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Bernoulli Wavelet Collocation Method for the Numerical Solution of Integral and Integro-Differential Equations

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Abstract Bernoulli wavelet collocation method for the numerical solution of Volterra, Fredholm, mixed Volterra-Fredholm integral and integro-differential equations and Abel's integral equations. The new technique is based upon Bernoulli polynomials, Bernoulli numbers and Bernoulli wavelet approximations. The properties of Bernoulli wavelet is first presented and the resulting Bernoulli wavelet matrices are utilized to reduce the integral and integro-differential equations into system of algebraic equations to get the required Bernoulli coefficients, which are computed by using Matlab. This technique is tested on some numerical examples and compared with the exact and existing methods (i.e., Legendre Wavelet and Hermite Wavelet). Error analysis is worked out, which shows efficiency of the new technique.
Keywords: Bernoulli wavelet, Collocation method, Integral equations, Integro-differential equations.

1. Introduction

Wavelets theory is a relatively new and an emerging tool in applied mathematical research area. It has been applied in a wide range of engineering disciplines; particularly, signal analysis for waveform representation and segmentations, time-frequency analysis and fast algorithms for easy implementation. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms. Since from 1991 the various types of wavelet method have been applied for numerical solution of different kinds of integral equation, a detailed survey on these papers can be found in [1].

Integral and integro-differential equations find its applications in various fields of science and engineering. There are several numerical methods for approximating the solution of integral and integro-differential equations are known and many different basic functions have been used. Such as Galerkin methods for the constructions of orthonormal wavelet bases approached by Liang et al. [2], Maleknejad et al. [3-7] used the continuous Legendre wavelets, a combination of Hybrid Taylor and block-pulse functions, Rationalized haar wavelet, Hermite Cubic splines, Coifman wavelet as scaling functions. Lepik et al. [8-13] applied the Haar Wavelets, Yousefi [14] have introduced a new CAS wavelet, Babolian et al. [15] derived the operational matrix for the product of two triangular orthogonal functions. Muthuvalu et al. [16] applied Half-sweep arithmetic mean method with composite trapezoidal scheme for the solution of Fredholm integral equations. Keshavarz et al. [17] applied Bernoulli wavelet operational matrix for the approximate solution of fractional

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order differential equations. In this paper, we introduced the Bernoulli wavelet collocation method for the numerical solution of integral and integro-differential equations.

2. Properties of Bernoulli Wavelets

Bernoulli wavelets are $B_{n,m} = B(k, \hat{n}, m, t)$ have four arguments; $\hat{n} = n-1$, $n = 1,2,3,\dots,2^{k-1}$, k is any positive integer, m is the order of Bernoulli polynomials and t is the normalized time. Then it can be defined [17] on the interval [0, 1) as follows,

$$B_{n,m}(t) = \begin{cases} 2^{\frac{k-1}{2}}\tilde{\beta}_m(2^{k-1}t-\hat{n}), \frac{\hat{n}}{2^{k-1}} \le t < \frac{\hat{n}+1}{2^{k-1}}, \\ 0, \qquad\qquad otherwise \end{cases} \tag{2.1}$$

with

$$\tilde{\beta}_m(t) = \begin{cases} 1, & m = 0, \\ \dfrac{1}{\sqrt{\dfrac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}}\beta_m(t), & m > 0, \end{cases}$$

where m = 0,1,2,...,M-1 and n = 1,2,...,2^{k-1}. The coefficient $\frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}}$ is for normality, $2^{-(k-1)}$is the dilation parameter, $\hat{n}2^{-(k-1)}$is the translation parameter and

$$\beta_m(t) = \sum_{i=0}^{m} \binom{m}{i} \alpha_{m-i} t^i$$

are the well-known Bernoulli polynomials of order m. Where $\alpha_i, i = 0,1,\dots,m$ are Bernoulli numbers. These numbers are a sequence of signed rational numbers which arise in the series expansion of trigonometric functions and can be defined by the identity,

$$\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} \alpha_i \frac{t^i}{i!}.$$

The first few Bernoulli numbers are

$$\alpha_0 = 1, \alpha_1 = \frac{-1}{2}, \alpha_2 = \frac{1}{6}, \alpha_4 = \frac{-1}{30}, \alpha_6 = \frac{1}{42}, \alpha_8 = \frac{-1}{30}, \alpha_{10} = \frac{5}{66}, \ . \ . \ .$$

With $\alpha_{2i+1} = 0, i = 1,2,3,\dots$

The first few Bernoulli Polynomials are,

$$\beta_0(t) = 1, \quad \beta_1(t) = t - \frac{1}{2}, \quad \beta_2(t) = t^2 - t + \frac{1}{6},$$
$$\beta_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t, \quad \beta_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30},$$
$$\beta_5(t) = t^5 - \frac{5}{2}t^4 + \frac{5}{3}t^3 - \frac{1}{6}t, \ \beta_6 = t^6 - 3t^5 + \frac{5}{2}t^4 - \frac{1}{2}t^2 + \frac{1}{42}, \dots$$

The six basis functions are given by:

$$\left.\begin{aligned} B_{10}(t) &= \sqrt{2} \\ B_{11}(t) &= \sqrt{6}(4t-1) \\ B_{12}(t) &= \sqrt{10}(24t^2-12t+1) \end{aligned}\right\} ; 0 \le t < \frac{1}{2},$$
$$\left.\begin{aligned} B_{20}(t) &= \sqrt{2} \\ B_{21}(t) &= \sqrt{6}(4t-3) \\ B_{22}(t) &= \sqrt{10}(24t^2-36t+13) \end{aligned}\right\} ; \frac{1}{2} \le t < 1$$

For k = 2 implies n = 1, 2 and M = 3 implies m = 0, 1, 2 then Eq. (2.1) gives the Bernoulli wavelet matrix of order ($N = 2^{k-1}M$) 6x6 as,

$$B_{6\times6} = \begin{bmatrix} 1.4142 & 1.4142 & 1.4142 & 0 & 0 & 0 \\ -1.6330 & 0 & 1.6330 & 0 & 0 & 0 \\ 0.5270 & -1.5811 & 0.5270 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.4142 & 1.4142 & 1.4142 \\ 0 & 0 & 0 & -1.6330 & 0 & 1.6330 \\ 0 & 0 & 0 & 0.5270 & -1.5811 & 0.5270 \end{bmatrix}$$

For k = 2 and M = 4 of order 8x8 as,

$$B_{8\times8} = \begin{bmatrix} 1.4142 & 1.4142 & 1.4142 & 1.4142 & 0 & 0 & 0 & 0 \\ -1.8371 & -0.6124 & 0.6124 & 1.8371 & 0 & 0 & 0 & 0 \\ 1.0870 & -1.2847 & -1.2847 & 1.0870 & 0 & 0 & 0 & 0 \\ 1.6811 & 1.2008 & -1.2008 & -1.6811 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.4142 & 1.4142 & 1.4142 & 1.4142 \\ 0 & 0 & 0 & 0 & -1.8371 & -0.6124 & 0.6124 & 1.8371 \\ 0 & 0 & 0 & 0 & 1.0870 & -1.2847 & -1.2847 & 1.0870 \\ 0 & 0 & 0 & 0 & 1.6811 & 1.2008 & -1.2008 & -1.6811 \end{bmatrix}$$

3. Bernoulli Wavelet Collocation Method of Solution

In this section, we present a Bernoulli wavelet (BW) collocation method for the numerical solution of integral and integro-differential equations,

3.1 Integral Equations

Fredholm Integral equations:

Consider the Fredholm integral equation,

$$u(t) = f(t) + \int_0^1 k_1(t,s)u(s)\,ds, \tag{3.1}$$

where $f(t) \in L^2[0,1)$, $k_1(t,s) \in L^2([0,1)\times[0,1))$ and $u(t)$ is an unknown function.

Let us approximate $f(t), u(t)$, and $k_1(t,s)$ by using the collocation points t_i as given in the above section 2.2. Then the numerical procedure as follows:

STEP 1: Let us first approximate $f(t) = X^T\Psi(t)$, and $u(t) \approx Y^T\Psi(t)$, (3.2)

Let the function $f(t) \in L^2[0,1]$ may be expanded as:

$$f(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}x_{n,m}B_{n,m}(t), \tag{3.3}$$

where

$$x_{n,m}=(f(t),B_{n,m}(t)). \tag{3.4}$$

In (3.4), (. , .) denotes the inner product.
If the infinite series in (3.3) is truncated, then (3.3) can be rewritten as:

$$f(t)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}x_{n,m}B_{n,m}(t)=X^T\Psi(t), \tag{3.5}$$

where X and $\Psi(t)$ are $N\times1$ matrices given by:

$$\begin{aligned} X &=[x_{10},x_{11},\dots,x_{1,M-1},x_{20},\dots,x_{2,M-1},\dots,x_{2^{k-1},0},\dots,x_{2^{k-1},M-1}]^T \\ &=[x_1,x_2,\dots,x_{2^{k-1}M}]^T, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \Psi(t) &=[B_{10}(t),B_{11}(t),\dots,B_{1,M-1}(t),B_{20}(t),\dots,B_{2,M-1}(t),\dots,B_{2^{k-1},0}(t),\dots,B_{2^{k-1},M-1}(t)]^T \\ &=[B_1(t),B_2(t),\dots,B_{2^{k-1}M}(t)]^T. \end{aligned} \tag{3.7}$$

STEP 2: Next, approximate the kernel function as: $k_1(t,s)\in L^2([0,1]\times[0,1])$

$$k_1(t,s)\simeq\Psi^T(t)K_1\Psi(s), \tag{3.8}$$

where K_1 is $2^{k-1}M\times2^{k-1}M$ matrix, with

$$[K_1]_{ij}=(B_i(t),(k_1(t,s),B_j(s))).$$

i.e., $K_1\simeq\left[\Psi^T(t)\right]^{-1}\cdot[k_1(t,s)]\cdot[\Psi(s)]^{-1}$ (3.9)

STEP 3: Substituting Eq. (3.2) and Eq. (3.8) in Eq. (3.1), we have:

$$\Psi^T(t)Y=\Psi^T(t)X+\int_0^1\Psi^T(t)K_1\Psi(s)\Psi^T(s)Yds$$

$$\Psi^T(t)Y=\Psi^T(t)X+\Psi^T(t)K_1\left(\int_0^1\Psi(s)\Psi^T(s)\,ds\right)Y$$

$$\Psi^T(t)Y=\Psi^T(t)(X+K_1Y),$$

Then we get a system of equations as,

$$(I-K_1)Y=X. \tag{3.10}$$

By solving this system obtain the Bernoulli wavelet coefficients 'Y' and substituting in step 4.
STEP 4: $u(t)\simeq Y^T\Psi(t)$
This is the required approximate solution of Eq. (3.1).

Volterra Integral equations:
Consider the Volterra integral equations with convolution but non-symmetrical kernel

$$u(t)=f(t)+\int_0^t k_2(t,s)u(s)\,ds,\qquad t\in[0,1] \tag{3.11}$$

where $f(t) \in L^2[0,1),\ k_2(t,s) \in L^2([0,1)\times[0,1))$ and $u(t)$ is an unknown function.

Let us approximate $f(t)$, $u(t)$, and $k_2(t,s)$ by using the collocation points t_i as given in the above section 2.2. Then the numerical procedure as follows:

STEP 1: This can be rewritten in Fredholm integral equations, with a modified kernel $\tilde{k}_2(t,s)$ and solved in Fredholm form [18] as,

$$u(t) = f(t) + \int_0^t \tilde{k}_2(t,s)\, y(s)\, ds, \qquad (3.12)$$

where, $\tilde{k}_2(t,s) = \begin{cases} k_2(t,s), & 0 \le s \le t \\ 0, & t \le s \le 1. \end{cases}$

STEP 2: Let us first approximate $f(t)$ and $u(t)$ as given in Eq. (3.2),

STEP 3: Next, we approximate the kernel function as: $\tilde{k}_2(t,s) \in L^2([0,1]\times[0,1])$

$$\tilde{k}_2(t,s) \approx \Psi^T(t)\cdot K_2 \cdot \Psi(s), \qquad (3.13)$$

where K_2 is $2^{k-1}M \times 2^{k-1}M$ matrix, with

$$(K_2)_{ij} = (B_i(t), (\tilde{k}_2(t,s), B_j(s))).$$

$$\text{i.e., } K_2 \approx \left[\Psi^T(t)\right]^{-1} \cdot \left[\tilde{k}_2(t,s)\right] \cdot \left[\Psi(s)\right]^{-1} \qquad (3.14)$$

STEP 4: Substituting Eq. (3.2) and Eq. (3.13) in Eq. (3.12), we have:

$$\Psi^T(t)Y = \Psi^T(t)X + \int_0^1 \Psi^T(t)K_2\Psi(s)\Psi^T(s)Yds$$

$$\Psi^T(t)Y = \Psi^T(t)X + \Psi^T(t)K_2\left(\int_0^1 \Psi(s)\Psi^T(s)\, ds\right)Y$$

$$\Psi^T(t)Y = \Psi^T(t)(X + K_2Y),$$

Then we get a system of equations as,

$$(I - K_2)Y = X. \qquad (3.15)$$

By solving this system obtain the Bernoulli wavelet coefficients '*Y*' and substituting in step 5.

STEP 5: $u(t) \approx Y^T\Psi(t)$

This is the required approximate solution of Eq. (3.11).

Fredholm-Volterra integral equations:

Consider the Fredholm-Volterra integral equation of the second kind,

$$u(t) = f(t) + \int_0^1 k_1(t,s)u(s)\, ds + \int_0^x k_2(t,s)u(s)\, ds, \qquad (3.16)$$

where $f \in L^2[0,1),\ k_1 \text{ and } k_2 \in L^2([0,1)\times[0,1))$ are known function and $u(t)$ is an unknown function.

Let us approximate $f(t)$, $u(t)$, $k_1(t,s)$ and $k_2(t,s)$ by using the collocation points as follows:

STEP 1: Let us first approximate $f(t)$ and $u(t)$ as given in Eq. (3.2),

STEP 2: Substituting Eq. (3.2), Eq. (3.9) and Eq. (3.14) in Eq. (3.16), we get a system of *N* equations with *N* unknowns,

i.e., $(I-K_1-K_2)Y = X.$ (3.17)

where, I is an identity matrix.

By solving this system we obtain the Bernoulli wavelet coefficient '*Y*' and substituting '*Y*' in step 3.

STEP 3: $u(t) \approx Y^T\Psi(t)$

This is the required approximate solution of Eq. (3.16).

Abel integral equations:

Consider the Abel integral equations,

$$f(t) = \int_0^t \frac{u(s)}{\sqrt{t-s}}\,ds, \quad 0 \le t\ s \le 1 \qquad (3.18)$$

we first approximate *u*(*t*) as truncated series defined in Eq. (3.5). That is,

$$u(t) = Y^T\Psi(t) \qquad (3.19)$$

where *Y* and $\Psi(t)$ are defined similarly to Eqs. (3.6) and (3.7). Then substituting Eq. (3.19) in Eq. (3.18), we get

$$f(t) = \int_0^t \frac{Y^T\Psi(s)}{\sqrt{t-s}}\,ds \qquad (3.20)$$

Now assume Eq. (3.20) is precise at the following collocation points t_i. Then we obtain

$$f(t_i) = \int_0^{t_i} \frac{Y^T\Psi(s)}{\sqrt{t_i-s}}\,ds \qquad (3.21)$$

Now, we get the system of algebraic equations with unknown coefficients.

$$f = Y^T K$$

By solving this system of equations, we get the Bernoulli wavelet coefficients '*Y*' and then substituting these coefficients in Eq. (3.19), we obtain the approximate solution of Eq. (3.18).

3.2 Integro-differential Equations

Fredholm Integro-differential equations:

Consider the Fredholm integro-differential equations,

$$u^{(n)}(t) = f(t) + \int_0^1 k_1(t,s)u(s)\,ds, \; u^{(l)} = b_l, \qquad (3.22)$$

where $f(t) \in L^2[0,1)$, $k_1(t,s) \in L^2([0,1)\times[0,1))$ and $u^{(n)}(t)$ is an unknown function.

where $u^{(n)}(t)$ is the n^{th} derivative of $u(t)$ with respect to t and b_l are constants that define the initial conditions.

Let us first, we convert the Fredholm integro-differential equation into Fredholm integral equation, then we reduce it into a system of algebraic equations as given in Eq. (3.10), using this system we solve the Eq. (3.22). Then we obtain the approximate solution of equation.

Volterra Integro-differential equations:

In this section, we concerned with converting to Volterra integral equations. We can easily convert the Volterra integro-differential equation to equivalent Volterra integral equation, provided the kernel is a difference kernel defined by $k(t, s) = k(t - s)$. This can be easily done by integrating both sides of the equation and using the initial conditions. To perform the conversion to a regular Volterra integral equation, we should use the well-known formula, which converts multiple integrals into a single integral [19]. i.e.,

$$\int_0^t\int_0^t \ldots\ldots\int_0^t u(t)dt^n = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}u(s)ds$$

Consider the Volterra integro-differential equations,

$$u^{(n)}(t) = f(t) + \int_0^t k_2(t,s)u(s)\,ds,\ u^{(l)} = b_l, \tag{3.23}$$

where $f(t) \in L^2[0,1)$, $k_2(t,s) \in L^2([0,1)\times[0,1))$ and $u^{(n)}(t)$ is an unknown function.

where $u^{(n)}(t)$ is the n^{th} derivative of $u(t)$ with respect to t and b_l are constants that define the initial conditions.

Let us first, we convert the Volterra integro-differential equation into Volterra integral equation, then we reduce it into a system of algebraic equations as given in Eq. (3.15), using this system we solve the Eq. (3.23). Then we obtain the approximate solution of equation.

4. Convergence Analysis

Theorem: The series solution $u(t) = \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} x_{p,q}B_{p,q}(t)$ defined in Eq. (3.5) using Bernoulli wavelet method converges to $u(t)$ as given in [20].

Proof: Let $L^2(R)$ be the Hilbert space and $B_{p,q}$ defined in Eq. (3.2) forms an orthonormal basis.

Let $u(t) = \sum_{i=0}^{M-1} x_{p,i}B_{p,i}(t)$ where $x_{p,i} = \langle u(t), B_{p,i}(t)\rangle$ for a fixed p.

Let us denote $B_{p,i}(t) = B(t)$ and let $\alpha_j = \langle u(t), B(t)\rangle$.

Now we define the sequence of partial sums S_p of $(\alpha_j B(t_j))$; Let S_p and S_q be the partial sums with $p \ge q$. We have to prove S_p is a Cauchy sequence in Hilbert space.

Let $S_p = \sum_{i=1}^{p} \alpha_j B(t_j)$.

Now $\left\langle u(t), S_p \right\rangle = \left\langle u(t), \sum_{i=1}^{p} \alpha_j B(t_j) \right\rangle = \sum_{j=1}^{p} \left|\alpha_j\right|^2$.

We claim that $\left\|S_p - S_q\right\|^2 = \sum_{j=q+1}^{p} \left|\alpha_j\right|^2, \quad p > q.$

Now

$$\left\| \sum_{j=q+1}^{p} \alpha_j L(t_j) \right\|^2 = \left\langle \sum_{j=q+1}^{p} \alpha_j B(t_j), \sum_{j=q+1}^{p} \alpha_j B(t_j) \right\rangle = \sum_{j=q+1}^{p} \left|\alpha_j\right|^2, \quad for \quad p > q.$$

Therefore, $\left\| \sum_{j=q+1}^{p} \alpha_j B(t_j) \right\|^2 = \sum_{j=1}^{p} \left|\alpha_j\right|^2, \quad for \quad p > q.$

From Bessel's inequality, we have $\sum_{j=1}^{p} \left|\alpha_j\right|^2$ is convergent and hence

$$\left\| \sum_{j=q+1}^{p} \alpha_j B(t_j) \right\|^2 \to 0 \quad as \quad q, p \to \infty$$

So, $\left\| \sum_{j=q+1}^{p} \alpha_j B(t_j) \right\| \to 0$ and $\{S_p\}$ is a Cauchy sequence and it converges to *s* (say).

We assert that $u(t) = s$.

Now $\left\langle s - u(t), B(t_j) \right\rangle = \left\langle s, B(t_j) \right\rangle - \left\langle u(t), B(t_j) \right\rangle = \left\langle \lim_{p\to\infty} S_p, B(t_j) \right\rangle - \alpha_j = \alpha_j - \alpha_j$,

This implies,

$$\left\langle s - u(t), B(t_j) \right\rangle = 0$$

Hence $u(t) = s$ and $\sum_{i=1}^{p} \alpha_j B(t_j)$. converges to $u(t)$ as $p \to \infty$ and proved.

5. Numerical experiments

In this section, we present Bernoulli wavelet (BW) collocation method for the numerical solution of integral and integro-differential equation in comparison with existing method to demonstrate the capability of the proposed method and error analysis are shown in tables and figures. Error function is presented to verify the accuracy and efficiency of the following numerical results:

$$E_{\max} = Error \ function = \left\| u_e(t_i) - u_a(t_i) \right\|_\infty = \sqrt{\sum_{i=1}^{n} \left(u_e(t_i) - u_a(t_i) \right)^2}$$

where u_e and u_a are the exact and approximate solution respectively.

Example 5.1 Let us consider the Fredholm integral equation of the second kind [4],

$$u(t) = \exp(t) - \frac{\exp(t+1) - 1}{t+1} + \int_0^1 \exp(ts) u(s)\, ds, \qquad 0 \le t \le 1 \tag{5.1}$$

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which has the exact solution $u(t)=\exp(t)$. Where $f(t)=\exp(t)-\dfrac{\exp(t+1)-1}{t+1}$ and kernel $k_1(t,s)=\exp(ts)$.

Firstly, we approximate $f(t)\simeq X^T\Psi(t)$, and $u(t)\simeq Y^T\Psi(t)$,

Next, approximate the kernel function as: $k_1(t,s)\in L^2([0,1]\times[0,1])$

$$k_1(t,s)\simeq \Psi^T(t)K_1\Psi(s),$$

where K_1 is $2^{k-1}M\times 2^{k-1}M$ matrix, with $[K_1]_{ij}=(H_i(t),(k_1(t,s),H_j(s)))$.

$$K_1\simeq\left[\Psi^T(t)\right]^{-1}\cdot\left[k_1(t,s)\right]\cdot\left[\Psi(s)\right]^{-1}$$

Next, substituting the function $f(t)$, $u(t)$, and $k_1(t,s)$ in Eq. (5.1), then using the collocation points, we get the system of algebraic equations with unknown coefficients for k = 2 and M = 4 (N = 8), as an order 8×8 as follows:

$$\Psi^T(t)Y=\Psi^T(t)X+\int_0^1\Psi^T(t)K_1\Psi(s)\Psi^T(s)Y ds$$

$$\Psi^T(t)Y=\Psi^T(t)X+\Psi^T(t)K_1\left(\int_0^1\Psi(s)\Psi^T(s)\,ds\right)Y$$

$$\Psi^T(t)Y=\Psi^T(t)(X+K_1Y),$$

$(I-K_1)Y=X$, where, $I=\int_0^1\Psi(s)\Psi^T(s)\,ds$ is the identity matrix.

We find, X = [-0.4978 0.0094 0.0027 0.0003 -0.4182 0.0400 0.0054 0.0005],

$$K_1=\begin{bmatrix}0.5331 & 0.0196 & 0.0004 & 0.0000 & 0.6072 & 0.0233 & 0.0005 & 0.0000\\ 0.0196 & 0.0118 & 0.0004 & 0.0000 & 0.0662 & 0.0152 & 0.0005 & 0.0000\\ 0.0004 & 0.0004 & 0.0001 & 0.0000 & 0.0033 & 0.0014 & 0.0001 & 0.0000\\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0002 & 0.0001 & 0.0000 & 0.0000\\ 0.6072 & 0.0662 & 0.0033 & 0.0002 & 0.8883 & 0.0982 & 0.0050 & 0.0003\\ 0.0233 & 0.0152 & 0.0014 & 0.0001 & 0.0982 & 0.0294 & 0.0024 & 0.0002\\ 0.0005 & 0.0005 & 0.0001 & 0.0000 & 0.0050 & 0.0024 & 0.0004 & 0.0000\\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0003 & 0.0002 & 0.0000 & 0.0000\end{bmatrix}$$

By solving this system of equations, we obtain the Bernoulli wavelet coefficients,
Y = [0.9170 0.1324 0.0085 0.0007 1.5122 0.2184 0.0140 0.0011]
and substituting these coefficients in $u(t)\simeq Y^T\Psi(t)$, we obtain the approximate solution $u(t)$ with exact solution are shown in table 1. Maximum Error analysis is shown in table 2 and compared with existing methods (Haar wavelet (HW) and Legendre wavelet (LW)).

Table 1: Numerical results of the example 5.1.

t	Exact	BW
0.0625	1.0645	1.0640
0.1875	1.2062	1.2057
0.3125	1.3668	1.3663
0.4375	1.5488	1.5483

0.5625	1.7551	1.7545
0.6875	1.9887	1.9882
0.8125	2.2535	2.2531
0.9375	2.5536	2.5532

Table 2: Maximum error analysis of the example 5.1.

N	$E_{\max}$(HW)	$E_{\max}$(LW)	$E_{\max}$(BW)
8	2.04e-02	5.58e-04	4.24e-05
16	5.48e-03	3.55e-05	2.65e-06
32	1.40e-03	2.23e-06	1.65e-07
64	3.54e-04	1.39e-07	1.03e-08
128	8.88e-05	8.74e-09	6.48e-10

Example 5.2 Let us consider the Fredholm integral equation of the second kind [11].

$$u(t) = t^2 + \int_0^1 (t+s)\ u(s)\ ds \qquad (5.2)$$

The solution $u(t)$ of Eq. (5.2) with the help of Bernoulli wavelet coefficients *Y* as [-1.7187 -0.1714 0.0023 2.56e-16 -2.2812 -0.1533 0.0023 3.02e-16 -2.7812 -0.1353 0.0023 3.50e-16 -3.2187 -0.1172 0.0023 4.21e-16] is obtained using the proposed method. The computational results for *k* = 3, M = 4 (*N* = 16) are compared with the exact $u(t) = t^2 - 5t - 17/6$ and existing method are given in table 4 and in figure 1, for *k* = 2 and M = 4 (*N* = 8). The maximum error is shown in table 3.

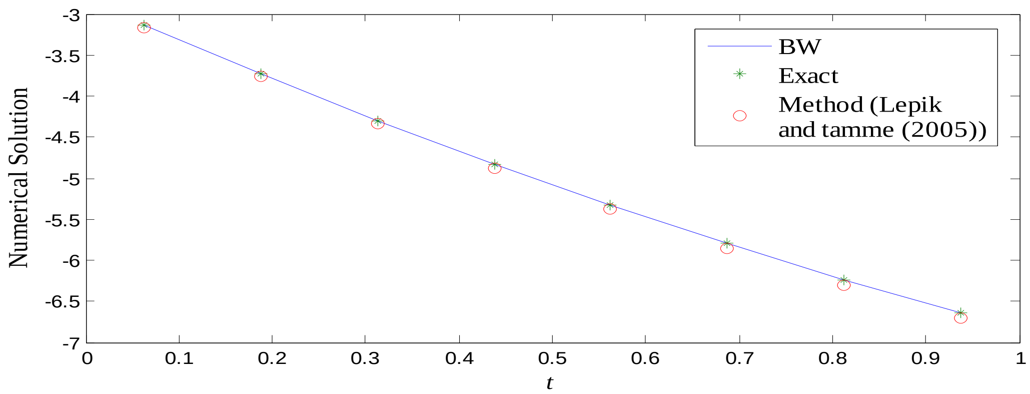

Fig. 1: Comparison of numerical solutions with exact solutions of the example 5.2, for *N* = 8.

Table 3: Error analysis of the example 5.2.

N	Method (Lepik and tame (2005b))	BW
8	6.71e-02	2.04e-14
16	1.69e-02	3.01e-14
32	4.27e-03	4.61e-14
64	1.07e-03	1.95e-14

128	2.68e-04	4.52e-14

Table 4: Numerical results of the example 5.2, for N = 16.

t	Exact	BW	Method (Lepik and tamme (2005b))	Error(BW)	Error (Method (Lepik and tamme (2005b)))
0.03125	-2.98861	-2.98861	-2.99453	1.24e-14	5.92e-03
0.09375	-3.29329	-3.29329	-3.29995	1.38e-14	6.66e-03
0.15625	-3.59017	-3.59017	-3.59756	1.55e-14	7.39e-03
0.21875	-3.87923	-3.87923	-3.88736	1.60e-14	8.13e-03
0.28125	-4.16048	-4.16048	-4.16935	1.60e-14	8.86e-03
0.34375	-4.43392	-4.43392	-4.44352	1.78e-14	9.60e-03
0.40625	-4.69954	-4.69954	-4.70988	1.95e-14	1.03e-02
0.46875	-4.95736	-4.95736	-4.96843	1.95e-14	1.10e-02
0.53125	-5.20736	-5.20736	-5.21916	2.13e-14	1.18e-02
0.59375	-5.44954	-5.44954	-5.46209	2.22e-14	1.25e-02
0.65625	-5.68392	-5.68392	-5.69722	2.40e-14	1.32e-02
0.71875	-5.91048	-5.91048	-5.92449	2.31e-14	1.40e-02
0.78125	-6.12923	-6.12923	-6.14398	2.75e-14	1.47e-02
0.84375	-6.34017	-6.34017	-6.35565	2.84e-14	1.54e-02
0.90625	-6.54329	-6.54329	-6.55951	3.02e-14	1.62e-02
0.96875	-6.73861	-6.73861	-6.75556	2.93e-14	1.69e-02

Example 5.3 Next, consider [6],

$$u(t) = \sin(2\pi t) + \int_0^1 \cos(t)\, u(s)\, ds \,. \tag{5.3}$$

which has the exact solution $u(t) = \sin(2\pi t)$. We applied the Bernoulli wavelets collocation method and solved Eq. (5.3), for k = 3 and M = 4 (N = 16) yields the values of $u(t)$ with the help of the Bernoulli wavelet coefficients Y as [0.3181 0.1441 -0.0314 -0.0077 0.3181 -0.1441 -0.0314 0.0077 -0.3181 -0.1441 0.0314 0.0077 -0.3181 0.1441 0.0314 -0.0077] is obtained using the proposed method. The maximum error analysis is shown in table 5 compared with existing method.

Example 5.4 Next, consider [6],

$$u(t) = \sin(2\pi t) + \int_0^1 (t^2 - t - s^2 + s)\, u(s)\, ds \tag{5.4}$$

which has the exact solution $u(t) = \sin(2\pi t)$. Applying the above method and solved Eq. (5.5.4) for k = 3 and M = 4 (N = 16) presents the values of $u(t)$ with the help of Bernoulli wavelet coefficients Y as [0.3181 0.1441 -0.0314 -0.0077 0.3181 -0.1441 -0.0314 0.0077 -0.3181 -0.1441 0.0314 0.0077 -0.3181 0.1441 0.0314 -0.0077] is obtained using the proposed method. The maximum error analysis is shown in table 5 compared with existing method.

Example 5.5 Next, consider [6],

$$u(t) = -2t^3 + 3t^2 - t + \int_0^1 (t^2 - t - s^2 + s)\,u(s)\,ds \qquad (5.5)$$

The approximate solution of $u(t)$ of Eq. (5.5) with the help of Bernoulli wavelet coefficients *Y* as [-0.0351 -0.0135 0.0052 -0.0005 -0.0273 0.0135 0.0017 -0.0005 0.0273 0.0135 -0.0017 -0.0005 0.0351 -0.0135 -0.0052 -0.0005] is obtained using the proposed method with the exact solution $u(t) = -2t^3 + 3t^2 - t$. The maximum error analysis is shown in table 5 compared with existing method.

Table 5: Comparison of the error analysis.

N	Example 5.3		Example 5.4		Example 5.5	
	BW	Method [6]	BW	Method [6]	BW	Method [6]
4	6.66e-16	2.84e-02	1.11e-16	2.84e-02	0	1.33e-10
8	4.44e-16	2.38e-03	2.77e-16	2.38e-03	4.16e-17	3.79e-10
16	4.16e-16	2.09e-04	2.22e-16	2.10e-04	4.16e-17	3.26e-10
32	1.33e-15	1.20e-04	4.44e-16	2.00e-04	5.55e-17	4.83e-10

Example 5.6 Next, consider the Volterra integral equation [18],

$$u(t) = t + \int_0^t \sin(t-s)\,u(s)\,ds, \qquad 0 \le t \le 1 \qquad (5.6)$$

which has the exact solution $u(t) = t + \frac{1}{6}t^3$.

Firstly, we approximate $f(t) \approx X^T\Psi(t)$, and $u(t) \approx Y^T\Psi(t)$,

Next, approximate the kernel function as: $k_2(t,s) \in L^2([0,1]\times[0,1])$

$$k_2(t,s) \approx \Psi^T(t) K_2 \Psi(s),$$

where K_2 is $2^{k-1}M \times 2^{k-1}M$ matrix, with $[K_2]_{ij} = (H_i(t),(k_2(t,s),H_j(s)))$.

$$K_2 \approx \left[\Psi^T(t)\right]^{-1} \cdot [k_2(t,s)] \cdot [\Psi(s)]^{-1}$$

Next, substituting the $f(t), u(t)$, and $k_2(t,s)$ in Eq. (5.6) using the collocation points t_i, we get the system of algebraic equations with unknown coefficients for *k* = 2 and M = 4 (*N* = 8), as an order 8×8 as follows:

$$\Psi^T(t)Y = \Psi^T(t)X + \int_0^1 \Psi^T(t)K_2\Psi(s)\Psi^T(s)Y ds$$

$$\Psi^T(t)Y = \Psi^T(t)X + \Psi^T(t)K_2\left(\int_0^1 \Psi(s)\Psi^T(s)\,ds\right)Y$$

$$\Psi^T(t)Y = \Psi^T(t)(X + K_2 Y),$$

$(I - K_2)Y = X$, where, $I = \int_0^1 \Psi(s)\Psi^T(s)\,ds$ is the identity matrix.

where, *X* = [0.1768 0.1021 0.0000 0 0.5303 0.1021 0.0000 -0.0000],

$$K_2 = \begin{bmatrix} 0.0398 & -0.0353 & 0.0082 & 0.0002 & 0 & 0 & 0 & 0 \\ 0.0353 & -0.0695 & -0.0003 & -0.0465 & 0 & 0 & 0 & 0 \\ 0.0082 & 0.0003 & -0.0113 & -0.0000 & 0 & 0 & 0 & 0 \\ -0.0002 & -0.0465 & 0.0000 & -0.0509 & 0 & 0 & 0 & 0 \\ 0.2348 & -0.0620 & -0.0022 & 0.0003 & 0.0398 & -0.0353 & 0.0082 & 0.0002 \\ 0.0620 & 0.0049 & -0.0006 & -0.0000 & 0.0353 & -0.0695 & -0.0003 & -0.0465 \\ -0.0022 & 0.0006 & 0.0000 & -0.0000 & 0.0082 & 0.0003 & -0.0113 & -0.0000 \\ -0.0003 & -0.0000 & 0.0000 & 0.0000 & -0.0002 & -0.0465 & 0.0000 & -0.0509 \end{bmatrix}$$

By solving this system of equations, we obtain the Bernoulli wavelet coefficients '*Y*'
Y = [0.1804 0.1016 0.0015 -0.0045 0.5852 0.1259 0.0045 -0.0057]
and substituting these coefficients in $u(t) = Y^T \Psi(t)$, we get the approximate solution $u(t)$ as shown in table 6. Maximum error analysis is compared with existing methods (Haar wavelet (HW) and Legendre wavelet (LW)) is shown in table 7.

Table 6: Numerical results of the example 5.6.

t	Exact	Bernoulli Wavelet
0.0625	0.0625	0.0625
0.1875	0.1886	0.1855
0.3125	0.3176	0.3208
0.4375	0.4515	0.4509
0.5625	0.5922	0.5915
0.6875	0.7417	0.7379
0.8125	0.9019	0.9059
0.9375	1.0748	1.0735

Table 7: Maximum Error analysis of example 5.6.

N	$E_{\max}$(HW)	$E_{\max}$(LW)	$E_{\max}$(BW)
8	2.57e-02	2.36e-03	4.02e-03
16	6.85e-03	6.12e-04	6.80e-04
32	1.77e-03	1.55e-04	1.50e-04
64	4.51e-04	3.90e-05	3.88e-05
128	1.14e-04	9.80e-06	9.78e-06

Example 5.7 Next, consider the Fredholm integro-differential equation [19],

$$u'(t) = 3 + 6t + t\int_0^1 s\,u(s)\,ds,\; u(0) = 0, \quad 0 \le t \le 1 \tag{5.7}$$

which has the exact solution $u(t) = 3t + 4t^2$.
Firstly, integrating Eq. (5.7) w.r.t *t*, we get Fredholm integral equations,

$$u(t) = 3t + 3t^2 + \frac{t^2}{2}\int_0^1 s\,u(s)\,ds, \tag{5.8}$$

Solving Eq. (5.8) using the present scheme, we get the approximate solution of $u(t)$ with the help of Bernoulli wavelet coefficients. Maximum error analysis is shown in table 8 compared with existing methods (Haar wavelet (HW) and Legendre wavelet (LW)).

Table 8: Maximum Error analysis of the example 5.6.

N	$E_{\max}(\mathrm{HW})$	$E_{\max}(\mathrm{LW})$	$E_{\max}(\mathrm{BW})$
8	4.42e-02	3.55e-15	1.77e-15
16	1.18e-02	1.77e-15	2.66e-15
32	3.06e-03	3.55e-15	2.66e-15
64	7.78e-04	3.55e-15	3.55e-15
128	1.96e-04	6.21e-15	4.44e-15

Example 5.8 Next, consider the Volterra integro-differential equation [21],

$$u'(t) = 2 - \frac{t^2}{4} + \frac{1}{4}\int_0^t u(s)\,ds,\ u(0) = 0, \quad 0 \le t \le 1 \tag{5.9}$$

which has the exact solution $u(t) = 2t$.

Firstly, integrating Eq. (5.9) w.r.t t, we get Volterra integral equations,

$$u(t) = 2t - \frac{t^3}{12} + \frac{1}{4}\int_0^t (t-s)\,u(s)\,ds, \tag{5.10}$$

Solving Eq. (5.10) using the present scheme, we get the approximate solution of $u(t)$ with the help of Bernoulli wavelet coefficients. Maximum error analysis is shown in table 9 compared with existing methods (Haar wavelet (HW) and Legendre wavelet (LW)).

Table 9: Maximum error analysis of the example 5.8.

N	$E_{\max}(\mathrm{HW})$	$E_{\max}(\mathrm{LW})$	$E_{\max}(\mathrm{BW})$
8	1.13e-02	9.74e-04	1.73e-03
16	3.03e-03	2.47e-04	2.68e-04
32	7.85e-04	6.24e-05	6.19e-05
64	1.99e-04	1.56e-05	1.56e-05
128	5.04e-05	3.92e-06	3.91e-06

Example 5.9 Next, consider the Volterra-Fredholm integral equation [22],

$$u(t) = t - 2\exp(t) + \exp(-t) + 1 + \int_0^t s\exp(s)\,u(s)\,ds + \int_0^1 \exp(t+s)\,u(s)\,ds, \quad 0 \le t \le 1 \tag{5.11}$$

which has the exact solution $u(t) = \exp(-t)$.

Let us approximate $f(t), u(t)$, $k_1(t,s)$ and $k_2(t,s)$ as given in Eq. (3.5), Eq. (3.9) and Eq. (3.14) using the collocation points, we get an system of *N* equations with *N* unknowns

i.e., $(I - K_1 - K_2)Y = X$. where, I is an identity matrix,
we find, X = [-0.3945 -0.2431 -0.0118 -0.0017 -1.4502 -0.3833 -0.0249 -0.0024],

$$K_1 = \begin{bmatrix} 0.8417 & 0.1215 & 0.0078 & 0.0006 & 1.3877 & 0.2003 & 0.0128 & 0.0010 \\ 0.1215 & 0.0175 & 0.0011 & 0.0001 & 0.2003 & 0.0289 & 0.0019 & 0.0001 \\ 0.0078 & 0.0011 & 0.0001 & 0.0000 & 0.0128 & 0.0019 & 0.0001 & 0.0000 \\ 0.0006 & 0.0001 & 0.0000 & 0.0000 & 0.0010 & 0.0001 & 0.0000 & 0.0000 \\ 1.3877 & 0.2003 & 0.0128 & 0.0010 & 2.2879 & 0.3302 & 0.0212 & 0.0016 \\ 0.2003 & 0.0289 & 0.0019 & 0.0001 & 0.3302 & 0.0477 & 0.0031 & 0.0002 \\ 0.0128 & 0.0019 & 0.0001 & 0.0000 & 0.0212 & 0.0031 & 0.0002 & 0.0000 \\ 0.0010 & 0.0001 & 0.0000 & 0.0000 & 0.0016 & 0.0002 & 0.0000 & 0.0000 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} 0.0393 & -0.0196 & -0.0306 & -0.0081 & 0 & 0 & 0 & 0 \\ 0.0425 & -0.2229 & -0.0358 & -0.2311 & 0 & 0 & 0 & 0 \\ 0.0151 & -0.0609 & -0.0259 & -0.0705 & 0 & 0 & 0 & 0 \\ 0.0036 & -0.1878 & 0.0034 & -0.2001 & 0 & 0 & 0 & 0 \\ 0.2674 & 0.1544 & 0.0000 & -0.0000 & 0.2863 & -0.2020 & -0.0681 & -0.0239 \\ 0.0386 & 0.0223 & 0 & -0.0000 & 0.2574 & -1.2269 & -0.0466 & -1.1544 \\ 0.0025 & 0.0014 & -0.0000 & -0.0000 & 0.0396 & -0.1792 & -0.1436 & -0.2554 \\ 0.0002 & 0.0001 & 0.0000 & 0.0000 & 0.0148 & -1.0549 & 0.0991 & -1.0762 \end{bmatrix}$$

By solving this system we obtain the Bernoulli wavelet coefficient,
Y = [0.5679 -0.0707 0.0087 0.0119 0.3431 -0.0328 0.0047 0.0188],
Then, substituting $u(t) \approx Y^T \Psi(t)$, we get the approximate solution of Eq. (5.11) are shown in table 11. Maximum error analysis is shown in table 10 compared with existing methods (Haar wavelet (HW) and Legendre wavelet (LW)).

Table 10: Maximum error analysis of the example 5.9.

N	$E_{\max}$(HW)	$E_{\max}$(LW)	$E_{\max}$(BW)
8	8.49e-02	2.41e-02	2.31e-02
16	5.72e-02	1.20e-02	1.19e-02
32	3.34e-02	6.73e-03	6.56e-03
64	1.82e-02	3.65e-03	3.55e-03

Table 11: Numerical results of the example 5.9.

t	Exact	Bernoulli Wavelet
0.0313	0.9692	0.9812
0.0938	0.9105	0.9217
0.1563	0.8553	0.8631
0.2188	0.8035	0.8126
0.2813	0.7548	0.7628
0.3438	0.7091	0.7190
0.4063	0.6661	0.6684
0.4688	0.6258	0.6316

0.5313	0.5879	0.5938
0.5938	0.5523	0.5620
0.6563	0.5188	0.5187
0.7188	0.4874	0.4912
0.7813	0.4578	0.4641
0.8438	0.4301	0.4417
0.9063	0.4040	0.4057
0.9688	0.3796	0.3846

Example 5.10 Let us consider the Abel's integral equation of first kind [23],

$$\frac{2}{105}\sqrt{t}(105-56t^2+48t^3)=\int_0^t \frac{u(s)}{\sqrt{t-s}}ds. \tag{5.12}$$

Firstly, consider

$$u(t)=Y^T\Psi(t) \tag{5.13}$$

substituting $u(t)$ in Eq. (5.12), we get

$$\frac{2}{105}\sqrt{t}(105-56t^2+48t^3)=\int_0^t \frac{Y^T\Psi(s)}{\sqrt{t-s}}ds. \tag{5.14}$$

Next, we collocate the point t_i and substitute in Eq. (5.14).

$$\frac{2}{105}\sqrt{t_i}(105-56t_i^{\,2}+48t_i^{\,3})=\int_0^{t_i} \frac{Y^T\Psi(s)}{\sqrt{t_i-s}}ds. \tag{5.15}$$

Now, we get the system of algebraic equations with unknown coefficients for *k* = 1 and M = 5 as given,

$$f=\begin{bmatrix}0.6294 & 1.0564 & 1.3065 & 1.4984 & 1.7100\end{bmatrix}$$

$$K=\begin{bmatrix} 0.6325 & 1.0954 & 1.4142 & 1.6733 & 1.8974 \\ -0.9494 & -1.1384 & -0.8165 & -0.1932 & 0.6573 \\ 0.8938 & 0.2156 & -0.6325 & -0.8681 & -0.0339 \\ 0.4727 & 1.2808 & 0.9759 & -0.0905 & -0.8107 \\ -4.0997 & -1.6545 & 3.5277 & 4.4946 & -0.5290 \end{bmatrix}$$

By solving this system of equations, we get the Bernoulli wavelet coefficients

$$Y=\begin{bmatrix}0.9167 & 0 & 0.0373 & 0.0345 & 0\end{bmatrix}$$

and then substituting these coefficients in Eq. (5.13), we get the accurate solution of Eq. (5.12) with exact solution $u(t)=t^3-t^2+1$ is shown in table 12 and the maximum error is 1.33e-15. Error analysis is shown in figure 3.

Table 12: Numerical results of the example 5.10.

t	Exact solution	Bernoulli wavelet (*k* = 1, *M* = 3)	Bernoulli wavelet (*k* = 1, *M* = 5)
0.1	0.991000000000000	0.990793650793651	0.990999999999999
0.2	0.968000000000000	0.955555555555555	0.968000000000000

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0.3	0.937000000000000	0.926031746031746	0.937000000000000
0.4	0.904000000000000	0.902222222222222	0.904000000000001
0.5	0.875000000000000	0.884126984126984	0.875000000000001
0.6	0.856000000000000	0.871746031746032	0.856000000000001
0.7	0.853000000000000	0.865079365079365	0.853000000000000
0.8	0.872000000000000	0.864126984126985	0.872000000000000
0.9	0.919000000000000	0.868888888888889	0.918999999999999

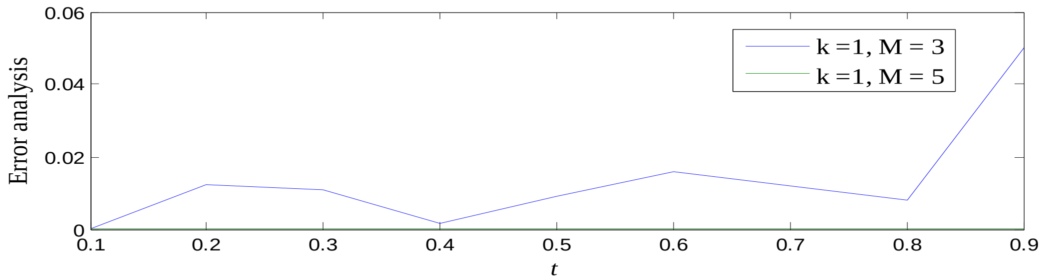

Fig. 3: Error analysis of the example 5.10.

Example 5.11 Next, consider [23],

$$t = \int_0^t \frac{u(s)}{\sqrt{t-s}} ds. \tag{5.16}$$

Applying the proposed method, we obtain the approximate solution $u(t)$ of Eq. (5.16) with the help of Bernoulli wavelet coefficients. Numerical solution is compared with exact solution $u(t) = \frac{2}{\pi}\sqrt{t}$ and existing methods is shown in table 13 and figure 4. Error analysis is shown in table 14 and figure 5 is compared with the existing method.

Table 13: Numerical results of the example 5.11.

t	Exact solution	Bernoulli wavelet (k = 1, M = 6)	Method [23]
0.1	0.201317	0.197156	0.200128
0.2	0.284705	0.284589	0.286092
0.3	0.348691	0.349102	0.347394
0.4	0.402634	0.402358	0.404161
0.5	0.450158	0.449889	0.449568
0.6	0.493124	0.493340	0.492704
0.7	0.532634	0.532707	0.532315
0.8	0.569410	0.568574	0.569156
0.9	0.603951	0.604356	0.603742

Table 14: Error analysis of the example 5.11.

t	Bernoulli wavelet k = 1, M = 6	Method [23]
0.1	4.16e-03	1.20e-03

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0.2	1.15e-04	1.40e-03
0.3	4.11e-04	1.30e-03
0.4	2.75e-04	1.50e-03
0.5	2.68e-04	5.90e-04
0.6	2.17e-04	4.20e-04
0.7	7.30e-05	3.19e-04
0.8	8.35e-04	2.54e-04
0.9	4.05e-04	2.09e-04

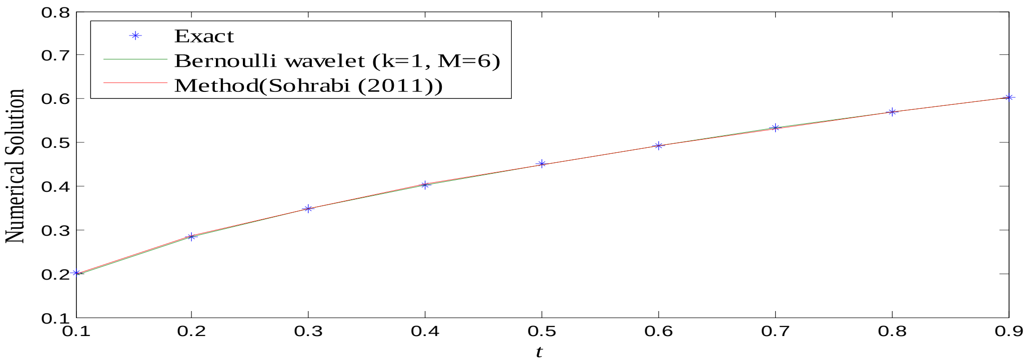

Fig. 4: Comparison of numerical results of the example 5.11.

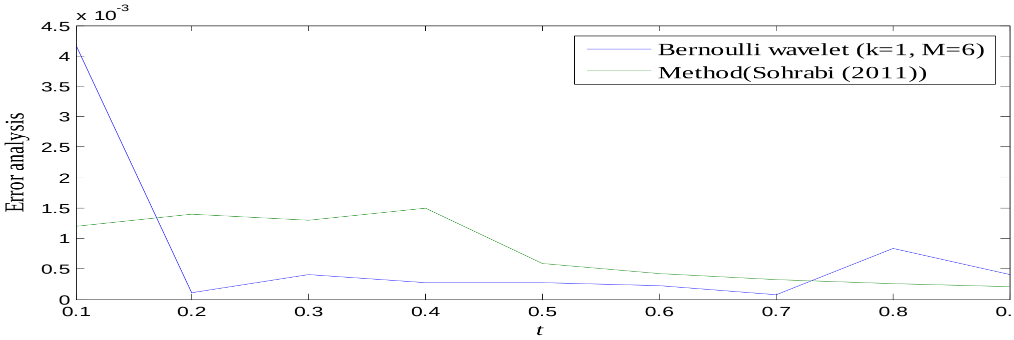

Fig. 5: Comparison of error analysis of the example 5.11.

6. Conclusion

In this paper, we proposed the Bernoulli wavelet collocation method for the numerical solution of integral and integro-differential equations. The given equations are reduced to system of algebraic equations. Our numerical results are compared with exact solutions and existing methods. Error analysis shows the accuracy and effectiveness of the proposed scheme as the level of resolution *N* increases for higher accuracy as compared to Legendre Wavelet and Hermite Wavelet, which are shown in tables and figures. Hence, the some of the illustrative examples are solved, which show the efficiency, validity and applicability of the present technique.

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