

Pseudo-differential type Operators and Their Multiplication involving Hankel type Convolution

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Abstract

In the present paper we have defined pseudo-differential type operators $A(x, D)$, $B(y, D)$ in terms of two symbols. Further the multiplication of these two operators also defined. It is also shown that the pseudo-differential type operator and multiplication of pseudo-differential type operators are bounded in certain Sobolev type space associated with the Hankel type transform. Finally some special cases are studied.

1 Introduction

The Hankel type transform of $\phi \in L^1(I)$, $I = (0, \infty)$ is defined by

$$(H_{\alpha,\beta}\phi)(x) = \int_0^\infty (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy)\phi(y)y^{4\alpha}dy, x \in I \quad (1.1)$$

where $J_{\alpha-\beta}$ denotes the Bessel type function of the first kind and order $(\alpha-\beta)$. Throughout this paper we assume that $(\alpha-\beta) \geq -1/2$. We note that if ϕ is a Lebegue measurable function on I and

$$\int_0^\infty x^{4\alpha}|\phi(x)|dx < \infty, \quad (1.2)$$

then as the function $t^{-(\alpha-\beta)}J_{\alpha-\beta}(t)$ is bounded on I , the Hankel type transformation $H_{\alpha,\beta}(\phi)$ is bounded on I . The inverse formula for (1.1) is given by

$$\phi(y) = \int_0^\infty \int_0^\infty (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy) (H_{\alpha,\beta}\phi)(x)x^{4\alpha}dx, y \in I \quad (1.3)$$

Following Zemanian [10], we introduce the space $H^{\alpha,\beta}$ as the space of all those complex valued and smooth functions ϕ defined on I such that for every $m, n \in \mathbb{N}_0$

$$\rho_{m,n}^{\alpha,\beta}(\phi) = \text{Sup}_{x \in I} (1 + x^2)^m \left| (x^{-1}D)^n x^{2\beta-1} \phi(x) \right| < \infty \quad (1.4)$$

On $H^{\alpha,\beta}$, the topology generated by the family $\{\rho_{m,n}^{\alpha,\beta}\}_{m,n \in \mathbb{N}_0}$ of seminorms. Then $H^{\alpha,\beta}$ is a Frechet space and Hankel type transformation $h_{\alpha,\beta}$ defined by

$$(h_{\alpha,\beta}\phi)(x) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\phi(y)dy, x \in I \quad (1.5)$$

We have following lemma:

Lemma 1 *The two forms $H_{\alpha,\beta}$ and $h_{\alpha,\beta}$ of Hankel type transforms are related through*

$$(H_{\alpha,\beta}\phi)(x) = x^{2\beta-1}h_{\alpha,\beta}(y^{2\alpha}\phi)(x), x \in I \quad (1.6)$$

Proof 1 *By (1.1), we have*

$$\begin{aligned} (H_{\alpha,\beta}\phi)(x) &= \int_0^\infty (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy)\phi(y)y^{4\alpha}dy, x \in I \\ &= \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\phi(y)x^{2\beta-1}y^{2\alpha}dy \\ &= x^{2\beta-1} \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\phi(y)(y^{2\alpha}\phi)(y)dy \\ &= x^{2\beta-1}h_{\alpha,\beta}(y^{2\alpha}\phi)(x) \end{aligned}$$

Thus lemma is proved.

Now for $1 \leq p < \infty$, we define the space $L_{\sigma,p}$ as the space of all those measurable functions ϕ on I such that

$$\|\phi\|_{L_{\sigma,p}} = \left[\int_0^\infty |\phi(x)|^p d\sigma(x) \right]^{1/p} < \infty \quad (1.7)$$

By $L_{\sigma,\infty}$ we represent the space of essentially (with respect to the measure $x^{4\alpha}dx$ or equivalently with respect to Lebesgue measure) bounded functions on I . The usual norm in $L_{\sigma,\infty}$ is denoted by $\|\cdot\|_{\sigma,\infty}$

If $f \in L_{\sigma,p}$ for some $1 \leq p < \infty$ then f defines an element of $H^{\alpha,\beta}$ through

$$\langle f, \phi \rangle = \int_0^\infty f(x)\phi(x)x^{4\alpha}dx, \phi \in H^{\alpha,\beta} \quad (1.8)$$

Following Hirschman [5] and haimo [4], we define the hankel type convolution on $L_{\sigma,p}$ by

$$(f\#g)(x) = \int_0^\infty f(y)(\tau_xg)(y)d\sigma(y), \quad (1.9)$$

where the Hankel type translation operator $\tau_x, x \in I$ is defined through

$$(\tau_x g)(y) = \int_0^\infty g(z) D_{\alpha, \beta}(x, y, z) d\sigma(z) \quad (1.10)$$

provided the above integral exists, where

$$D_{\alpha, \beta}(x, y, z) = [2^{\alpha-\beta} \Gamma(3\alpha + \beta)]^2 \int_0^\infty (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt) (yt)^{-(\alpha-\beta)} J_{\alpha-\beta}(yt) (zt)^{-(\alpha-\beta)} J_{\alpha-\beta}(zt) t^{4\alpha} dt \quad (1.11)$$

$$j_{\alpha-\beta}(x) = 2^{\alpha-\beta} \Gamma(3\alpha + \beta) x^{-(\alpha-\beta)} J_{\alpha-\beta}(x) \quad (1.12)$$

$$d\sigma(x) = \frac{x^{4\alpha}}{2^{\alpha-\beta} \Gamma(3\alpha + \beta)} dx \quad (1.13)$$

The following interchange formula holds,

$$H_{\alpha, \beta}(f \# g) = H_{\alpha, \beta}(f) H_{\alpha, \beta}(g) \quad (1.14)$$

and

$$(f \# g) \# h(x) = f \# (g \# h), f, g, h \in L_{\sigma, p}$$

If $f \in L_{\sigma, 1}, g \in L_{\sigma, p}$ then the integral defining $f \# g(x)$ converges for all x and

$$\|f \# g\|_{L_{\sigma, p}} \leq \|f\|_{L_{\sigma, 1}} \|g\|_{L_{\sigma, p}} \quad (1.15)$$

From [9] we have,

$$(H_{\alpha, \beta, a} \phi)(x) = \int_0^\infty (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy) a(x, y) y^{4\alpha} (H_{\alpha, \beta} \phi)(y) dy \quad (1.16)$$

where

$$(H_{\alpha, \beta} \phi)(x) = \int_0^\infty (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy) \phi(y) y^{4\alpha} dy, (\alpha - \beta) \geq -1/2 \quad (1.17)$$

According to Rodino [6] the symbol $a(x, y)$ is defined to be the complex valued infinitely differentiable function on $I \times I$ which satisfies

$$\left| (x^{-1} D_x)^a (y^{-1} D_y)^b a(x, y) \right| \leq C^{a+b+1} a! b! (1 + y)^{m-\rho b+\delta a} \quad (1.18)$$

for all $a, b \in \mathbb{N}_0$, where C is a constant and ρ, δ are real numbers such that $0 \leq \delta < \rho < 1$ and m is a fixed real number. The class of such symbols are denoted by $H(m)$.

Following [1], we define the the Bessel type operator as:

$$\Delta_{\alpha, \beta} = x^{4\beta-2} D x^{4\alpha} D \quad (1.19)$$

The Bessel type operator $\Delta_{\alpha,\beta}$ can also be written as

$$\Delta_{\alpha,\beta} = 4\alpha x^{4(\alpha+\beta)-3} D_x + x^{4(\alpha+\beta)-2} D_x^2 \quad (1.20)$$

From [1], for $1 \leq p < \infty$, we say that a measurable function $f \in I$ is in $H_{\alpha,\beta,p}$ if for every $k \in \mathbb{N}_0$, $\Delta_{\alpha,\beta}^k f \in L_{\alpha,\beta,p}$ such that

$$\langle \Delta_{\alpha,\beta}^k f, \phi \rangle = \int_0^\infty (\Delta_{\alpha,\beta}^k \phi)(x) f(x) x^{4\alpha} dx, \phi \in H^{\alpha,\beta} \quad (1.21)$$

and if $f \in H_{\alpha,\beta,p}$ with $1 \leq p \leq 2$ then from [[11] lemma 5.4] we have

$$H_{\alpha,\beta}(\Delta_{\alpha,\beta}^k f) = (-y^2)^k H_{\alpha,\beta}(f) \quad (1.22)$$

and from [9] we have

$$(x^{-1} D_x)^k (\phi\psi) = \sum_{r=0}^k kCr (x^{-1} D_x)^r \phi (x^{-1} D_x)^{k-r} \psi \quad (1.23)$$

We require following definition:

Definition 1 (Sobolev type space) The space $G_{\alpha,\beta,p}^s(I)$, $s \in \mathbb{R}$, $(\alpha - \beta) \in \mathbb{R}$ is defined to be the set of all those elements $\phi \in H_1^{\alpha,\beta}$ which satisfy

$$\|\phi\|_{G_{\alpha,\beta,p}^s} = \|(1 + \eta^2)^s H_{\alpha,\beta} \phi\|_{L_{\sigma,p}}, 1 \leq p \leq \infty \quad (1.24)$$

2 Pseudo-differential type operator $A(x, D)$

Definition 2 We define the pseudo-differential type operator $A(x, D)$ as

$$A(x, D)\phi(x) = \int_0^\infty (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt) a(x, t) (H_{\alpha,\beta} \phi)(t) t^{4\alpha} dt \quad (2.1)$$

where $\phi \in H^{\alpha,\beta}(I)$, $I = (0, \infty)$, $(\alpha - \beta) \geq -1/2$ and

$$a(x, t) = \int_0^\infty (x\lambda)^{(\alpha-\beta)} J_{\alpha-\beta}(x\lambda) b(\lambda, t) \lambda^{4\alpha} d\lambda \quad (2.2)$$

with the condition that for all $\lambda \in I$, $t \in I$,

$$|b(\lambda, t)| \leq k(\lambda) \in L_{\sigma,1}(I) \quad (2.3)$$

Theorem 1 Let $(\alpha - \beta) \geq -1/2$ then

$$\|A(x, D)\phi(x)\|_{G_{\alpha,\beta,1}^0} \leq \|k\|_{L_{\sigma,1}} \|\phi\|_{G_{\alpha,\beta,1}^0}, \phi \in H^{\alpha,\beta}(I)$$

Proof 2 From (2.1) and (2.2), we have

$$A(x, D)\phi(x) = \int_0^\infty (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt) \left(\int_0^\infty (x\lambda)^{-(\alpha-\beta)} J_{\alpha-\beta}(x\lambda) b(\lambda, t) \lambda^{4\alpha} d\lambda \right) \times (H_{\alpha,\beta}\phi)(t) t^{4\alpha} dt$$

Therefore

$$\begin{aligned} [H_{\alpha,\beta}(A(x, D))\phi(x)](z) &= \int_0^\infty (zx)^{-(\alpha-\beta)} J_{\alpha-\beta}(zx) A(x, D)\phi(x) x^{4\alpha} dx \\ &= \int_0^\infty \int_0^\infty \int_0^\infty (zx)^{-(\alpha-\beta)} J_{\alpha-\beta}(zx) (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt) \\ &\quad \times (x\lambda)^{-(\alpha-\beta)} J_{\alpha-\beta}(x\lambda) b(\lambda, t) (H_{\alpha,\beta}\phi)(t) (x\lambda)^{4\alpha} dt d\lambda dx \\ &= \int_0^\infty \int_0^\infty b(\lambda, t) \lambda^{4\alpha} (H_{\alpha,\beta}\phi)(t) t^{4\alpha} \int_0^\infty (zx)^{-(\alpha-\beta)} J_{\alpha-\beta}(zx) \\ &\quad \times (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt) (x\lambda)^{-(\alpha-\beta)} J_{\alpha-\beta}(x\lambda) x^{4\alpha} dx dt d\lambda \end{aligned}$$

Now using inequality (1.11), the last expression can be written as

$$\begin{aligned} [H_{\alpha,\beta}(A(x, D))\phi(x)](z) &= \frac{1}{[2^{\alpha-\beta}\Gamma(3\alpha + \beta)]^2} \int_0^\infty \int_0^\infty b(\lambda, t) \lambda^{4\alpha} (H_{\alpha,\beta}\phi)(t) t^{4\alpha} \\ &\quad \times D_{\alpha,\beta}(t, \lambda, z) dt d\lambda \\ &= \int_0^\infty \int_0^\infty b(\lambda, t) (H_{\alpha,\beta}\phi)(t) D_{\alpha,\beta}(t, \lambda, z) d\sigma(t) d\sigma(\lambda). \end{aligned}$$

Now using inequality (2.3), we obtain

$$\begin{aligned} |[H_{\alpha,\beta}(A(x, D))\phi(x)](z)| &\leq k(\lambda) \left(\int_0^\infty |(H_{\alpha,\beta}\phi)(t)| D_{\alpha,\beta}(t, \lambda, z) d\sigma(t) d\sigma(\lambda) \right) \\ &\leq k(\lambda) (\tau_z(H_{\alpha,\beta}\phi))(\lambda) d\sigma(\lambda) \end{aligned}$$

In view of [11] and [4], we have

$$|[H_{\alpha,\beta}(A(x, D))\phi(x)](z)| \leq (k\# |H_{\alpha,\beta}\phi|)(z).$$

Thus

$$\int_0^\infty |[H_{\alpha,\beta}(A(x, D))\phi(x)](z)| d\sigma(z) \leq \int_0^\infty (k\# |(H_{\alpha,\beta}\phi)(t)|)(z) d\sigma(z).$$

or

$$\|H_{\alpha,\beta}(A(x, D))\phi(x)\|_{L_{\sigma,1}} \leq \|k\# |(H_{\alpha,\beta}\phi)(t)|\|_{L_{\sigma,1}}$$

Using the inequality (1.15), we have

$$\|H_{\alpha,\beta}(A(x, D))\phi(x)\|_{L_{\sigma,1}} \leq \|k\|_{L_{\sigma,1}} \|H_{\alpha,\beta}\phi\|_{L_{\sigma,1}}$$

Now applying definition (1.1) we obtain

$$\|A(x, D)\phi(x)\|_{G_{\alpha,\beta,1}^0} \leq \|k\|_{L_{\sigma,1}} \|\phi\|_{G_{\alpha,\beta,1}^0}, \phi \in H^{\alpha,\beta}(I)$$

The proof is completed.

3 Pseudo-differential type operator $B(y, D)$

Definition 3 We define the pseudo-differential type operator $B(y, D)$ as

$$B(y, D)\phi(y) = \int_0^\infty (ys)^{-(\alpha-\beta)} J_{\alpha-\beta}(ys) e(y, s) (H_{\alpha,\beta}\phi)(s) s^{4\alpha} ds, \phi \in H^{\alpha,\beta}(I) \quad (3.1)$$

where

$$e(y, s) = \int_0^\infty (yb)^{-(\alpha-\beta)} J_{\alpha-\beta}(yb) f(b, s) b^{4\alpha} db \quad (3.2)$$

with the condition that for all $b, s \in I$ and $(\alpha - \beta) \geq -1/2$,

$$|f(b, s)| \leq l(b) \in L_{\sigma,1}(I) \quad (3.3)$$

Theorem 2 Let $(\alpha - \beta) \geq -1/2$, then

$$\|B(y, D)\phi\|_{G_{\alpha,\beta,1}^0} \leq \|k\|_{L_{\sigma,1}} \|\phi\|_{G_{\sigma,1}^0}, \phi \in H^{\alpha,\beta}(I)$$

Proof 3 Proceeding as in the proof of theorem 1 we find that

$$\begin{aligned} [H_{\alpha,\beta}(B(y, D))\phi(y)](z) &= \left| \int_0^\infty (zy)^{-(\alpha-\beta)} J_{\alpha-\beta}(yz) [B(y, D)\phi(y)] y^{4\alpha} dy \right| \\ &= \int_0^\infty (zy)^{-(\alpha-\beta)} J_{\alpha-\beta}(yz) \int_0^\infty (ys)^{-(\alpha-\beta)} J_{\alpha-\beta}(ys) \\ &\quad \times e(y, s) (H_{\alpha,\beta}\phi)(s) s^{4\alpha} ds y^{4\alpha} dy \\ &= \int_0^\infty \int_0^\infty (zy)^{-(\alpha-\beta)} J_{\alpha-\beta}(yz) (ys)^{-(\alpha-\beta)} J_{\alpha-\beta}(ys) \\ &\quad \times \int_0^\infty (yb)^{-(\alpha-\beta)} J_{\alpha-\beta}(yb) f(b, s) b^{4\alpha} db (H_{\alpha,\beta}\phi)(s) s^{4\alpha} ds y^{4\alpha} dy \\ &= \int_0^\infty \int_0^\infty \int_0^\infty (zy)^{-(\alpha-\beta)} J_{\alpha-\beta}(yz) (ys)^{-(\alpha-\beta)} J_{\alpha-\beta}(ys) \\ &\quad \times (yb)^{-(\alpha-\beta)} J_{\alpha-\beta}(yb) f(b, s) (ybs)^{4\alpha} (H_{\alpha,\beta}\phi)(s) db ds dy \end{aligned}$$

Now by using (1.7), (1.10), (1.11) and (3.3) we get

$$|[H_{\alpha,\beta}(B(y, D))\phi(y)](z)| \leq (l\# |H_{\alpha,\beta}\phi|)(z) \quad (3.4)$$

From which the assertion follows. Thus proof is completed.

4 Multiplication of Pseudo-differential type operators

Definition 4 The multiplication of two pseudo-differential type operators $A(x, D)$ and $B(y, D)$ associated with symbols $a(x, t)$ and $b(y, s)$ respectively is defined by

$$A(x, D)B(y, D)\phi(x) = \int_0^\infty (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt)a(x, t)H_{\alpha,\beta}(B(y, D)\phi)(t)t^{4\alpha} dt \quad (4.1)$$

From (2.2), (3.1) and (3.2), we have

$$\begin{aligned} A(x, D)B(y, D) &= \int_0^\infty (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt) \int_0^\infty (x\lambda)^{(\alpha-\beta)} J_{\alpha-\beta}(x\lambda)b(\lambda, t)\lambda^{4\alpha} d\lambda \\ &\quad \times \int_0^\infty (ty)^{-(\alpha-\beta)} J_{\alpha-\beta}(ty)B(y, D)\phi(y)y^{4\alpha} dyt^{4\alpha} dt \\ &= \int_0^\infty \int_0^\infty \int_0^\infty (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt)t^{4\alpha}(x\lambda)^{(\alpha-\beta)} J_{\alpha-\beta}(x\lambda)b(\lambda, t)\lambda^{4\alpha} \\ &\quad \times (ty)^{-(\alpha-\beta)} J_{\alpha-\beta}(ty) \int_0^\infty (ys)^{-(\alpha-\beta)} J_{\alpha-\beta}(ys)e(y, s)(H_{\alpha,\beta}\phi)(s) \\ &\quad \times s^{4\alpha} ds\phi(y)y^{4\alpha} dyd\lambda dt \\ &= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt)(x\lambda)^{(\alpha-\beta)} J_{\alpha-\beta}(x\lambda) \\ &\quad \times b(\lambda, t)(ty)^{-(\alpha-\beta)} J_{\alpha-\beta}(ty)(ys)^{-(\alpha-\beta)} J_{\alpha-\beta}(ys) \\ &\quad \times \int_0^\infty (sb)^{-(\alpha-\beta)} J_{\alpha-\beta}(sb)f(b, s)b^{4\alpha} db(H_{\alpha,\beta}\phi)(s) \\ &\quad \times (y\lambda ts)^{4\alpha} \phi(y)dyd\lambda dt ds \\ &= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt)(x\lambda)^{(\alpha-\beta)} J_{\alpha-\beta}(x\lambda) \\ &\quad \times (ty)^{-(\alpha-\beta)} J_{\alpha-\beta}(ty)(ys)^{-(\alpha-\beta)} J_{\alpha-\beta}(ys)(sb)^{-(\alpha-\beta)} J_{\alpha-\beta}(sb) \\ &\quad \times b(\lambda, t)f(b, s)(H_{\alpha,\beta}\phi)(s)(y\lambda tsb)^{4\alpha} \phi(y)dyd\lambda dt ds db \end{aligned}$$

provided multiple integral exists.

Theorem 3 Let $(\alpha - \beta) \geq -1/2$ then

$$\|A(x, D)B(x, D)\phi(x)\|_{G_{\alpha,\beta,1}^0} \leq \|k\|_{L_{\sigma,1}} \|\phi\|_{G_{\alpha,\beta,1}^0} \quad (4.2)$$

Proof 4 From definition (4.1) and (2.2), we have

$$\begin{aligned} A(x, D)B(x, D)\phi(x) &= \int_0^\infty (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt) a(x, t) H_{\alpha,\beta}(B(y, D)\phi)(t) t^{4\alpha} dt \\ &= \int_0^\infty (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt) \int_0^\infty (x\lambda)^{-(\alpha-\beta)} J_{\alpha-\beta}(x\lambda) \\ &\quad \times b(\lambda, t) \lambda^{4\alpha} d\lambda H_{\alpha,\beta}(B(y, D)\phi)(t) t^{4\alpha} dt \end{aligned}$$

Therefore

$$\begin{aligned} H_{\alpha,\beta}(A(x, D)B(x, D)\phi(x))(z) &= \int_0^\infty (xz)^{-(\alpha-\beta)} J_{\alpha-\beta}(xz) (A(x, D)B(y, D)\phi(x)) \\ &\quad \times x^{4\alpha} dx \\ &= \int_0^\infty (xz)^{-(\alpha-\beta)} J_{\alpha-\beta}(xz) \int_0^\infty (xt)^{-(\alpha-\beta)} \\ &\quad \times J_{\alpha-\beta}(xt) a(x, t) H_{\alpha,\beta}(B(y, D)\phi)(x) t^{4\alpha} dt x^{4\alpha} dx \\ &= \int_0^\infty \int_0^\infty (xz)^{-(\alpha-\beta)} J_{\alpha-\beta}(xz) (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt) \\ &\quad \times \int_0^\infty (x\lambda)^{-(\alpha-\beta)} J_{\alpha-\beta}(x\lambda) b(\lambda, t) \lambda^{4\alpha} d\lambda H_{\alpha,\beta} \\ &\quad \times (B(y, D)\phi(x)) (xt)^{4\alpha} dt dx \\ &= \int_0^\infty \int_0^\infty \int_0^\infty b(\lambda, t) H_{\alpha,\beta}(B(y, D)\phi(x)) \\ &\quad \times (xz)^{-(\alpha-\beta)} J_{\alpha-\beta}(xz) (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt) \\ &\quad \times (x\lambda)^{-(\alpha-\beta)} J_{\alpha-\beta}(x\lambda) (xt\lambda)^{4\alpha} d\lambda dt dx \end{aligned}$$

Now using (1.11) and (1.13), we have

$$\begin{aligned} &\leq \frac{1}{(2^{\alpha-\beta} \Gamma(3\alpha + \beta))^2} \int_0^\infty \int_0^\infty b(\lambda, t) H_{\alpha,\beta}(B(y, D)\phi(x)) D_{\alpha,\beta}(z, t, \lambda) (\lambda t)^{4\alpha} dt d\lambda \\ &\quad \leq \int_0^\infty \int_0^\infty b(\lambda, t) H_{\alpha,\beta}(B(y, D)\phi(x)) D_{\alpha,\beta}(z, t, \lambda) (\lambda t)^{4\alpha} d\sigma(t) d\sigma(\lambda) \end{aligned}$$

By using inequalities (2.3) and (3.4) we have

$$H_{\alpha,\beta}(A(x, D)B(x, D)\phi(x))(z) \leq \int_0^\infty \int_0^\infty k(\lambda) (l\# |H_{\alpha,\beta}\phi|)(t) D_{\alpha,\beta}(z, t, \lambda) d\sigma(t) d\sigma(\lambda)$$

Now by applying equations (1.11) and (1.10) we have

$$H_{\alpha,\beta} (A(x, D)B(x, D)\phi(x)) (z) \leq \int_0^\infty k(\lambda)\tau_z (l\# |H_{\alpha,\beta}\phi|) (\lambda)d\sigma(\lambda) \quad (4.3)$$

$$\leq k\# (l\# |H_{\alpha,\beta}\phi|) (z) \quad (4.4)$$

Hence

$$\begin{aligned} \int_0^\infty H_{\alpha,\beta} (A(x, D)B(x, D)\phi(x)) (z)d\sigma(z) &\leq \int_0^\infty k\# (l\# |H_{\alpha,\beta}\phi|) (z)d\sigma(z) \\ &\leq \|k\|_{L_{\sigma,1}} \|l\# |H_{\alpha,\beta}\phi|\|_{L_{\sigma,1}} \end{aligned}$$

So that

$$\|H_{\alpha,\beta} (A(x, D)B(x, D))\|_{L_{\sigma,1}} \leq \|k\|_{L_{\sigma,1}} \|l\|_{L_{\sigma,1}} \|H_{\alpha,\beta}\phi\|_{L_{\sigma,1}}$$

Using theorem 1 we have

$$\|(A(x, D)B(x, D))\|_{G_{\alpha,\beta,1}^0} \leq \|l\|_{L_{\sigma,1}} \|\phi\|_{G_{\alpha,\beta,1}^0}$$

Thus proof is completed.

5 Some Special Cases

In this section, we study some special cases as following theorems

Theorem 4 (i) Let

$$b(\lambda, t) = b_1(\lambda)b_2(t) \quad (5.1)$$

then

$$\|A(x, D)\phi(x)\|_{G_{\alpha,\beta,1}^0} \leq \|b_1\|_{L_{\sigma,1}} \|b_2H_{\alpha,\beta}\phi\|_{L_{\sigma,1}}$$

and

$$A(x, D)\phi(x) = (H_{\alpha,\beta}b_1) (x) (H_{\alpha,\beta}b_2H_{\alpha,\beta}\phi) (x)$$

(ii) Let

$$f(b, s) = f_1(b)f_2(s) \quad (5.2)$$

then

$$\|B(y, D)\phi(y)\|_{G_{\alpha,\beta,1}^0} \leq \|f_1\|_{L_{\sigma,1}} \|f_2H_{\alpha,\beta}\phi\|_{L_{\sigma,1}}$$

and

$$B(y, D)\phi(x) = (H_{\alpha,\beta}f_1) (y) (H_{\alpha,\beta}f_2H_{\alpha,\beta}\phi) (y)$$

Proof 5 here we prove (ii). (i) can be proved similarly. From (3.1), (3.2) and (5.2), we have

$$B(y, D)\phi(y) = \int_0^\infty (ys)^{-(\alpha-\beta)} J_{\alpha-\beta}(ys) \int_0^\infty (yb)^{-(\alpha-\beta)} J_{\alpha-\beta}(yb) f_1(b) f_2(s) b^{4\alpha} db \tag{5.3}$$

$$\times H_{\alpha,\beta}\phi(s) s^{4\alpha} ds \tag{5.4}$$

$$= \int_0^\infty \int_0^\infty (ys)^{-(\alpha-\beta)} J_{\alpha-\beta}(ys) (yb)^{-(\alpha-\beta)} J_{\alpha-\beta}(yb) f_1(b) f_2(s) \tag{5.5}$$

$$\times H_{\alpha,\beta}\phi(s) (bs)^{4\alpha} db ds \tag{5.6}$$

From equation (1.11)

$$D_{\alpha,\beta}(s, b, z) = [2^{\alpha-\beta}\Gamma(3\alpha + \beta)]^2 \int_0^\infty (ys)^{-(\alpha-\beta)} J_{\alpha-\beta}(ys) \int_0^\infty (yb)^{-(\alpha-\beta)} J_{\alpha-\beta}(yb) \times (zy)^{-(\alpha-\beta)} J_{\alpha-\beta}(yz) y^{4\alpha} dy$$

$$\int_0^\infty (yz)^{-(\alpha-\beta)} J_{\alpha-\beta}(yz) D_{\alpha,\beta}(s, b, z) z^{4\alpha} dz = [2^{\alpha-\beta}\Gamma(3\alpha + \beta)]^2 (ys)^{-(\alpha-\beta)} J_{\alpha-\beta}(ys) \tag{5.7}$$

$$\times \int_0^\infty (yb)^{-(\alpha-\beta)} J_{\alpha-\beta}(yb) \tag{5.8}$$

Thus from above, we have

$$\begin{aligned} B(y, D)\phi(y) &= \frac{1}{[2^{\alpha-\beta}\Gamma(3\alpha + \beta)]^2} \int_0^\infty (yz)^{-(\alpha-\beta)} J_{\alpha-\beta}(yz) \int_0^\infty \int_0^\infty f_1(b) f_2(s) \\ &\times H_{\alpha,\beta}\phi(s) (bs)^{4\alpha} D_{\alpha,\beta}(s, b, z) z^{4\alpha} dz db ds \\ &= \frac{1}{[2^{\alpha-\beta}\Gamma(3\alpha + \beta)]^2} \int_0^\infty (yz)^{-(\alpha-\beta)} J_{\alpha-\beta}(yz) \int_0^\infty f_1(b) \\ &\times \int_0^\infty (f_2 H_{\alpha,\beta}\phi)(s) D_{\alpha,\beta}(s, b, z) d\sigma(s) (bz)^{4\alpha} dz db \\ &= \int_0^\infty (yz)^{-(\alpha-\beta)} J_{\alpha-\beta}(yz) \int_0^\infty f_1(b) \tau_z (f_2(H_{\alpha,\beta}\phi)(b) d\sigma(b)) z^{4\alpha} dz \\ &= \int_0^\infty (yz)^{-(\alpha-\beta)} J_{\alpha-\beta}(yz) (f_1 \# f_2(H_{\alpha,\beta}\phi))(z) z^{4\alpha} dz \end{aligned}$$

An application of the inverse Hankel type transform yields

$$\int_0^\infty (yz)^{-(\alpha-\beta)} J_{\alpha-\beta}(yz) B(y, D)\phi(y) y^{4\alpha} dy = (f_1 \# f_2(H_{\alpha,\beta}\phi))(z)$$

or in other words,

$$H_{\alpha,\beta} [B(y, D)\phi(y)](z) = (f_1 \# f_2(H_{\alpha,\beta}\phi))(z) \tag{5.9}$$

Thus we have

$$\int_0^\infty H_{\alpha,\beta} [B(y, D)\phi(y)] (z) d\sigma(z) = \int_0^\infty (f_1 \# f_2 (H_{\alpha,\beta} \phi)) (z) d\sigma(z)$$

Now by applying definition(1), inequality (1.15) we get

$$\begin{aligned} \|H_{\alpha,\beta} [B(y, D)\phi(y)]\|_{L_{\sigma,1}} &= \|(f_1 \# f_2 (H_{\alpha,\beta} \phi))\|_{L_{\sigma,1}} \\ \|B(y, D)\phi(y)\|_{G_{\alpha,\beta,1}^0} &= \|f_2 H_{\alpha,\beta} \phi\|_{L_{\sigma,1}} \end{aligned}$$

Now from (5.6), using Hankel type inversion formula, we obtain

$$B(y, D)\phi(y) = H_{\alpha,\beta} (f_1 \# f_2 H_{\alpha,\beta} \phi) (y)$$

By using (1.14) we get $B(y, D)\phi(y) = (H_{\alpha,\beta} f_1) (y) (H_{\alpha,\beta} f_2 H_{\alpha,\beta} \phi) (y)$

Thus the proof is completed.

Theorem 5 (i) Let $b(\lambda, t) = b_1(\lambda)b_2(t)$ and $f(b, s) = f_1(b)f_2(s)$, where $b_2 = A$ and $f_2 = B$ are assumed to be constants. Assume further that $b_1(\lambda) \in L_{\sigma,1}(I)$ and $f_1(b) \in L_{\sigma,1}$. Then

$$\|A(x, D)B(x, D)\phi(x)\|_{G_{\alpha,\beta,1}^0} \leq \|b_1\|_{L_{\sigma,1}} \|f\|_{L_{\sigma,1}} \|\phi\|_{G_{\sigma,1}^0}$$

and

$$A(x, D)B(y, D)\phi(x) = AB (H_{\alpha,\beta} b_1) (x) (H_{\alpha,\beta} f_1) (x)\phi(x).$$

Proof 6 By definition (4.1) and (2.1) we have

$$\begin{aligned} A(x, D)B(x, D) &= \int_0^\infty (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}(xt) \int_0^\infty (x\lambda)^{-(\alpha-\beta)} J_{\alpha-\beta}(x\lambda) b_1(\lambda) b_2(t) \lambda^{4\alpha} d\lambda \\ &\quad \times (H_{\alpha,\beta} B(y, D)\phi(y)) (t) t^{4\alpha} dt \end{aligned}$$

Now using relation (1.11), we obtain

$$\begin{aligned} A(x, D)B(x, D) &= \int_0^\infty \int_0^\infty b_1(\lambda) b_2(t) (H_{\alpha,\beta} B(y, D)\phi(y)) (t) (\lambda t)^{4\alpha} d\lambda dt \\ &\quad \times \frac{1}{[2^{\alpha-\beta} \Gamma(3\alpha + \beta)]^2} \int_0^\infty (xz)^{-(\alpha-\beta)} J_{\alpha-\beta}(xz) \\ &\quad \times \int_0^\infty \int_0^\infty (b_2(t) (f_1 \# f_2 H_{\alpha,\beta} \phi)) (t) b_1(\lambda) D_{\alpha,\beta}(t, \lambda, z) (t\lambda)^{4\alpha} z^{4\alpha} d\lambda dt dz \\ &= \frac{1}{[2^{\alpha-\beta} \Gamma(3\alpha + \beta)]} \int_0^\infty (xz)^{-(\alpha-\beta)} J_{\alpha-\beta}(xz) \int_0^\infty \int_0^\infty b_2(f_1 \# f_2 H_{\alpha,\beta} \phi) \\ &\quad \times D_{\alpha,\beta}(t, \lambda, z) \frac{1}{[2^{\alpha-\beta} \Gamma(3\alpha + \beta)]} b_1(\lambda) \lambda^{4\alpha} t^{4\alpha} z^{4\alpha} d\lambda dt dz \\ &= \int_0^\infty (xz)^{-(\alpha-\beta)} J_{\alpha-\beta}(xz) \int_0^\infty \tau_z (b_2(f_1 \# f_2 H_{\alpha,\beta} \phi)) (\lambda) b_1(\lambda) \\ &\quad \times \frac{1}{[2^{\alpha-\beta} \Gamma(3\alpha + \beta)]} \lambda^{4\alpha} d\lambda z^{4\alpha} dz \\ &= \int_0^\infty (xz)^{-(\alpha-\beta)} J_{\alpha-\beta}(xz) [b_1 \# b_2 (f_1 \# f_2 H_{\alpha,\beta} \phi)] (z) z^{4\alpha} dz \end{aligned}$$

Now an application of the inverse Hankel type transform yields,

$$\int_0^\infty (xz)^{-(\alpha-\beta)} J_{\alpha-\beta}(xz) A(x, D) B(x, D) \phi(x) x^{4\alpha} dx = [b_1 \# b_2 (f_1 \# f_2 H_{\alpha, \beta} \phi)](z)$$

Therefore

$$[H_{\alpha, \beta} A(x, D) B(x, D) \phi(x)](z) = [b_1 \# A(f_1 \# B H_{\alpha, \beta} \phi)](z)$$

Thus

$$\int_0^\infty |H_{\alpha, \beta} A(x, D) B(x, D) \phi(x)| (z) d\sigma(z) = AB \int_0^\infty [|b_1| \# |f_1| \# |H_{\alpha, \beta} \phi|] (z) d\sigma(z)$$

Hence,

$$\begin{aligned} \|H_{\alpha, \beta} A(x, D) B(x, D) \phi(x)\|_{L_{\sigma, 1}} &= AB \| |b_1| \# (|f_1| \# |H_{\alpha, \beta} \phi|) \|_{L_{\sigma, 1}} \\ &\leq AB \| |b_1| \|_{L_{\sigma, 1}} \| (|f_1| \# |H_{\alpha, \beta} \phi|) \|_{L_{\sigma, 1}} \end{aligned}$$

Now we can use definition (1.1), equations (1.7) and (1.15) to obtain

$$\|A(x, D) B(x, D) \phi(x)\|_{G_{\alpha, \beta, 1}^0} \leq AB \| |b_1| \|_{L_{\sigma, 1}} \| |f_1| \|_{L_{\sigma, 1}} \| \phi \|_{G_{\alpha, \beta, 1}^0}$$

Finally from we get

$$\begin{aligned} A(x, D) B(x, D) \phi(x) &= H_{\alpha, \beta} [b_1 \# b_2 (f_1 \# f_2 H_{\alpha, \beta} \phi)](x) \\ &= (H_{\alpha, \beta} b_1)(x) H_{\alpha, \beta} b_2 (f_1 \# f_2 H_{\alpha, \beta} \phi)(x) \\ &= (H_{\alpha, \beta} b_1)(x) H_{\alpha, \beta} (b_2 f_1)(x) H_{\alpha, \beta} (f_2 H_{\alpha, \beta} \phi)(x) \\ &= AB (H_{\alpha, \beta} b_1)(x) (H_{\alpha, \beta} f_1)(x) H_{\alpha, \beta} (H_{\alpha, \beta} \phi)(x) \\ &= AB (H_{\alpha, \beta} b_1)(x) (H_{\alpha, \beta} f_1)(x) \phi(x). \end{aligned}$$

This completes the proof.

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