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Fixed Points of Geraghty Contractions with Rational Type Expressions

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Abstract (10pt)

In this paper, we prove the existence of fixed and common fixed point results of generalized Geraghtycontractions of self maps with altering distance function φ invoving rational type expressions in partially ordered metric spaces. These results extend the some known results. Examples are provided in support of our results.

Keywords:

Fixed point;
Partially ordered metric space;
Rational type contraction mappings;
Geraghty contraction;

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1. Introduction and Preliminaries

Banach contraction principle is one of the fundamental result in fixed point theory for which several authors generalized and etended it both in terms of considering more general contraction codition and a more general ambient space. Now-a-days, fixed point theory gained lot of interest in the direction of proving the existence of fixed points in partilly ordered metric spaces. Existence of fixed points in partially ordered sets has been considered by Ran and Reurings[14]. For more works on the existence of fixed points in partially ordered sets, we refer [9,10,11] and [15].

Khan, Swaleh and Sessa [13] studied the existence of fixed points in metric spaces by using altering distance functions.

Definition 1.1 ([13]) A function $\psi : R^+ \to R^+$, $R^+ = [0,\infty)$ is said to be an *altering distance function* if the following conditions hold:

(i) ψ is continuous,
(ii) ψ is non-decreasing, and

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(iii) $\psi(t) = 0$ if and only if $t = 0$.

Geraghty contractions depends on the class of functions

$$S = \{\beta : [0,\infty) \to [0,1) / \beta(t_n) \to 1 \Rightarrow t_n \to 0\}$$

Defnition 1.2.[7] Let (X, d) be a metric space. A selfmap $f : X \to X$ is said to be *a Geraghty contraction* if there exists $\beta \in S$ such that

$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y)$ for all $x, y \in X$.

Theorem 1.3.[7] Let (X, d) be a complete metric space. Let $f : X \to X$ be a selfmap. If there exists $\beta \in S$ such that

$$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y) \text{ for all } x, y \in X,$$

then f has a unique xed point in X.

In 2013, Cabrera, Harjani and Sadarangani [5] proved the above theorem in the context of partially ordered metric spaces as follows.

Theorem 1.4.[5]. Let $(X, \preccurlyeq)$ be a partially ordered set and supposethat there exists a metric d on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and non-decreasing mapping such that (1.1.1) is satised for all $x, y \in X$ with $x \preccurlyeq y$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$ then T has a xed point.

Theorem 1.5.[5]. Let $(X, \preccurlyeq)$ be a partially ordered set and supposethat there exists a metric d on X such that (X, d) is a complete metric space. Assume that if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x_n \preccurlyeq x$, for all $n \in N$. Let $T : X \to X$ be a non-decreasing mapping such that (1.1.1) is satised for all $x, y \in X$ with $x \preccurlyeq y$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$ then T has a xed point.

Theorem 1.6.[5]. In addition to the hypotheses of Theorem 1.3 (orTheorem 1.4), suppose that for every $x, y \in X$, there exists $u \in X$ such that $u \preccurlyeq x$ and $u \preccurlyeq y$. Then T has a unique xed point.

De nition 1.7. [3] Let $(X, \preccurlyeq, d)$ be a partially ordered metric space and let $f : X \to X$ be a selfmap. Let $\psi \in$. If there exist $\beta \in S$ and $L \geq 0$ such that

$$\psi(d(f(x), f(y))) \leq \beta(\psi(M(x, y)))\psi(M(x, y)) + L.N(x, y)$$

where

$M(x, y) = max\{d(x, y), \frac{1}{2}(d(x, f(x))+d(y, f(y))), \frac{1}{2}(d(x, f(y))+d(y, f(x)))\}$ $N(x, y) = min\{d(x, f(x)), d(x, f(y)), d(y, f(x))\}$ for all $x, y \in X$ with $x \succcurlyeq y$ then we call f is a *ψ-weak generalized Geraghty contraction.*

Theorem 1.8.[3] *Let $(X, \preccurlyeq, d)$ be a partially ordered complete metric space. Let $f : X \to X$ be a non-decreasing mapping such that there exists $x_0 \in X$ with $x_0 \preccurlyeq f(x_0)$. Assume that f is ψ-weak generalized Geraghty contraction.*

Furthermore, assume that either

(i) *f is continuous; (or)*

(ii) *X is such that if $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to x$, then $x_n \preccurlyeq x$ for all $n \geq 1$.*

Further, if for any $s > 0$, $\lim\sup\beta(t) = \beta(s)$ *then f has a fixed point in X.*

In 1975, Dass and Gupta [6] extended the Banach contraction prin-ciple through rational expression as follows.

Theorem 1.9.[6]. Let (X, d) be a complete metric space and $T: X \to X$ a mapping such that there exist $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ satisfying

$$d(Tx,Ty) \leq \alpha \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} + \beta\, d(x,y) \text{ for all } x, y \in X.$$

Then T has a unique fixed point.
The following Lemma, which we use in our main theorem, can be easily established.

Lemma 1.10.[2] Let (X, d) be metric space. Let $\{x_n\}$ be a sequencein X such that $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ and $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$. For each $k > 0$, corresponding to m(k), we can choose n(k) to be the smallest integer such that $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$ and $d(x_{m(k)}, x_{n(k)\,1}) < \epsilon$. It can be shown that the following identities are satisfied.

(i) $\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon$ (ii) $\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)+1}) = \varepsilon$,

(iii) $\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}) = \varepsilon$, and (iv) $\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)+1}) = \varepsilon$.

In Section 2 , we prove the theorems of fixed point results satisfying a generalized Geraghty contractions of selfmaps with altering distance function φ involving rational type expressions.

2. MAIN RESULTS

Notation :

$\Phi = \{\varphi : R^+ \to R^+ / \varphi$ is non-decreasing, continuous and $\varphi(t) = 0 \Leftrightarrow t = 0\}$.

Theorem 2.1.Let $(X, \preccurlyeq)$ be a partially ordered set and (X, d) be a complete metric space.

Let $T : X \to X$ be a non-decreasing mapping. Suppose there exist $\varphi \in \Phi$ such that,

for all $x, y \in X$ with $x \preccurlyeq y$,

$$\varphi(d(Tx, Ty)) \leq \beta(\varphi(M(x, y))).\varphi(M(x, y)) + L.min\,.\varphi(N(x, y)) \qquad (2.1.1)$$

where

$$M(x,y) = \max\left\{ \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(x,Tx)[1+d(y,Ty)]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty)]}{1+d(x,y)}, d(x,y) \right\}$$

and

$$N(x,y) = \max\left\{ \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty)]}{1+d(x,y)}, d(x,y) \right\}$$

If there exists $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$, then the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \ldots$ is a Cauchy sequence.

*Proof.*Let $x_0 \in X$ be such that $x_0 \preccurlyeq Tx_0$. (by hypothesis)

We define $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for each $n = 0, 1, 2, \ldots,$.

Since $x_0 \preccurlyeq Tx_0$ and T is a non-decreasing function, by mathematical induction it follows that

$$x_0 \preccurlyeq Tx_0 \preccurlyeq Tx_1 \preccurlyeq Tx_2 \preccurlyeq \ldots \preccurlyeq Tx_{n1} \preccurlyeq Tx_n \preccurlyeq \ldots$$

$$i.e.,\ x_0 \preccurlyeq x_1 \preccurlyeq x_2 \ldots \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \ldots$$

so that $x_n \preccurlyeq x_{n+1}$ for each $n = 0, 1, 2, \ldots,$.

If $x_n = x_{n+1}$ for some $n \in N$ then $x_n = Tx_n = x_{n+1}$.
Hence $x_{n+2} = Tx_{n+1} = Tx_n = x_n$.
Then $x_n = x_{n+1} = x_{n+2} = \ldots$.
Hence $\{x_n\}$ is a Cauchy sequence.
Hence without loss of generality, we assume that $x_n \neq x_{n+1}$ for each n.

Since $x_n \preccurlyeq x_{n+1}$ for each $n \geq 0$ from (2.1.1), we have

$$\varphi(d(x_n,\ x_{n+1}) = \varphi(d(Tx_{n-1},\ Tx_n)$$

$$\leq \beta(\varphi(M(x_{n-1},\ x_n)))\varphi(M(x_{n-1},\ x_n)) + L\min\varphi(N(x_{n-1},\ x_n)) \qquad 2.1.2$$

$$M(x_{n-1},\ x_n) = \max\left\{\frac{d(x_n,Tx_n)[1+d(x_{n-1},Tx_{n-1})]}{1+d(x_{n-1},x_n)},\ \frac{d(x_{n-1},Tx_{n-1})[1+d(x_n,Tx_n)]}{1+d(x_{n-1},x_n)},\ \frac{d(x_n,Tx_{n-1})[1+d(x_{n-1},Tx_n)]}{1+d(x_{n-1},x_n)},\ d(x_{n-1},x_n)\right\}$$

$$= \max\left\{\frac{d(x_n,x_{n+1})[1+d(x_{n-1},x_n)]}{1+d(x_{n-1},x_n)},\ \frac{d(x_{n-1},x_n)[1+d(x_n,x_n+1)]}{1+d(x_{n-1},x_n)},\ \frac{d(x_n,x_n)[1+d(x_{n-1},x_n+1)]}{1+d(x_{n-1},x_n)},\ d(x_{n-1},x_n)\right\}$$

$$= \max\left\{d(x_n,x_{n+1}),\ \frac{d(x_{n-1},x_n)[1+d(x_n,x_{n+1})]}{1+d(x_{n-1},x_n)},\ d(x_{n-1},x_n)\right\}$$

$$N(x_{n-1},\ x_n) = \min\left\{\frac{d(x_n,Tx_n)[1+d(x_{n-1},Tx_{n-1})]}{1+d(x_{n-1},x_n)},\ \frac{d(x_n,Tx_{n-1})[1+d(x_{n-1},Tx_n)]}{1+d(x_{n-1},x_n)},\ d(x_{n-1},x_n)\right\}$$

=0

Suppose $\max\left\{d(x_n,x_{n+1}),\ d(x_{n-1},x_n)\right\} = d(x_n,x_{n+1})$, ,

then $\max\left\{d(x_n,x_{n+1}),\ \frac{d(x_{n-1},x_n)[1+d(x_n,x_{n+1})]}{1+d(x_{n-1},x_n)},\ d(x_{n-1},x_n)\right\} = d(x_n,x_{n+1})$.

Therefore from (2.1.2),

$$\varphi(d(x_n,x_{n+1}) < \varphi(d(x_n,x_{n+1}) \qquad (2.1.3)$$

Which is contradiction.

So $\max\left\{d(x_n,x_{n+1}),\ d(x_{n-1},x_n)\right\} = d(x_{n-1},x_n)$

Therefore from 2.1.2 we have $\varphi(d(x_n,x_{n+1}) < \varphi(d(x_{n-1},x_n)$ (2.1.4)

Thus it follows that $\{\varphi(d(x_n, x_{n+1}))\}$ is a strictly decreasing sequence of positive real numbers and so $\lim_{n\to\infty} \varphi(d(x_n, x_{n+1}))$ exists and it is r (say). *i.e.,* $\lim_{n\to\infty} \varphi(d(x_n, x_{n+1})) = r \geq 0$.

From (2.1.4), since φ is non-decreasing, it follows that $\{d(x_n, x_{n+1})\}$ is also a strictly decreasing sequence of positive real numbers and so $\lim_{n\to\infty} d(x_n, x_{n+1})$ exists and it is s (say). *i.e.,* $\lim_{n\to\infty} d(x_n, x_{n+1}) = s \geq 0$.

We now show that $s = 0$.

Suppose that s>0.

From (2.1.2)

$$0 \leq \varphi(d(x_n,x_{n+1}) \leq \beta(\varphi(d(x_{n-1},x_n)\ \varphi(d(x_{n-1},x_n) \to 0 \text{ as } n\to\infty$$

So that $\lim_{n\to\infty} \varphi(d(x_n, x_{n+1})) = r = 0$ and hence s = 0.

Now, we show that $\{x_n\}$ is Cauchy.
Suppose that $\{x_n\}$ is not a Cauchy sequence and from lemma 1.10

Suppose n(k) > m(k) ,), we have $x_{n(k)-1} > x_{m(k)-1}$

$$\varphi(d(x_{m(k)},\ x_{n(k)})) = \varphi(d(Tx_{m(k)-1},\ Tx_{n(k)-1}))$$

$$\leq \beta(\varphi(M(x_{m(k)-1}, x_n(k)-1)\ \varphi(M(x_{m(k)-1}, x_{n(k)-1}) + L\min \varphi(N(x_{m(k)-1}, x_{n(k)-1})\ ) \quad (2.1.5)$$

$$M(x_{m(k)-1}, x_n(k)-1) = \max\left\{\frac{d(x_{n(k)}, Tx_{n(k)-1})[1+d(x_{m(k)-1}, Tx_{m(k)-1})]}{1+d(x_{m(k)-1}, x_{n(k)-1})}, \frac{d(x_{m(k)-1}, Tx_{m(k)-1})[1+d(x_{n(k)-1}, Tx_n(k))]}{1+d(x_{m(k)-1}, x_{n(k)-1})}, \frac{d(x_{n(k)-1}, Tx_{m(k)-1})[1+d(x_{m(k)-1}, Tx_n(k)-1)]}{1+d(x_{m(k)-1}, x_{n(k)-1})}, d(x_{m(k)-1}, x_{n(k)-1})\right\}$$

$$= \max\left\{\frac{d(x_{n(k)}, x_{n(k)})[1+d(x_{m(k)-1}, x_{m(k)})]}{1+d(x_{m(k)-1}, x_{n(k)-1})}, \frac{d(x_{m(k)-1}, x_{m(k)})[1+d(x_{n(k)-1}, x_{n}(k))]}{1+d(x_{m(k)-1}, x_{n(k)-1})}, \frac{d(x_{n(k)-1}, x_{m(k)})[1+d(x_{m(k)-1}, x_n(k))]}{1+d(x_{m(k)-1}, x_{n(k)-1})}, d(x_{m(k)-1}, x_{n(k)-1})\right\}$$

On letting $k \to \infty$,

$\lim_{k\to\infty} M\ (x_{n(k)-1}, x_{m(k)-1}) =$ max (0, 0, $\frac{\varepsilon(1+\varepsilon)}{1+\varepsilon}$, 0)= ε

Similarly $\lim_{k\to\infty} N(x_{n(k)-1}, x_{m(k)-1}) =$ min (0, 0, ε) = 0

Therefore from 2.1.5, we have

$$\varphi(d(x_{m(k)},\ x_{n(k)})) \leq \beta(\varphi(M(x_{m(k)-1}, x_n(k)-1)\ \varphi(d(x_{m(k)-1}, x_{n(k)-1})$$

and hence $\dfrac{\varphi(d(x_{m(k)},\ x_{n(k)}))}{\varphi(d(x_{m(k)-1}, x_{n(k)-1})} \leq \beta(\varphi(M(x_{m(k)-1}, x_n(k)-1) < 1$

On letting $k \to \infty$, and from Lemma 1.10, we get

$$1 = \frac{\varphi(\varepsilon)}{\varphi(\varepsilon)} \leq \beta(\varphi(M(x_{m(k)-1}, x_n(k)-1) \leq 1$$

So that $\beta(\varphi(M(x_{m(k)-1}, x_n(k)-1)) \to 1$ as $k \to \infty$.

Since $\beta \in S$, $\varphi(M(x_{m(k)-1}, x_n(k)-1) \to 0$.

i.e., $\varphi(\varepsilon) = 0$ and is continuous, it follows that ε = 0, a contradiction .

Therefore $\{x_n\}$ is a Cauchy sequence in *X*.

Theorem 2.2.In addition of the hypothesis of Theorem 2.1 supposethat is continuous.
Then *T* has a fixed point.

*Proof.*Let $\{x_n\}$ be as in theorem 2.1 then , by theorem 2.1 $\{x_n\}$ is aCauchy sequence in *X*.
Since *X* is complete, there exists *z* such that $\lim x_n = z$ as $n \to \infty$.

Since *T* is continuous, $T x_n \to T z$ that implies $x_{n+1} \to T z$.
But $x_{n+1} \to z$. Therefore by the uniqueness of the limit, $T z = z$.

Lemma 2.3.Under the hypothesis of Theorem 2.2 suppose that *z* is a *fi*xed point of *T* and $z \prec u$ for some $u \in X$ and $\{T^n u\}$ converges. Then $T^n(u) \to z$.

*Proof.*Now $z \prec u$ that implies $T z \preccurlyeq T u$ so that $z \preccurlyeq T u$.

By induction, $z \preccurlyeq T^n u$ for every n.
We have

$\varphi(d(z, T^{n+1}(u))) = \varphi(d(T^{n+1}(z), T^{n+1}(u)))$

$= \varphi(d(T(T^n(z), T(T^n(u)))$

$= \varphi(d(T(z), T(T^n(u))))$

$\leq \beta(\varphi(M(z, T^n(u))))\varphi(M(z, T^n(u))) + L. \min \varphi(N(z, T^n(u)))$ *(2.3.1)*

$$M(z,T^n(u)) = \max\left\{\frac{d(T^n u,T^{n+1}u)[1+d(z,Tz)]}{1+d(z,T^n u)}, \frac{d(z,Tz)[1+d(T^n u,T^{n+1}u)]}{1+d(z,T^n u)}, \frac{d(T^n u,Tz)[1+d(z,T^{n+1}u)]}{1+d(z,T^n u)}, d(z,T^n u)\right\}$$

$$= \max\left\{\frac{d(T^n u,T^{n+1}u)[1+d(z,z)]}{1+d(z,T^n u)}, \frac{d(z,z)[1+d(T^n u,T^{n+1}u)]}{1+d(z,T^n u)}, \frac{d(T^n u,z)[1+d(z,T^{n+1}u)]}{1+d(z,T^n u)}, d(z,T^n u)\right\}$$

$$= \max\left\{\frac{d(T^n u,T^{n+1}u)}{1+d(z,T^n u)}, 0, \frac{d(T^n u,z)[1+d(z,T^{n+1}u)]}{1+d(z,T^n u)}, d(z,T^n u)\right\}$$

$$= \max\left\{\frac{d(T^n u,z)[1+d(z,T^{n+1}u)]}{1+d(z,T^n u)}, d(z,T^n u)\right\}$$

$= d(z,T^n u)$

Similarly $N(z,T^n(u)) = 0$

From 2.3.1 $\varphi(d(z,T^{n+1}(u)) \leq \beta(\varphi(M(z,T^n(u)))\ \varphi(z,T^n(u)) + L.0$ (2.3.2)

Now suppose that $\lim_{n\to\infty} T^n(u) = v \neq z$.

Then d (z, $T^n(u)$) > 0 for large n consequently φ (d (z, $T^n(u)$) > 0 for large n.

Therefore from 2.3.2 $\varphi(d(z,T^{n+1}(u))) < \varphi(d(z,T^n(u)))$.

Hence $d(z,T^{n+1}(u))) < d(z,T^n(u))$ for large n'

Therefore $\{\varphi(d(z, T^{n+1}(u)))\}$ is a decreasing sequence and converges to (say) and
$\{d(z, T^{n+1}(u))\}$ is also decreasing sequence and converges to s (say).
From (2.3.2)
Now $\beta(\varphi(M(z, T^n(u))) \to 1$ then by the property of β, we have $\varphi(d(z, T^n u)) \to 0$ and hence $r = 0$.
Therefore $\varphi(d(z, T^n u)) \to 0$ and hence $d(z, T^n u) \to 0$.

Therefore $d(z, v) = 0$ *i.e.*, $\lim T^n(u) = z$ so that $T^n(u) \to z$.

Similarly we can prove the following lemma.

Lemma 2.5.Under the hypothesis of Theorem 2.2 , suppose that z is a fixed point of T and z is comparable with u for some $u \in X$ and $\{T^n u\}$ converges.
Then $T^n(u) \to z$.

*Proof.*Let $z \prec u$ and $\{T^n u\}$ converge. Then by lemma 2.4 $\{T^n u\}$ converges to z.
Let $z \succ u$ and $\{T^n u\}$ converge.
Then by lemma 2.4 $\{T^n u\}$ converges to z.
Therefore z is comparable to u and $\{T^n u\}$ converges to z.

Theorem 2.6.In addition to the hypotheses of Theorem 2.2 we assumethe following:
for every $u, v \in X$, there exists $z \in X$ which is comparable to both u and v".
Then T has a unique fixed point in X.

*Proof.*Let u and v be two fixed d points of T .

Suppose z is comparable to both u and v .

Since z is comparable to both u *then by Lemma 2.5* $T^n(z) \to u$.

Since z is comparable to both v *then by Lemma 2.5* $T^n(z) \to v$.

Now we prove the existence of common fixed point for a pair of selfmaps.

Theorem 2.7. Let $(X, d, \preccurlyeq)$ *be a partially ordered complete metric space. Let* $S, T : X \to X$ *be self maps of* X *and* T *is* S *non-decreasing. Sup-pose there exist* $\varphi \in \Phi$ *such that*

$$\varphi(d(Tx, Ty)) \leq \beta(\varphi(M(x, y)))\varphi(M(x, y)) + L \min \varphi(N(x, y)), \qquad (2.7.1)$$

where

$$M(x, y) = \max\left\{\frac{d(Sy,Ty)[1+d(Sx,Tx)]}{1+d(Sx,Sy)}, \frac{d(Sx,Tx)[1+d(Sy,Tyg)]}{1+d(Sx,Sy)}, \frac{d(Sy,Tx)[1+d(Sx,Tyg)]}{1+d(Sx,Sy)}, d(Sx, Sy)\right\}$$

and

$$N(x, y) = \min\left\{\frac{d(Sy,Ty)[1+d(Sx,Tx)]}{1+d(Sx,Sy)}, \frac{d(Sy,Tx)[1+d(Sx,Tyg)]}{1+d(Sx,Sy)}, d(Sx, Sy)\right\},$$ *for all* $x, y \in X$ *with* $Sx \preccurlyeq Sy$.

Further, assume that

(i) $T(X) \subseteq S(X)$;

(ii) there exists $x_0 \in X$ *such that* $Sx_0 \preccurlyeq Tx_0$;

(iii) $S(X)$ *is a complete subset of* X; *and*

(iv) if any non-decreasing sequence $\{x_n\}$ *in* X *converges to* x *then* $\{x_n\} \preccurlyeq x$ *for all* $n = 0, 1, 2, \dots$.

Then S *and* T *have a coincident point in* X.

Proof. By (*ii*), let $x_0 \in X$ such that $Sx_0 \preccurlyeq Tx_0$. Since $T(X) \subseteq S(X)$, we choose $x_1 \in X$ such that $Sx_1 = Tx_0$. Since $Sx_0 \preccurlyeq Tx_0$ and T is S non-decreasing, we have $Sx_0 \preccurlyeq Sx_1$, so that Tx_1

By using the similarly argument we choose a sequence $\{x_n\}$ in X with $Sx_{n+1} = Tx_n$ for each $n = 0, 1, 2, \dots$.

Further, since $Tx_1 \preccurlyeq Tx_2$ and T is S non-decreasing, we have $Sx_1 \preccurlyeq Sx_2$ so that $Tx_2 \preccurlyeq Tx_3$. On continuing this process, we get $Sx_n \preccurlyeq Sx_{n+1}$ for all $n = 0, 1, 2, \dots$.

If $Sx_n = Sx_{n+1}$ for some $n \in N$ then $Sx_n = Tx_n$ so that x_n is a coin-cidence point of S and T .

Hence, w. l. g., we assume that $Sx_n \neq Sx_{n+1}$ for each n then we have $d(Sx_n, Sx_{n+1}) > 0$ for all n.

$\varphi(d(Sx_n, Sx_{n+1})) = \varphi(d(Tx_{n-1}, Tx_n))$

$$\leq \beta(\varphi(M(x_{n-1}, x_n)))\varphi(M(x_{n-1}, x_n)) + L.\min \varphi(N(x_{n-1}, x_n)) \qquad (2.7.1)$$

$$M(x_{n-1},\ x_n)=\max\left\{\frac{d(Sx_n,Tx_n)[1+d(Sx_{n-1},Tx_{n-1})]}{1+d(Sx_{n-1},Sx_n)},\ \frac{d(Sx_{n-1},Tx_{n-1})[1+d(Sx_n,Tx_n)]}{1+d(Sx_{n-1},Sx_n)},\ \frac{d(Sx_n,Tx_{n-1})[1+d(Sx_{n-1},Sx_{n+1})]}{1+d(Sx_{n-1},Sx_n)},\ d(Sx_{n-1},Sx_n)\right\}$$

$$=\max\left\{d(Sx_n,Sx_{n+1}),\ \frac{d(Sx_{n-1},Sx_n)[1+d(Sx_n,Sx_{n+1})]}{1+d(Sx_{n-1},Sx_n)},\ d(Sx_{n-1},Sx_n)\right\}$$

And

$$N(x_{n-1},\ x_n)=\min\left\{\frac{d(Sx_n,Tx_n)[1+d(Sx_{n-1},Tx_{n-1})]}{1+d(Sx_{n-1},Sx_n)},\ \frac{d(Sx_n,Tx_{n-1})[1+d(Sx_{n-1},Sx_{n+1})]}{1+d(Sx_{n-1},Sx_n)},\ d(Sx_{n-1},Sx_n)\right\}$$

$$=\min\left\{d(Sx_n,Sx_{n+1}),\ \frac{d(Sx_n,Sx_n)[1+d(Sx_{n-1},Sx_{n+1})]}{1+d(Sx_{n-1},Sx_n)},\ d(Sx_{n-1},Sx_n)\right\}=0$$

If $M(x_{n-1},\ x_n)=\max\left\{d(Sx_n,Sx_{n+1}),\ \frac{d(Sx_{n-1},Sx_n)[1+d(Sx_n,Sx_{n+1})]}{1+d(Sx_{n-1},Sx_n)},\ d(Sx_{n-1},Sx_n)\right\}=d(Sx_n,Sx_{n+1})$

Then from 2.7.1

$$\varphi(d(Sx_n,\ Sx_{n+1})\le\beta(\varphi(M(x_{n-1},\ x_n)))\varphi(M(x_{n-1},\ x_n))\ +L.0\ <\varphi(d(Sx_n,\ Sx_{n+1})$$

Which is contradiction .

Hence $\max\left\{d(Sx_n,\ Sx_{n+1}),\ \ d(Sx_{n-1},Sx_n)\right\}=\ d(Sx_{n-1},Sx_n)$

Therefore

$$M(x_{n-1},\ x_n)=\max\left\{\frac{d(Sx_{n-1},Sx_n)[1+d(Sx_n,Sx_{n+1})]}{1+d(Sx_{n-1},Sx_n)},\ \ d(Sx_{n-1},Sx_n)\right\}=d(Sx_{n-1},Sx_n)$$

Therefore from 2.7.2

We get

$$\varphi(d(Sx_n,\ Sx_{n+1})\le\beta(\varphi(M(x_{n-1},\ x_n)))\varphi(d(Sx_{n-1},\ Sx_n))\ +L.0\ <\ \varphi(d(Sx_{n-1},\ Sx_n))$$

(2.7.3)

Thus it follows that $\{\varphi(d(Sx_n, Sx_{n+1}))\}$ is a strictly decreasing sequence of positive real numbers and so $\lim\varphi(d(Sx_n, Sx_{n+1}))$ exists and it is r (say).

i.e., $\lim \varphi(d(Sx_n, Sx_{n+1})) = r \ge 0$.

sinceφ is non- decreasing, it follows that $\{d(Sx_n, Sx_{n+1})\}$ is a strictly decreasing sequence of positive real numbers and so $\lim d(Sx_n, Sx_{n+1})$ exists and it is r' (say).

i.e., $\lim d(Sx_n, Sx_{n+1}) = r' \ge 0$.

Suppose that $r' > 0$.

From 2.7.3 $\varphi(d(Sx_n, Sx_{n+1})) \le \beta(\varphi(d(Sx_{n-1}, Sx_n)))\varphi(d(Sx_{n-1}, Sx_n))$.

Taking limit supermum on both sides, we have

$\lim\varphi(d(Sx_n, Sx_{n+1})) \le \lim\beta(\varphi(d(Sx_{n-1}, Sx_n)))\varphi(d(Sx_{n-1}, Sx_n)) \ \to 0$

$n \to \infty$

So that

$\lim_{n\to\infty}\varphi(d(Sx_{n-1}, Sx_n)) = 0$. which is contradiction , so that $r' = 0$

Now, we show that $\{Sx_n\}$ is Cauchy.
Suppose that $\{Sx_n\}$ is not a Cauchy sequence. Then by lemma 1.10

$$\varphi(d(Sx_{m(k)}, Sx_{n(k)}) = \varphi(d(Tx_{m(k)-1}, Tx_{n(k)-1})$$

$$\leq \beta(\varphi(M(x_{m(k)-1}, x_{n(k)-1})))\varphi(M(x_{m(k)-1}, x_{n(k)-1})) + L\min \varphi(N(x_{m(k)-1}, x_{n(k)-1}))$$
(2.7.4)

$$M(x_{m(k)-1}, x_{n(k)-1}) = \max\left\{\frac{d(Sx_{n(k)-1}, Tx_{n(k)-1})[1+d(Sx_{m(k)-1}, Tx_{m(k)-1})]}{1+d(Sx_{m(k)-1}, Sx_{n(k)-1})}, \frac{d(Sx_{m(k)-1}, Tx_{m(k)-1})[1+d(Sx_{n(k)-1}, Tx_{n(k)-1})]}{1+d(Sx_{m(k)-1}, Sx_{n(k)-1})}, \frac{d(Sx_{n(k)-1}, Tx_{m(k)-1})[1+d(Sx_{m(k)-1}, Tx_{n(k)-1})]}{1+d(Sx_{m(k)-1}, Sx_{n(k)-1})}, d(Sx_{m(k)-1}, Sx_{n(k)-1})\right\}$$

On letting $k\to\infty$, *we get* $M(x_{m(k)-1}, x_{n(k)-1}) = \in$,

$$N(x_{m(k)-1}, x_{n(k)-1}) = 0$$

From 2.7.4 and taking limit supremum, we have
$\varphi(\epsilon) = \lim \varphi(d(Sx_{m(k)}, Sx_{n(k)})) \leq \lim \beta(\varphi(M(x_{m(k)} {}_1, x_{n(k)} {}_1)))\varphi(\epsilon)$

and it implies that
$\lim\varphi(M(x_{m(k)-1}, x_{n(k)-1})) = 0$.

Since $\beta \in S$, $\varphi(M(x_{m(k)-1}, x_{n(k)-1})) \to 1$ as $k\to\infty$. i,e., $\varphi(\epsilon)$ = 0, and φ is continuous, it follows that $\epsilon = 0$, a contradiction.
Therefore $\{Sx_n\}$ is a Cauchy sequence in X.
Since $S(X)$ is complete, there exists $z \in S(X)$ such that
$\lim_{n!1} Sx_{n+1} = \lim_{n!1} Tx_n = Sy = z$ for some $y \in X$.
Now we show that $Sy = Ty$.
Suppose that $Sy \neq Ty$, *i.e.*, $d(Sy, Ty) > 0$.
Now, suppose that the condition (*iv*) holds. Since $\{Sx_n\}$ is a non-decreasing sequence and $Sx_n\to Sy$ for some $y \in X$, we have $Sx_n \preccurlyeq Sy$ for all $n \geq 0$.
Now, from (2.7.1), we have
$\varphi(d(Tx_n, Ty)) \leq \beta(\varphi(M(x_n, y)))\varphi(M(x_n, y)) + L\min(N(x_n, y))$ *(2.7.5)*

$$M(x_n, y) = \max\left\{\frac{d(Sy,Ty)[1+d(Sx_n, Sx_{n+1})]}{1+d(Sx_n,Sy)}, \frac{d(Sx_n,Sx_{n+1})[1+d(Sy,Ty)]}{1+d(Sx_n,Sy)}, \frac{d(Sy,Sx_{n+1})[1+d(Sx_n,Ty)]}{1+d(Sx_n,Sy)}, d(Sx_n,Sy)\right\}$$

$$N(x_n, y) = \max\left\{\frac{d(Sy,Ty)[1+d(Sx_n, Sx_{n+1})]}{1+d(Sx_n,Sy)}, \frac{d(Sx_n,Sx_{n+1})[1+d(Sy,Ty)]}{1+d(Sx_n,Sy)}, d(Sx_n,Sy)\right\}$$

On letting $n\to\infty$. we get

$M(x_n, y) = 0$, and $N(x_n, y) = 0$.
On letting $n\to\infty$ in (2.7.5), we get
$\varphi(d(Sy, Ty)) \leq \beta(\varphi(d(Sy, Ty)))\varphi(d(Sy, Ty)) + L.0$, which implies that

$\varphi(d(Sy, Ty)) = 0$.

Hence $Ty = Sy$ so that T and S have a coincidence point y.

Theorem 2.8. *In addition to the hypotheses of Theorem 2.7, if T and S are weakly compatible, and T is continuous then T and S have a unique common fixed point in X.*

Proof. From the proof of Theorem 2.7, we have $\{Sx_n\}$ is non-decreasingsequence that converges to Sx.
Let $w = Tz = Sz$.

Since T and S are weakly compatible, $Tw = TSz = STz = Sw$ and $Sz \preccurlyeq SSz = Sw$.
Suppose that $w = Tw$.
Consider
$\varphi(d(w, Tw)) = \varphi(d(Tz, TTz))$
$\leq \beta(\varphi(M(z, Tz)))\varphi(M(z, Tz)) + L \min \varphi(N(z, Tz))$
where

$$M(z, Tz) = \max\left\{\frac{d(STz,TTz)[1+d(Sz,Tz)]}{1+d(Sz,STz)}, \frac{d(Sz,Tz)[1+d(STz,TSz)]}{1+d(Sz,STz)}, \frac{d(STz,Tz)[1+d(Sz,TTz)]}{1+d(Sz,STz)}, d(Sz, STz)\right\}$$

$$= \max\left\{\frac{d(Sw,TTz)}{1+d(Sz,Sw)}, 0, \frac{d(Sw,Tz)[1+d(Sz,TTz)]}{1+d(Sz,Sw)}, d(Sz, Sw)\right\}$$

$$= \max\left\{\frac{d(Tw,TTz)}{1+d(w,Tw)}, 0, \frac{d(Tw,w)[1+d(w,TTz)]}{1+d(w,Tw)}, d(w, Tw)\right\}$$

$$= \max\left\{\frac{d(Tw,Tw)}{1+d(w,Tw)}, 0, \frac{d(Tw,w)[1+d(w,Tw)]}{1+d(w,Tw)}, d(w, Tw)\right\}$$

$$= d(w, Tw).$$

$$N(z, Tz) = \min\left\{\frac{d(STz,TTz)[1+d(Sz,Tz)]}{1+d(Sz,STz)}, \frac{d(Sz,Tz)[1+d(STz,TSz)]}{1+d(Sz,STz)}, d(Sz, STz)\right\}$$

$$= \min\left\{\frac{d(Sw,TTz)}{1+d(Sz,Sw)}, 0, d(Sz, Sw)\right\}$$

$$= \min\left\{\frac{d(Tw,TTz)}{1+d(w,Tw)}, 0, d(w, Tw)\right\}$$

$$= \min\left\{\frac{d(Tw,Tw)}{1+d(w,Tw)}, 0, d(w, Tw)\right\} = 0.$$

from (2.2.1) $\varphi(d(w, Tw)) < \varphi(d(w, Tw))$,
a contradiction, so that $w = Tw$. Hence $w = Tw = Sw$. Therefore w is a common xed point of T and S. Uniqueness:

Let z, and w be two xed points of T and S with $z \neq w$.
$\varphi(d(z, w)) = \varphi(d(Tz, Tw))$
$\leq \beta(\varphi(M(z, w)))\varphi(M(z, w)) + L \min(N(z, w))$
Where

$$M(z, w) = \max\left\{\frac{d(Sw,Tw)[1+d(Sz,Tz)]}{1+d(Sz,Sw)}, \frac{d(Sz,Tz)[1+d(Sw,Tw)]}{1+d(Sz,Sw)}, d(Sw,Tz)[1+d(Sz,Tw)], d(Sz, Sw)\right\}$$

$$= \max\{d(w,w), 0, \frac{d(w,z)[1+d(z,w)]}{1+d(z,w)} \frac{}{1+d(z,w)}, d(z,w)\}$$

$= \max\{0, 0, d(z, w), d(z, w)\}$

$= d(z, w)$.

$$N(z,w) = \min\{\frac{d(Sw,Tw)[1+d(Sz,Tz)]}{1+d(Sz,Sw)}, \frac{d(Sz,Tz)[1+d(Sw,Tw)]}{1+d(Sz,Sw)}, d(Sz,Sw)\}$$

$= \min\{\frac{d(w,w)}{1+d(z,z)}, 0, d(z, w)\}$

$= \min\{0, 0, , d(z, w)\}$

$= 0.$

from (2.2.1) $\varphi(d(z, w)) \leq \beta(\varphi(d(z, w)))\varphi(d(z, w)) + L.0\ \varphi(d(z, w)) < \varphi(d(z, w))$

acontradiction, so that z = w Therefore T and S have a unique common xed point in X.

The following is an example in support of our main Theorem 2.1.

Example 2.9. Let $X = \left\{0, \frac{1}{4}, \frac{1}{2}, 1\right\}$ with the usual metric.

We de ne partial order $\preccurlyeq$ on X as follows:

$$\preccurlyeq := \left\{(0, 0), (\frac{1}{4}, \frac{1}{4}), (\frac{1}{2}, \frac{1}{2}), (1,1), (\frac{1}{4}, \frac{1}{2})\right\}$$

Clearly (X, d) is a metric space and $(X, \preccurlyeq)$ is a partially ordered set.

We de ne $T : X \to X$ by $T(0) = \frac{1}{4}, T(\frac{1}{4}) = \frac{1}{2}, T(\frac{1}{2}) = 1,$ and $T(1) = 1$.

Moreover, we choose $x_0 = \frac{1}{4} \in X$ then $x \preccurlyeq T(x)$.

We de ne $\beta : [0,\infty) \to [0, 1)$ by $\beta(t) = \frac{1}{1+t}$ We now verify the inequality (2.1.1) for the elements $(\frac{1}{4}, \frac{1}{2})$ and in the remaining cases the inequality (2.1.1) holds trivially.

Case(i) : $(x, y) = (\frac{1}{4}, \frac{1}{2})$

In this case $\varphi(d(T(\frac{1}{4}, \frac{1}{2}))) = \varphi(d(\frac{1}{2}, 1)) = \varphi(\frac{1}{2}) = \frac{1}{4}$, and

$M(\frac{1}{4}, \frac{1}{2})) = \frac{3}{5}$, and $N(\frac{1}{4}, \frac{1}{2}) = \frac{1}{4}$

$$\varphi(d(T(\frac{1}{4}, \frac{1}{2}))) = \frac{1}{4} \leq \beta(\varphi(\frac{3}{5}))\varphi(\frac{3}{5}) + L\ \varphi(\frac{1}{4})$$

holds for $L \geq 3$.

Therefore Tsatises all the conditions of Theorem 2.1 and Thas a uniquefixed point 1.

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