tilde{c}_0 & \tilde{c}_1 & 0 & 0 & \dots 0 & 0 \\ 0 & \dots & 0 & \tilde{d}_{-1} & \tilde{d}_0 & \tilde{d}_1 & \tilde{d}_2 & \tilde{d}_3 & 0 \dots & 0 \end{array} \right)^T_{\frac{N}{2} \times N}.$$

Modified Discrete Biorthogonal Spline wavelet transform (MDBSWT) matrix:

Here, we developed MDBSWT matrix from DBSWT matrix in which by adding rows and columns consecutively with diagonal element as 1, which is built as,

New decomposition matrix:

$$MD_W = \left(\begin{array}{cccccccccccccc} c_{-1} & 0 & c_0 & 0 & c_1 & 0 & c_2 & 0 & \dots & 0 & 0 & 0 & c_{-2} & 0 \\ 0 & 1 & 0 & 0 & \dots & \dots & & \dots & 0 & 0 & 0 & 0 & 0 & \\ d_1 & 0 & d_2 & 0 & \dots & \dots & & \dots & 0 & 0 & 0 & d_0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 & \\ \vdots & & \ddots & & \dots & & \dots & & \ddots & & \vdots & & & \\ c_1 & 0 & c_2 & 0 & \dots & 0 & 0 & 0 & c_{-2} & 0 & c_{-1} & 0 & c_0 & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 & 0 & 1 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & d_0 & 0 & d_1 & 0 & d_2 & 0 & \\ 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 & 0 & 1 & & & & \end{array} \right)_{N \times N}$$

New reconstruction matrix:

$$MR_W = \left(\begin{array}{cccccccccccccc} \tilde{c}_0 & 0 & \tilde{c}_1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \tilde{c}_{-1} & 0 & & \\ 0 & 1 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 & 0 & & & \\ \tilde{d}_0 & 0 & \tilde{d}_1 & 0 & \tilde{d}_2 & 0 & \tilde{d}_3 & \dots & 0 & 0 & 0 & \tilde{d}_{-1} & 0 & \\ 0 & 0 & 0 & 1 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & & & \\ \vdots & & \ddots & & \dots & \dots & & \ddots & & \vdots & & & & \\ 0 & 0 & \dots & 0 & \tilde{c}_{-1} & 0 & \tilde{c}_0 & 0 & \tilde{c}_1 & 0 & & & & \\ 0 & 0 & 0 & \dots & \dots & \dots & 0 & 1 & 0 & 0 & & & & \\ \tilde{d}_2 & 0 & \tilde{d}_3 & \dots & 0 & 0 & 0 & \tilde{d}_{-1} & 0 & \tilde{d}_0 & 0 & \tilde{d}_1 & 0 & \\ 0 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 & 1 & & & & \end{array} \right)_{N \times N}$$

Modified Biorthogonal Spline wavelet operators:

Using the above matrices, we introduced a new biorthogonal spline wavelet restriction and prolongation operators respectively as,

New biorthogonal spline wavelet restriction operator:

$$MBSWT_R = \left(\begin{array}{cccccccccccccc} c_{-1} & 0 & c_0 & 0 & c_1 & 0 & c_2 & 0 & 0 & 0 & \dots & 0 & c_{-2} & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & 0 & & & \\ d_1 & 0 & d_2 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & 0 & d_0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & 0 & & \\ 0 & 0 & c_{-2} & 0 & c_{-1} & 0 & c_0 & 0 & c_1 & 0 & c_2 & 0 & \dots & 0 \; 0 \\ 0 & 0 & d_0 & 0 & d_1 & 0 & d_2 & 0 & 0 & \dots \dots & 0 & 0 & 0 & \\ \vdots & & \ddots & & \ddots & & \ddots & & \ddots & & \ddots & & & \\ 0 & 0 & \dots & 0 & d_0 & 0 & d_1 & 0 & d_2 & 0 & 0 & \dots & 0 & 0 \end{array} \right)_{\frac{N}{2} \times N}$$

New biorthogonal spline wavelet prolongation operator:

$$MBSWT_P = \begin{pmatrix} \tilde{c}_0 & 0 & \tilde{c}_1 & 0 & & 0 & 0 & \dots & 0 & 0 & \tilde{c}_{-1} & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & \dots & \dots & & 0 & 0 \\ \tilde{d}_0 & 0 & \tilde{d}_1 & 0 & \tilde{d}_2 & 0 & \tilde{d}_3 & 0 & \dots & 0 & \tilde{d}_{-1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & \dots & & 0 \\ 0 & 0 & \tilde{c}_{-1} & 0 & \tilde{c}_0 & 0 & \tilde{c}_1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & & \ddots & & & \ddots & & & \ddots & & & \vdots \\ 0 & \dots & 0 & \tilde{d}_{-1} & 0 & \tilde{d}_0 & 0 & \tilde{d}_1 & 0 & \tilde{d}_2 & 0 & \tilde{d}_3 \; 0 \dots 0 \end{pmatrix}^T_{\frac{N}{2}\times N}$$

3. Biorthogonal Spline Wavelet Transform Method (BSWTM) of solution

In this section, we approximate solution containing some error. There are many approaches to minimize the error. Some of them are multigrid (MG) method, biorthogonal spline wavelet transform method (BSWTM) and modified biorthogonal spline wavelet transform method (MBSWTM).

Consider the Volterra integral equations of the second kind,

$$u(t) = f(t) + \int_0^t k(t,s)u(s)ds \quad 0 \le t, s \le 1, \tag{3.1}$$

Consider the Fredholm integral equation of the second kind,

$$u(t) = f(t) + \int_0^1 k(t,s)u(s)ds \quad 0 \le t, s \le 1, \tag{3.2}$$

where $f(t)$ and the kernels $k(t,s)$ are assumed to be in $L^2(R)$ on the interval $0 \le t, s \le 1$. After discretizing the integral equation through the trapezoidal discretization method (TDM) [27], we get system of algebraic equations. Through this system we can write the system as

$$\text{i.e.,} \quad (I-K)u = f \Rightarrow Au = b \tag{3.3}$$

where $A = (I-K)$ is $N \times N$ coefficient matrix, b is $N \times 1$ matrix and u is $N \times 1$ matrix to be determined.

Solving the system of equation (3.3) through the iterative method, we get the approximate solution v of u. i.e. $u = e + v \Rightarrow v = u - e$, where e is ($N \times 1$ matrix) error to be determined.

Now we are discussing about the method of solution as follows.

From equation (3.3), we get the approximate solution v of u. Now we find the residual as

$$r_{N\times1} = [b]_{N\times1} - [A]_{N\times N}[v]_{N\times1} \tag{3.4}$$

We reduce the matrices in the finer ($N^{th} = 2^J$) level to coarsest level using Wavelet Restriction operator ($BSWT_R$) and then construct the matrices back to finer level from the coarsest level using Wavelet Prolongation operator ($BSWT_P$).

From (3.4),

$$r_{N/2\times1} = [BSWT_R]_{N/2\times N}[r]_{N\times1} \tag{3.5}$$

and $$[A]_{N/2\times N/2} = [BSWT_R]_{N/2\times N}[A]_{N\times N}[BSWT_P]_{N\times N/2}$$

Residual equation becomes, $[A]_{N/2\times N/2}[e]_{N/2\times1} = [r]_{N/2\times1}$

where $e_{N/2\times1}$ is to be determined. Solve $e_{N/2\times1}$ with initial guess '0'.

From (3.5),

$$r_{N/4\times1} = [BSWT_R]_{N/4\times N/2}[r]_{N/2\times1} \tag{3.6}$$

and $$[A]_{N/4\times N/4} = [BSWT_R]_{N/4\times N/2}[A]_{N/2\times N/2}[BSWT_P]_{N/2\times N/4}$$

Then residual equation becomes, $[A]_{N/4\times N/4}[e]_{N/4\times1} = [r]_{N/4\times1}$

Solve $e_{N/4\times1}$ with initial guess '0'.
Continue the procedure up to the coarsest level, we have,

$$r_{1\times1} = [BSWT_R]_{1\times2}[r]_{2\times1} \tag{3.7}$$

and $$[A]_{1\times1} = [BSWT_R]_{1\times2}[A]_{2\times2}[BSWT_P]_{2\times1}$$

Residual equation is, $[A]_{1\times1}[e]_{1\times1} = [r]_{1\times1}$

Solve $e_{1\times1}$ exactly.
Now correct the solution

$$u_{2\times1} = [e]_{2\times1} + [BSWT_P]_{2\times1}[e]_{1\times1}$$

Solve $[A]_{2\times2}[u]_{2\times1} = [r]_{2\times1}$ with initial guess $u_{2\times1}$.
Correct the solution

$$u_{4\times1} = [e]_{4\times1} + [BSWT_P]_{4\times2}[u]_{2\times1}$$

Solve $[A]_{4\times4}[u]_{4\times1} = [r]_{4\times1}$ with initial guess $u_{4\times1}$.
Continue the procedure up to the finer level,
Correct the solution

$$u_{N\times1} = [v]_{N\times1} + [BSWT_P]_{N\times N/2}[u]_{N/2\times1}$$

Solve $[A]_{N\times N}[u]_{N\times1} = [b]_{N\times1}$ with initial guess $u_{N\times1}$.

$u_{N\times1}$ is the required solution of system (3.3).

Similarly, the same procedure is applied for modified biorthogonal spline wavelet transform method (MBSWTM) as explained in the above. Here, we use the modified wavelet intergrid operators as given in section 2, instead of multigrid intergrid operators.

4. Illustrative examples

In this section, we present numerical solution of integral and integro-differential equation to demonstrate the capability of the proposed scheme using biorthogonal spline wavelet transform method. The error function is presented to verify the accuracy and efficiency of the following numerical results:

$$\text{Error function} = E_{\max} = \|u_e(t_i) - u_a(t_i)\|_{\max} = \sqrt{\sum_{i=1}^{n}\left(u_e(t_i) - u_a(t_i)\right)^2}$$

where u_e and u_a are exact and approximate solution respectively.

Test problem 4.1 Let us consider the linear Volterra integral equation [28],

$$u(t) = \sin(t) - \int_0^t (t-s)u(s)\,ds \qquad 0 \le t, s \le 1, \tag{4.1}$$

which has the exact solution $u(t) = \frac{1}{2}(\sin(t) + \sinh(t))$. The numerical solutions of Eq. (4.1) is obtained through the method as explained in section 3 compared with the exact and existing method are shown in table 1 and in the figure 1 for N=64. Maximum error and CPU time are shown in table 2.

Table 1. Numerical results of test problem 4.1, for N = 8.

t	MG	BSWTM	MBSWTM	Exact

0	0.0000	0.0000	0.0000	0
0.1428	0.1423	0.1423	0.1423	0.1428
0.2857	0.2847	0.2847	0.2847	0.2857
0.4285	0.4271	0.4271	0.4271	0.4286
0.5714	0.5698	0.5698	0.5698	0.5719
0.7142	0.7132	0.7132	0.7132	0.7158
0.8571	0.8577	0.8577	0.8577	0.8609
1	1.0043	1.0043	1.0043	1.0083

Table 2. Maximum error and CPU time (in seconds) of the methods of test problem 4.1.

N	Method	E_{max}	Setup time	Running time	Total time
16	MG	8.72e-04	0.2599	0.0296	0.2895
	BSWTM	8.72e-04	0.0470	0.0107	0.0578
	MBSWTM	8.72e-04	0.0462	0.0030	0.0492
32	MG	2.04e-04	0.1722	0.0321	0.2043
	BSWTM	2.04e-04	0.0975	0.0098	0.1073
	MBSWTM	2.04e-04	0.0648	0.0029	0.0677
64	MG	4.95e-05	0.1848	0.0334	0.2182
	BSWTM	4.95e-05	0.1385	0.0109	0.1494
	MBSWTM	4.95e-05	0.0894	0.0032	0.0926
128	MG	1.21e-05	0.3431	0.0117	0.3548
	BSWTM	1.21e-05	0.2226	0.0310	0.2536
	MBSWTM	1.21e-05	0.1798	0.0041	0.1839

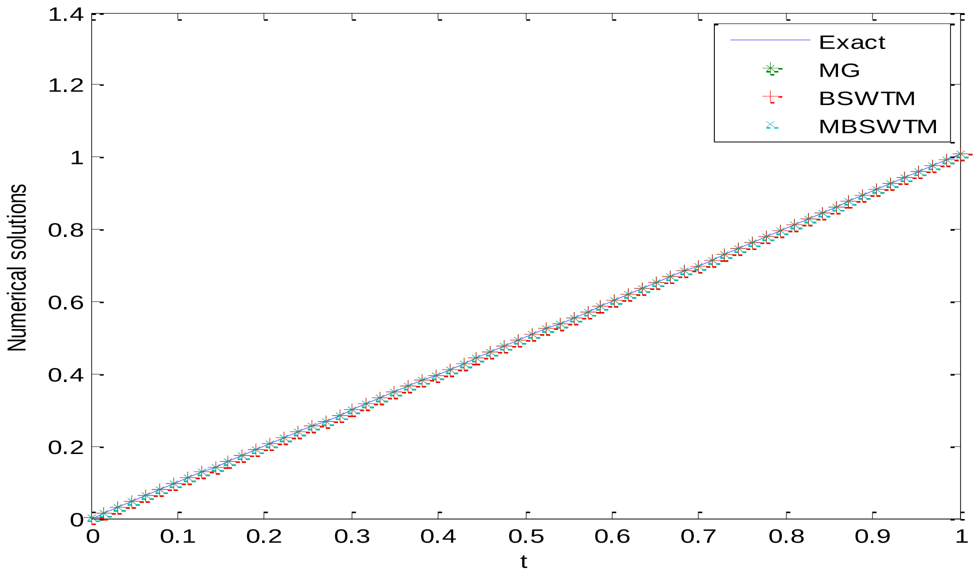

Figure 1. Comparison of numerical solutions with exact solution of test problem 4.1, for N=64.

Test problem 4.2 Next, consider the Volterra integro-differential equations [29]

$$u'(t) = 1 - 2t\sin(t) + \int_0^t u(s)\,ds, \quad u(0) = 0, \; 0 \le t \le 1 \tag{4.2}$$

which has the exact solution $u(t) = t\cos(t)$. We convert the Volterra integro-differential equation to equivalent Volterra integral equation by using the well-known formula, which converts multiple integrals into a single integral.
i.e.,

$$\int_0^t \int_0^t .... \int_0^t u(t)dt^n = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} u(s)ds$$

Integrating Eq. (4.2) on both sides from 0 to t and using the initial condition and also converting the double integral to the single integral, we obtain

$$u(t) = f(t) + \int_0^t k(t,s)u^2(s)ds, \tag{4.3}$$

where $k(t,\ s) = (t-s)$ and $f(t) = t - 2\sin(t) + 2t\cos(t)$.

The numerical solutions of Eq. (4.2) are presented in table 3 for N = 16 and in figure 2 for N = 64. Maximum error analysis and CPU time is shown in table 4.

Table 3. Numerical results of test problem 4.2, for N = 16.

t	MG	BSWTM	MBSWTM	Exact
0	0.0000	0.0000	0.0000	0
0.0666	0.0664	0.0664	0.0664	0.0665
0.1333	0.1320	0.1320	0.1320	0.1321
0.2000	0.1958	0.1958	0.1958	0.1960
0.2666	0.2570	0.2570	0.2570	0.2572
0.3333	0.3147	0.3147	0.3147	0.3149
0.4000	0.3681	0.3681	0.3681	0.3684
0.4666	0.4164	0.4164	0.4164	0.4167
0.5333	0.4588	0.4588	0.4588	0.4592
0.6000	0.4947	0.4947	0.4947	0.4952
0.6666	0.5234	0.5234	0.5234	0.5239
0.7333	0.5443	0.5443	0.5443	0.5448
0.8000	0.5568	0.5568	0.5568	0.5573
0.8666	0.5604	0.5604	0.5604	0.5610
0.9333	0.5548	0.5548	0.5548	0.5554
1	0.5396	0.5396	0.5396	0.5403

Table 4. Maximum error and CPU time (in seconds) of the methods of test problem 4.2.

N	Method	E_{max}	Setup time	Running time	Total time
16	MG	6.90e-04	0.2644	0.0296	0.2939
	BSWTM	6.90e-04	0.0486	0.0125	0.0611
	MBSWTM	6.90e-04	0.0494	0.0054	0.0548
32	MG	1.61e-04	0.1716	0.0307	0.2022
	BSWTM	1.61e-04	0.0700	0.0117	0.0817
	MBSWTM	1.61e-04	0.0623	0.0042	0.0664
64	MG	3.91e-05	0.1914	0.0308	0.2222
	BSWTM	3.91e-05	0.1824	0.0122	0.1946
	MBSWTM	3.91e-05	0.1028	0.0038	0.1066
128	MG	9.65e-06	0.3804	0.0116	0.3920
	BSWTM	9.65e-06	0.2439	0.0357	0.2796
	MBSWTM	9.65e-06	0.1912	0.0031	0.1943

Test problem 4.3 Next, consider the linear Fredholm integral equation [30],

$$u(t) = t^3 - (6 - 2\exp(1))\exp(t) + \int_0^1 \exp(t+s)u(s)\,ds, \qquad 0 \le t \le 1 \tag{4.4}$$

which has the exact solution $u(t) = t^3$. The numerical solutions of Eq. (4.4) is obtained through the method as explained in section 3 compared with the exact and existing method are shown in table 5 and in the figure 3 for N=64. Maximum error and CPU time are shown in table 6.

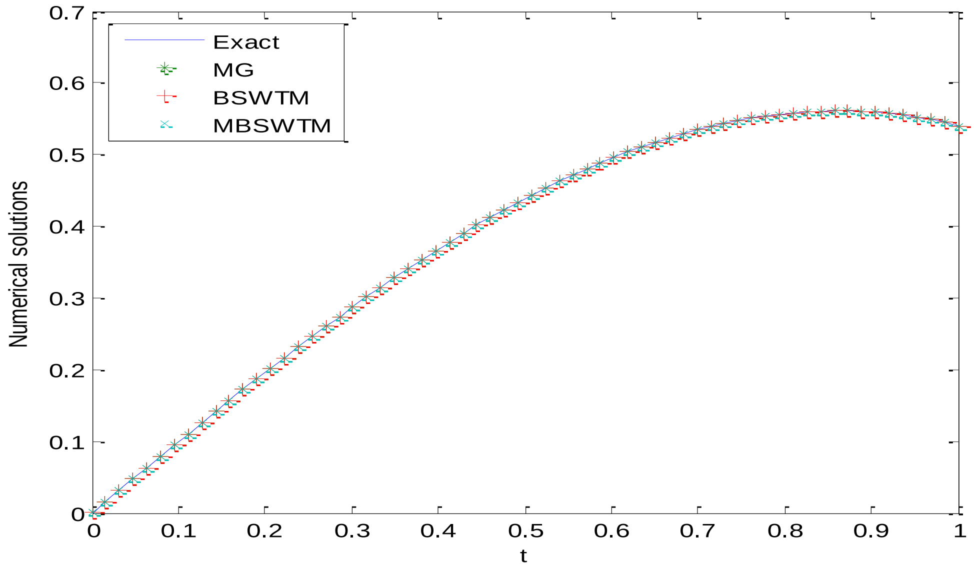

Figure 2. Comparison of numerical solutions with exact solution of test problem 4.2, for N=64.

Table 5. Numerical results of test problem 4.3, for N = 16.

t	MG	BSWTM	MBSWTM	Exact
0	-0.0018	-0.0018	-0.0018	0
0.0666	-0.0016	-0.0016	-0.0016	0.0002
0.1333	0.0002	0.0002	0.0002	0.0023
0.2000	0.0057	0.0057	0.0057	0.0080
0.2666	0.0165	0.0165	0.0165	0.0189
0.3333	0.0344	0.0344	0.0344	0.0370
0.4000	0.0612	0.0612	0.0612	0.0640
0.4666	0.0987	0.0987	0.0987	0.1016
0.5333	0.1485	0.1485	0.1485	0.1517
0.6000	0.2126	0.2126	0.2126	0.2160
0.6666	0.2927	0.2927	0.2927	0.2962
0.7333	0.3905	0.3905	0.3905	0.3943
0.8000	0.5079	0.5079	0.5079	0.5120
0.8666	0.6466	0.6466	0.6466	0.6509
0.9333	0.8083	0.8083	0.8083	0.8130
1	0.9950	0.9950	0.9950	1

Table 6. Maximum error and CPU time (in seconds) of the methods of test problem 4.3.

N	Method	E_{max}	Setup time	Running time	Total time
16	MG	4.97e-03	0.2854	0.0330	0.3184
	BSWTM	4.97e-03	0.0621	0.0106	0.0727
	MBSWTM	4.97e-03	0.0543	0.0030	0.0574
32	MG	1.16e-03	0.2933	0.0308	0.3241
	BSWTM	1.16e-03	0.0666	0.0116	0.0782
	MBSWTM	1.16e-03	0.0665	0.0033	0.0698
64	MG	2.82e-04	0.2651	0.0306	0.2957
	BSWTM	2.82e-04	0.1506	0.0107	0.1613
	MBSWTM	2.82e-04	0.1064	0.0045	0.1109
128	MG	6.95e-05	0.4965	0.0120	0.5085
	BSWTM	6.95e-05	0.4575	0.0335	0.4910
	MBSWTM	6.95e-05	0.2596	0.0040	0.2636

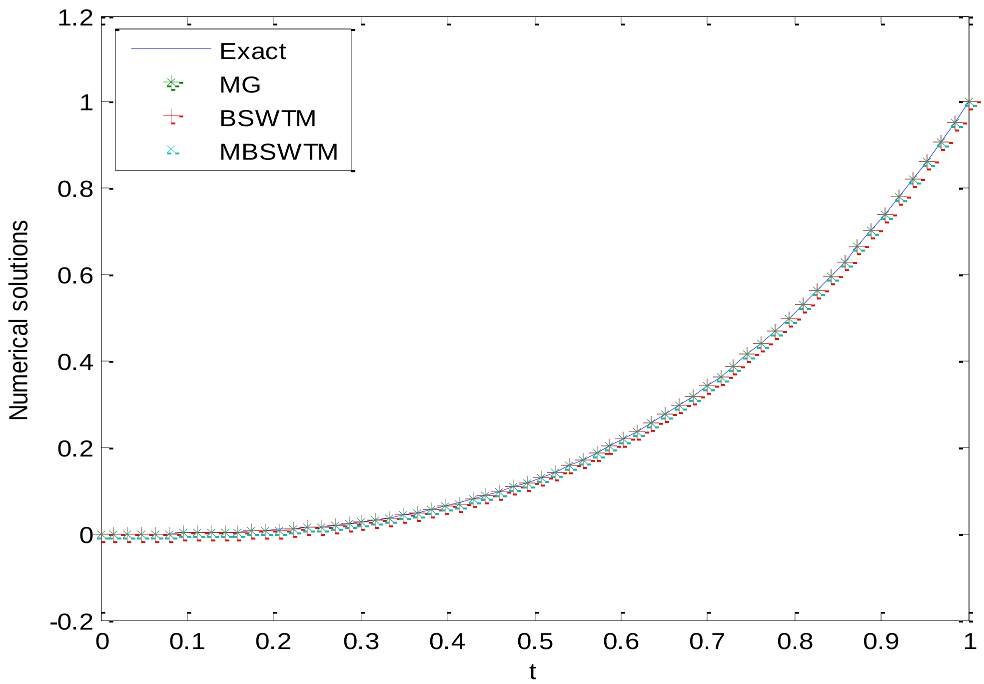

Figure 3. Comparison of numerical solutions with exact solution of test problem 4.3, for N=64.

Test problem 4.4. Next, consider the linear Fredholm integro-differential equation [31],

$$u''(t) = \exp(t) - t + t\int_0^1 s\,u(s)\,ds, \quad u(0)=1, \quad u'(0)=1, \qquad 0 \le t \le 1 \tag{4.5}$$

which has the exact solution $u(t) = e^t$.

Integrating the Eq. (4.5) twice w.r.to t and using the initial conditions, we get

$$u(t) = \exp(t) - \frac{t^3}{6} + \frac{t^3}{6}\int_0^1 s\,u(s)\,ds,$$

Solving this equation, we obtain the numerical solutions of Eq. (4.5) through the present method as explained in section 3 compared with the exact and existing methods are shown in table 7 and in the figure 4 for N=64. Maximum error and CPU time are shown in table 8.

Table 7. Numerical results of test problem 4.4, for N = 16.

t	MG	BSWTM	MBSWTM	Exact
0	0.0000	0.0000	0.0000	0
0.0666	-0.0622	-0.0622	-0.0622	-0.0622
0.1333	-0.1155	-0.1155	-0.1155	-0.1155
0.2000	-0.1600	-0.1600	-0.1600	-0.1600
0.2666	-0.1955	-0.1957	-0.1957	-0.1957
0.3333	-0.2222	-0.2224	-0.2224	-0.2224
0.4000	-0.2400	-0.2403	-0.2403	-0.2403
0.4666	-0.2488	-0.2493	-0.2493	-0.2493
0.5333	-0.2488	-0.2494	-0.2494	-0.2494
0.6000	-0.2400	-0.2405	-0.2405	-0.2405
0.6666	-0.2222	-0.2227	-0.2227	-0.2227
0.7333	-0.1955	-0.1959	-0.1959	-0.1959
0.8000	-0.1600	-0.1602	-0.1602	-0.1602
0.8666	-0.1155	-0.1156	-0.1156	-0.1156
0.9333	-0.0622	-0.0622	-0.0622	-0.0622
1	0.0000	0.0000	0.0000	0

Table 8. Maximum error and CPU time (in seconds) of the methods of test problem 4.4.

N	Method	E_{max}	Setup time	Running time	Total time
16	MG	2.83e-04	0.2993	0.0295	0.3288
	BSWTM	2.83e-04	0.0463	0.0088	0.0551
	MBSWTM	2.83e-04	0.0498	0.0042	0.0540
32	MG	6.63e-05	0.1960	0.0302	0.2263
	BSWTM	6.63e-05	0.0728	0.0102	0.0830
	MBSWTM	6.63e-05	0.0589	0.0046	0.0635
64	MG	1.60e-05	0.2054	0.0308	0.2362
	BSWTM	1.60e-05	0.1655	0.0145	0.1800
	MBSWTM	1.60e-05	0.1015	0.0042	0.1057
128	MG	3.95e-06	0.3600	0.0121	0.3721
	BSWTM	3.95e-06	0.2226	0.0329	0.2554
	MBSWTM	3.95e-06	0.1905	0.0034	0.1939

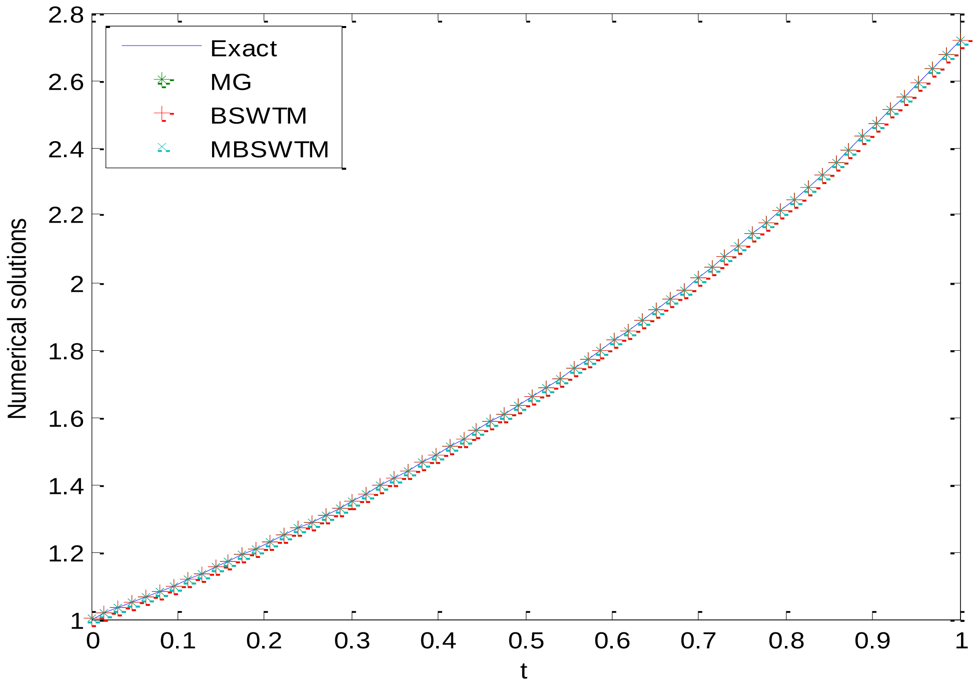

Figure 4. Comparison of numerical solutions with exact solution of test problem 4.4, for N=64.

5. Conclusions

We proposed a biorthogonal spline wavelet transform method using wavelet intergrid operators based on biorthogonal spline wavelet filter coefficients for the numerical solution of integral and integro-differential equations. Wavelet intergrid operators of prolongation and restrictions are defined, a MBSWTM, has been shown to be effective and versatile. Test problems are justified through the error analysis, as the level of resolution N increases for higher accuracy. The numerical solutions obtained agree well with the exact ones, as increasing the number N used. In this paper, the multigrid method, BSWTM and MBSWTM shows the error's are same, but the CPU time changes. The standard multigrid method and BSWTM converges slowly with larger computational cost as compared to MBSWTM ensure such slower convergence with lesser computational cost as shown in the tables. The numerical implementation from the tables demonstrates the accuracy of the approximations and super convergence phenomena with less CPU time. Hence, the proposed scheme is very convenient and efficient than the existing standard methods.

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