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# EFFECT OF THERMAL BOUNDARY CONDITIONS ON BÉNARD-MARANGONI CONVECTION WITH TEMPERATURE DEPENDENT VISCOSITY AND DENSITY

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**Abstract**

The effect of thermal boundary conditions on Bénard-Marangoni convection in a horizontal fluid layer with temperature dependent viscosity and density is investigated. The lower boundary is considered to be rigid which is either at fixed temperature or at fixed heat flux condition, while the temperature dependence of the surface tension at the upper free boundary and subject to a general thermal boundary condition. The problem is solved numerically using the Galerkin technique. It is found that the critical thermal Rayleigh number or critical Marangoni number making the onset of Bénard-Marangoni convection is greatest for fixed temperature and least for fixed heat flux condition. The effect of increasing in Biot number, temperature dependent viscosity and density is to delay the onset of convection, while increasing in the Marangoni number due to surface tension effect at the free boundary is to hasten the convection. Some known results are recovered as special cases.

**Keywords:** temperature dependent viscosity, temperature dependent density, Bénard- Marangoni, fixed heat flux, heat transfer coefficient, Galerkin technique

## 1. Introduction

The convective instability and the spontaneous formation of cell patterns in horizontal layer fluid are important problems in physiochemical hydrodynamics, heat and mass transport, and the theory of self-organization in open systems. Chandrasekhar [6] initiated the study of the Rayleigh-Bénard convective instability induced by buoyancy in a magnetic field. Another effect is due to local variation in surface tension force. This type of convective instability is referred as Marangoni instability and was first theoretically analyzed by Pearson [19]. The instability of Bénard–Marangoni convection is due to the combined effects of thermal buoyancy force and surface tension force. As these two kinds of instability take place at the same time, the instability mechanism is known as the Bénard–Marangoni instability. Nield [17] first analyzed the Bénard–Marangoni convective instability in a horizontal fluid layer with a non-deformable free surface. Rudraiah [22] has studied the linear stability analysis on the onset of Marangoni convection in a thin horizontal fluid layer subjected to rotation about a vertical axis. Microgravity science applications of thermocapillary flows are motivated the study of surface tension induced instability known as the Marangoni-Bénard problem in both one-g and microgravity environments (Legros et al. [14]. Rabin [20] has studied the threshold values for the onset of Bénard-Marangoni convection in a fluid layer heated from below using linear perturbation techniques. Celli et al. [5] have studied the effect of viscous dissipation on Marangoni instability of thin liquid film flowing over a plate by considering the impermeable and adiabatic boundary conditions.

In this study, the viscosity effect plays an significant role. In many fluids with large Prandtl number and in particular, in some oils, the fluids posses a temperature dependent viscosity as viscosity is more sensitive to temperature variations than heat capacity and thermal conductivity and the effects are important on the onset of convection in viscous fluid layer. Palm [18] investigated theoretically the cells in steady state convection approach hexagonal form, and the occurrence of ascent or descent in the middle of cell depends on how the kinematic viscosity varies with temperature and starting from his study, effects of variable viscosity is discussed in some of the problems (Torrance [25]; Booker [2]; Booker [3]; Stengel [24]). The studies almost exclusively consider the buoyancy induced instability problem. Lebon and Cloot [12] have investigated the effect of variable viscosity on Marangoni instability in a horizontal fluid layer under microgravity effect. Char and Chen [7] have studied the onset of stationary Bénard-Marangoni convective instabilities in a fluid layer with thermally dependent surface tension and viscosity. Chiang [8] have investigated the effect of a non-uniform basic temperature gradient on the stationary and oscillatory stability analysis of the onset of Bénard-Marangoni convection in a horizontal fluid layer.

Kechil and Hashim [1] have investigated the effect of control strategy on Marangoni instability in a temperature-dependent-viscosity fluid layer with deformable free surface.

This study is also concerned with fluid having a linear relationship between density and temperature, which is standard for ordinary fluids at temperatures not too low. It is known that clean water has a density maximum which can be approximately taken as $4^0$C at the atmospheric pressure. The density–temperature dependence for temperatures from $0^0$C to $10^0$C can be approximated by the quadratic function with the maximum at $4^0$C. Convection in the water layer with the quadratic density–temperature dependence was mathematically formulated in Veronis [26] for the first time. Legros et al. [13] have studied experimentally the thermal stability of horizontal layer heated from below with upper boundary at temperatures between $0^0$C and $20^0$C. Their results are good agreement with those obtained by Veronis. Merker et al. [15] studied the linear stability analysis on the onset of convection in a water layer near temperature where the density anomaly. Merker and Straub [16] have also carried out experimental studies on maximum density effects on the heat transfer through a horizontal water layer cooled from below with 0°C. Kuznetsova and Sibgatullin [10] have studied the effect of quadratic temperature dependent density in a horizontal layer of fluid. Large and Andereck [11] have studied the onset conditions of convective flows due to imposed temperature differences in the presence of a density maximum in water near 4 °C.

Thus the purpose of this paper is to study analytically the combined effects of temperature dependent viscosity and density on Bénard-Marangoni convection with thermal boundary conditions. In particular, consideration is given to the case of arbitrary heat transfer coefficient at upper free boundaries corresponding to either fixed temperature or to fixed heat flux condition at a lower rigid boundary. The critical stability parameters are obtained numerically for the thermal Rayleigh and Marangoni numbers. The effects of various physical parameters on the stability of the system are analyzed.

The paper is organized as follows. The mathematical formulation of the problem is described in section 2. The basic state equations are derived in section 3. The boundary conditions are also specified in this section. The linear stability analysis that employs the normal mode analysis procedure for the perturbations is handled in this section. The eigenvalue problem involving the system of linear ordinary differential equations is handled numerically using a eight-order Galerkin weighted residual method in the same section. The numerical results so obtained are discussed and explained in detail in section 4. Finally, the possible conclusions from the numerical study are documented in section 5.

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## 2. Mathematical Formulation

We consider an infinite, horizontal layer of viscous fluid of uniform thickness *d* (see Fig.1). A Cartesian coordinate system (*x*, *y*, *z*) is chosen with the origin at the bottom of the layer. The z direction is opposite to the gravitational acceleration. The combined system, which is heated from below, is infinite in the horizontal direction *x*. The lower and upper boundaries of the layer are maintained at constant temperatures $T_o$ and $T_1 = T_0 - \Delta T$ $(\Delta T > 0)$ respectively, The upper free boundary is considered to be flat and subjected to a surface tension $\sigma$ which varies linearly with temperature as $\sigma = \sigma_0 - \sigma_T (T - T_0)$, where $\sigma_0$ is the unperturbed value and $\sigma_T$ is the rate of change of surface tension with temperature. The fluid viscosity is assumed to vary as linear function of temperature:

$$\mu = \mu_0 [1 - \eta (T - T_0)] \tag{1}$$

where $\mu_0$ and $\eta$ are positive constants.

The continuity equation for an incompressible Boussinesq fluid is

$$\nabla \cdot \vec{q} = 0. \tag{2}$$

where, $\vec{q} = (u, v, w)$ is the velocity vector.

The momentum equation for an incompressible fluid with viscous force $2\nabla \cdot [\eta \ \underline{\underline{D}}]$ is

$$\rho_0 \left[ \frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} \right] = -\nabla p + \rho \vec{g} + +2\nabla \cdot [\mu \ \underline{\underline{D}}], \tag{3}$$

where $\rho_0$ is the density at reference temperature $T = T_o$, $\vec{g} = (0, 0, -g)$ is the gravitational acceleration, *t* is the time, *T* is the temperature, *p* is the pressure and $\underline{\underline{D}} = [\nabla \vec{q} + (\nabla \vec{q})^T]/2$ is the rate of strain tensor.

The energy equation for an incompressible fluid which obeys Fourier's law is

$$\frac{\partial T}{\partial t} + (\vec{q} \cdot \nabla) T = \kappa \nabla^2 T, \tag{4}$$

where $\kappa$ is the effective thermal diffusivity and $\nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 + \partial^2 / \partial z^2$ is the Laplacian operator. The thermal expansion coefficient variation included by assuming the density to be quadratic in temperature as follows

$$\rho = \rho_0 [1 - \alpha_1 (T - T_0) - \alpha_2 (T - T_0)^2], \tag{5}$$


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where $T_0 = 4^0 C$, $\rho_0$ is the density at $4^0 C$, $\alpha_1 \approx 64.817 \times 10^{-6}$ and $\alpha_2 \approx 7.798 \times 10^{-6}$ are the first and second thermal expansion coefficients, respectively, and the subscript '0'denote the reference state.

**3. Basic state and linear stability analysis**

The basic state is assumed to be quiescent and is described by

$$[\vec{q},\ \rho,\ T,\ p,\ \mu] = [0,\ \rho_b(z),\ T_b(z),\ p_b(z),\ \mu_b(z)], \tag{6}$$

where the subscript *b* denotes the basic state.

Substituting Eq.(6) in Eqs.(2)-(4), we get

$$\frac{\partial p_b}{\partial z} + \rho_0 \left[1 + \alpha_1 \beta z - \alpha_2 \beta^2 z^2\right] g = 0, \tag{7}$$

$$T_b(z) = T_0 - \beta d, \tag{8}$$

where $\beta = \Delta T / d$ is the temperature gradient.

To make a linear stability analysis of the flow system, we now perturb the basic state with infinitesimal disturbances as follows:

$$[\vec{q},\ p,\ T,\ \rho,\ \mu] = [0,\ p_b,\ T_b,\ \rho_b,\ \mu_b] + [\vec{q}',\ p',\ T',\ \rho',\ \mu'], \tag{9}$$

where the primed quantities are infinitesimal disturbances.

Substituting Eq.(9) in Eqs.(2)-(4) and using the basic state Eqs. (7) and (8), linearizing the resulting equations by neglecting the products of primed quantities, eliminating the pressure from the momentum equation by operating curl twice and retaining the vertical component, we get the governing equations for the perturbations (after neglecting the primes for simplicity)

$$\rho_0 \frac{\partial}{\partial t}\left(\nabla^2 w\right) - \mu(z)\nabla^4 w = -\frac{\mu_0 \eta \beta}{\kappa}\frac{\partial w}{\partial z} + 2\mu_0 \eta \beta \frac{\partial}{\partial z}\left(\nabla^2 w\right) + \rho_0 g\left(\alpha_1 - 2\alpha_2 \beta z\right)\nabla_h^2 T \tag{10}$$

$$\left(\frac{\partial}{\partial t} - \kappa \nabla^2\right) T = \frac{\Delta T}{d} w, \tag{11}$$

where $\mu(z) = \mu_0\left(1 + \eta \beta z\right)$ and $\nabla_h^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two-dimensional Laplacian operator.

The perturbation quantities in terms of normal modes are

$$\{w, T\} = \{W(z), \Theta(z)\} \exp\left[i\left(\ell x + my + \omega t\right)\right], \tag{12}$$

where $\ell$ and *m* are wave numbers in the *x* and *y* directions, respectively, while $W(z)$ and $\Theta(z)$ are the amplitudes of z-component of perturbed velocity and perturbed temperature respectively, and $\omega$ is the


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growth rate. We apply this expression in the linearized version of the Eqs. (10) and (11) and non-dimensionalizing the equations using the following definition:

$$(x^*, y^*, z^*) = \left(\frac{x}{d}, \frac{y}{d}, \frac{z}{d}\right), \quad W^* = \frac{W}{(\kappa/d)}, \quad t^* = \frac{t}{(d^2/\kappa)}, \quad \Theta^* = \frac{\Theta}{\Delta T} \quad \omega^* = \frac{\omega}{(\kappa/d^2)} \tag{13}$$

we obtain (after ignoring the asterisks)

$$\left[(1+\eta z)(D^2-a^2) - \frac{\omega}{\Pr}\right](D^2-a^2)W + 2\eta\, D(D^2-a^2)W - a^2 R_t\,\Theta + 2R_t a^2 \delta z \Theta = 0 \tag{14}$$

$$[(D^2-a^2) - \omega Pr]\theta + W = 0 \tag{15}$$

where $D \equiv d/dz$ is the differential operator, $a = \sqrt{\ell^2 + m^2}$ is the horizontal wave number, $Pr = \frac{\nu}{\kappa}$ is the Prandtl number, $R_t = \alpha_1 g \Delta T d^3 / \nu\kappa$ is the thermal Rayleigh number and $\delta = \alpha_2 \Delta T / \alpha_1$ is the variable thermal expansion co-efficient. Since the principle of exchange of stability is valid (Bragard and Velarde 1998) and rendering the Eqs. (14) and (15) yield

$$(1+\eta z)(D^2-a^2)^2 W + 2\eta\, D(D^2-a^2)W - a^2 R_t\,\Theta + 2R_t a^2 \delta z \Theta = 0 \tag{16}$$

$$(D^2-a^2)\Theta + W = 0 \tag{17}$$

Equations (16) and (17) form an eigenvalue problem for thermal Rayleigh number ($R_t$) and is to be solved using the following boundary combinations:

At $z = 0$, the lower boundary is rigid and either perfectly conducting (ie., fixed temperature) or perfectly insulating (i.e., fixed heat flux):

$$W = DW = \Theta = 0 \tag{18a}$$

or

$$W = DW = D\Theta = 0 \tag{18b}$$

At $z = 1$, the upper free boundary is non-deformable with surface tension and the general thermal boundary condition ;

$$W = (1+\eta)D^2 W + a^2 Ma\,\Theta = D\Theta + Bi\,\Theta = 0. \tag{18c}$$

where, $Ma = \sigma_T \Delta T d / \mu\kappa$ is the Marangoni number. We remark that the Marangoni instability is a capillarity effect and it arises from the variation of the surface tension $\sigma_T$ at the upper surface with temperature. In addition, the Biot number, $Bi = h d / k$, arises from the Newton's heat transfer law due to


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cooling at the upper boundary. Note that $Bi$ for a perfectly heat conducting boundary tends to infinity, and for perfectly insulated boundary tends to zero.

## 4. Method of solution

Equations (16) and (17) together with boundary conditions (18a,b,c) constitute an eigenvalue problem with $R_t$ or $Ma$ as an eigenvalue. To solve the resulting eigenvalue problem, using the eight-order Galerkin method. The dependent variables are written as a series of linearly independent functions (basis functions) :

$$W = \sum_{i=1}^{8} A_i W_i \,, \quad \Theta = \sum_{i=1}^{8} C_i \Theta_i \,, \tag{19}$$

where $A_i$ and $C_i$ are constants and the basis functions $W_i$ and $\Theta_i$ will be represented by the power series satisfying the respective boundary conditions. Substituting Eq.(19) into Eqs. (16) and (17), multiplying the resulting momentum Eq.(16) by $W_j(z)$ and energy Eq.(17) by $\Theta_j(z)$; performing the integration by parts with respect to z between z = 0 and z = 1 and using the boundary conditions given by (18a,b,c), we obtain the following system of linear homogeneous algebraic equations:

$$\begin{bmatrix} E_{ji} & F_{ji} \\ G_{ji} & H_{ji} \end{bmatrix} \begin{bmatrix} A_i \\ C_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{20}$$

where

$$E_{ji} = <(1+\eta z)[D^2W_j D^2W_i + 2a^2 DW_j DW_i + a^4 W_j W_i]>,$$

$$F_{ji} = a^2\, Ma\, DW_j(1)\Theta_i(1) - a^2 R_t < W_j\Theta_i > + 2a^2 R_t \delta < z W_j \Theta_i >,$$

$$G_{ji} = < \Theta_j W_i >,$$

$$H_{ji} = - < D\Theta_j D\Theta_i + a^2 \Theta_j \Theta_i > - Bi\, \Theta_j(1)\Theta_i(1).$$

In the above equations, the inner product is defined as $< fg > = \int_0^1 fg dz$ .

The above set of homogeneous algebraic equations has a non-trivial solution if and only if

$$\begin{vmatrix} E_{ji} & F_{ji} \\ G_{ji} & H_{ji} \end{vmatrix} = 0. \tag{21}$$

The eigenvalue has to be extracted from the characteristic Eq.(20). We select the trial functions as


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$$W_i = (2z^{i+3} - 5z^{i+2} + 3z^{i+1})T_i^* \ , \quad \Theta_i = (z^i - z^{i+1}/2)T_i^* \tag{22}$$

where $T_i^*$'s are the modified Chebyshev polynomials ($i = 0, 1, 2, ......$) of the second kind. It seen that the velocity trial function satisfy but the temperature trial function do not satisfy the their respective boundary condition, namely $(1+\eta)D^2W + Ma\,a^2\,\Theta = 0 = D\Theta + Bi\,\Theta$ at $z = 1$. However the residual from this condition is included as residual from the differential equations. Equation (20) leads to characteristic equation from which the critical Marangoni number, $Ma_c$, as a function of wave number $a$ is extracted numerically for various values of physical parameters $R_t$, $Bi$, $\eta$ and $\delta$.

## 5. Results and Discussion

The classical linear stability analysis has been carried out to investigate the effects of variable viscosity and thermal expansion coefficient on the onset of convection in a horizontal fluid layer. Attention is focused on two types of thermal boundary conditions keeping in mind the laboratory and engineering problems. The lower boundary is assumed to be rigid at fixed temperature or at fixed heat flux, while the upper boundary is stress-free and subjected to a general thermal boundary condition with temperature-dependent surface tension. The critical physical stability parameters are computed numerically by Galerkin technique. The critical thermal Rayleigh number ($R_{tc}$) or critical Marangoni number ($Ma_c$) and the corresponding critical wave number ($a_c$) for various physical parameters ($Bi$, $\eta$ and $\delta$) are used to characterize the stability of the system.

To validate the numerical procedure used and to make a comparison with some existing results, we have applied the method to the problems studied by Sparrow et al. (1964) under limiting conditions. Our results for values of the critical thermal Rayleigh number, $R_{tc}$, and the corresponding critical wave number, $a_c$, compare very well with those of Sparrow et al. (1964) are plotted in Figs 2 and 3, and are also listed in Table 1 for different values of $Bi$ in the absence of surface tension force ($Ma = 0$) with fixed values of $\eta = 0$ and $\delta = 0$. Besides for fixed temperature of lower rigid boundary and general thermal boundary condition of upper free boundary to assure the validity of solution procedure, the critical results $[Ma_c, a_c]$ are compared with those of Nield (1965) for selected values of $R_t$ and $Bi$ in the absence of temperature dependent viscosity ($\eta = 0$) and thermal expansion coefficient ($\delta = 0$).The comparison is listed in Table 2 and found in good agreement.

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For the classical Marangoni convection ( $R_t = 0$ ) with fixed heat flux of upper ( $Bi = 0$ ) and lower boundary, it is found that the critical Marangoni number $Ma_c = 48$ and the corresponding critical wave number $a_c = 0$ . These rapidly converge to the well known critical values (Pearson, (1958)). Also, it is instructive to know the process of convergence of result as the number of terms in the Galerkin approximation increases for the problem considered. Hence, the various levels of approximation to $Ma_c$ and the corresponding $a_c$ are also obtain for different values of $Bi$ for fixed values of $\delta = 0.2$, $\eta = 0.5$, $R_t = 100$ and results are tabulated in Table 3 for lower boundary at constant temperature and Table 4 for lower boundary at fixed heat flux conditions. It is seen that with an increase in the number of terms in Galerkin approximations, $Ma_c$ goes on increasing and finally for the order $i = j = 8$ the results converge. This clearly shows the accuracy of the numerical procedure employed in solving the problem.

In Figs.4-9, there are three solid curves corresponding to lower boundary at constant temperature and three dotted curves corresponding to lower boundary at fixed heat flux condition for different thermal boundary conditions. Figure 4 shows the locus of critical thermal Rayleigh number $R_{tc}$ as a function of critical Marangoni number $Ma_c$ by considering different values of heat transfer coefficient *Bi* (i.e., Biot number) with other parameters are fixed ( $\eta = 0.2$ and $\delta = 0.2$ ). In Fig.3 reveals that the fixing of the surface temperature should provide a stronger constraint against perturbation of the temperature profile than does the fixing of the temperature gradient at the surface. Besides, the observed effect of *Bi* seen in Fig. 3 may be attributed to the fact that with increasing *Bi*, the thermal disturbances can easily dissipate into the ambient surrounding due to a better convective heat transfer coefficient at the upper surface and hence higher heating is required to make the system unstable. The variation of critical wave number $a_c$ as a function of $Bi$ is shown in Fig. 5 for two types of thermal boundary conditions with different values of $R_t$ . From the figure it is seen that the critical wave number $a_c$ increases as the values of Biot number $Bi$ increase and hence its effect is to contract the convection cell size. On the contrary, the presence of buoyancy force ( $R_t$ ) is to decrease the critical wave number ( $a_c$ ).

Figure 6 represents the variation of $R_{tc}$ as a function of $Ma_c$ for small to moderate different values of temperature dependent viscosity parameter $\eta$ with two types of thermal boundary conditions when $Bi = 2$ and $\delta = 0.2$ . It can be observed that an increasing $\eta$, the critical stability parameters ( $R_{tc}$


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and $Ma_c$) increases and thus it has a stabilizing effect on the system. That is, the effect of increasing $\eta$ is to delay the onset of Bénard-Marangoni convection. This is the good agreement of the result found by Stengel (1982) and White (1988). In Fig. 7 an increase in the value of $\eta$ is to increase $a_c$ and hence its effect is to decrease the size of convection cells.

Figure 8 illustrates the variation of $Ma_c$ verses temperature dependent density parameter $\delta$ for different values of $R_t$ when $\eta = 0.2$ and $Bi = 2$. It is noted that for a lower rigid at constant temperature/ fixed heat flux conditions of $Ma_c$ increases, the parameter $\delta$ as increases. Thus it has a stabilizing effect on the system. It is important to note that for the case of fixed heat flux condition at lower rigid, the critical Marangoni number $Ma_c$ decreases with increase of $R_t$ for $\delta(< 0.5)$ is to destabilizing the system whereas for $\delta(> 0.5)$ this trend reverses**.** Nonetheless, in the absence of buoyancy force ($R_t = 0$), the critical value $Ma_c = 170.763$ for fixed temperature of lower boundary and $Ma_c = 147.458$ for fixed heat flux condition of lower boundary with fixed values of $\eta = 0.2$, $Bi = 2$. Thus it has no effect on Marangoni convection on the stability system. The critical wave number $a_c$ verses Biot number $Bi$ is plotted in Fig. 9 shows that the value of $a_c$ increases as thermal expansion coefficient $\delta$ increases and thus its effect is to diminish the dimension of convection cells in the case of lower boundary at fixed heat flux. However at constant temperature of lower surface reveals that the variation in $a_c$ with $\delta$ is insignificant on the system.

To know the simultaneous presence of buoyancy and surface tension forces on the stability of the system, the locus of $Ma_c$ and $R_{tc}$ is exhibited in Figs.4 and 6 for two types of thermal boundary conditions. From the figures, it is observed that there is a strong coupling between the critical thermal Rayleigh number $R_{tc}$ and the critical Marangoni number $Ma_c$. If there is increase in the value of one, then there is a decrease in the value of other. On seeing the result of Figs. 4 and 6 it is obvious that increasing the strengths of surface tension force leads to destabilization of the Rayleigh-Bénard configuration. This result is true for both thermal boundary conditions considered.

The variation of critical wave number $a_c$ on Bénard-Marangoni convection under variable viscosity and thermal expansion coefficient effects for different values of the physical parameters $Bi$, $\eta$

and $\delta$, respectively, in Figs. 5, 7 and 9 essentially reiterate the observation made in the context of classical Rayleigh-Bénard convection in horizontal fluid layer. The general results is true for $(a_c)_{\text{fixed temperature of lower boundary}} > (a_c)_{\text{fixed heat flux of lower boundary}}$.

## 5 . Conclusions

The effect of variable viscosity and thermal expansion coefficient on the onset of convection in viscous fluid layer heated uniformly from below is investigated for two types of thermal boundary conditions. The resulting eigenvalue problem is solved using the Galerkin method. Based on the present investigations the following conclusions are made:

(i) The heat transfer coefficient $Bi$ of the free upper surface is to enhance to the onset convection.

(ii) The critical Rayleigh number and the critical Marangoni number increase as the temperature dependent viscosity parameter $\eta$ and density parameter $\delta$ increases and thus effect is to delay the onset of convection.

(iii) The critical Marangoni number $Ma_c$, decreases with an increase in thermal Rayleigh number $R_t$ i.e., when buoyancy is predominant and surface tension is negligible.

(iv) The system is more stable for "constant temperature" compared to "fixed heat flux" conditions. It is noted that $(R_{tc} \text{ and } Ma_c)_{\text{constant temperature}} > (R_{tc} \text{ and } Ma_c)_{\text{fixed } heat\ flux}$.

(v) The critical wave number increases with an increase in $\eta$, $\delta$, $Bi$ and decrease in $R_t$. Thus their effect is to contract the size of convection cells.

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**Table 1:** Comparison of $R_{tc}$ and $a_c$ for different thermal boundary conditions with various values of $Bi$ when $\eta = 0$ and $\delta = 0$

| $B_i$ | lower boundary at fixed temperature | | | | lower boundary at fixed heat flux | | | |
|---|---|---|---|---|---|---|---|---|
| | Sparrow et al (1964) | | Present study | | Sparrow et al (1964) | | Present study | |
| | $R_{tc}$ | $a_c$ | $R_{tc}$ | $a_c$ | $R_{tc}$ | $a_c$ | $R_{tc}$ | $a_c$ |
| **0** | 669.001 | 2.09 | 668.98 | 2.086 | 320.00 | 0 | 320 | 0.648 |
| **0.1** | 682.361 | 2.115 | 682.36 | 2.116 | 381.665 | 1.015 | 381.665 | 1.015 |
| **0.3** | 706.365 | 2.17 | 706.39 | 2.169 | 428.29 | 1.3 | 428.29 | 1.299 |
| **1** | 770.569 | 2.3 | 770.57 | 2.293 | 513.792 | 1.64 | 513.79 | 1.644 |
| **3** | 872.506 | 2.46 | 872.53 | 2.452 | 619.666 | 1.92 | 619.666 | 1.921 |
| **30** | 1055.345 | 2.65 | 1055.5 | 2.648 | 780.240 | 2.18 | 780.237 | 2.176 |
| **100** | 1085.893 | 2.67 | 1085.9 | 2.672 | 804.973 | 2.2 | 804.972 | 2.203 |
| $\infty$ | 1100.657 | 2.68 | 1100.5 | 2.682 | 816.748 | 2.21 | 816.744 | 2.215 |

**Table 2:** Comparison of critical values of pure Marangoni number $Ma_c$ (i.e., $R_t = 0$) and pure Rayleigh number (i.e., $Ma = 0$) for the corresponding wave numbers $a_c$ for different values of $Bi$ for $\delta = 0$ and $\eta = 0$ (i.e., for viscous fluids)

| $Bi$ | Pure Marangoni convection ($R_t = 0$) | | | | Pure Benard convection ($Ma = 0$) | | | |
|---|---|---|---|---|---|---|---|---|
| | Nield (1964) | | Present study | | Nield (1964) | | Present study | |
| | $Ma_c$ | $a_c$ | $Ma_c$ | $a_c$ | $Ma_c$ | $a_c$ | $Ma_c$ | $a_c$ |
| 0 | 79.607 | 1.993 | 79.607 | 1.992 | 669.000 | 2.086 | 668.998 | 2.085 |
| $10^{-2}$ | 79.991 | 1.997 | 79.991 | 1.996 | 670.38 | 2.089 | 670.380 | 2.088 |
| $10^{-1}$ | 83.427 | 2.028 | 83.427 | 2.028 | 682.36 | 2.117 | 682.360 | 2.116 |
| 1 | 116.127 | 2.246 | 116.127 | 2.246 | 770.57 | 2.293 | 770.569 | 2.292 |
| 10 | 413.443 | 2.743 | 413.443 | 2.742 | 989.49 | 2.589 | 989.491 | 2.588 |
| $10^2$ | 3303.872 | 2.976 | 3303.872 | 2.975 | 1085.90 | 2.672 | 1085.897 | 2.671 |
| $10^3$ | 32170.5 | 3.010 | 32170.5 | 3.009 | 1099.12 | 2.681 | 1099.123 | 2.681 |
| $\infty$ | $32.0730 \times 10^{10}$ | 3.014 | $32.0730 \times 10^{10}$ | 3.014 | 1100.65 | 2.682 | 1085.897 | 2.682 |



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**Table 3:** Critical values of $Ma_c$ and $a_c$ for different values of Biot number $Bi$ when $\eta = 0.5$, $\delta = 0.2$ and $R_t = 100$: lower boundary at constant temperature (i.e., $\Theta = 0$)

| $Bi$ | $i=j=1$ | | $i=j=3$ | | $i=j=5$ | | $i=j=7$ | | $i=j=8$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $Ma_c$ | $a_c$ | $Ma_c$ | $a_c$ | $Ma_c$ | $a_c$ | $Ma_c$ | $a_c$ | $Ma_c$ | $a_c$ |
| **---0** | 89.29 | 2.017 | 95.17 | 1.957 | 95.81 | 1.954 | 95.86 | 1.954 | 95.86 | 1.954 |
| **0.01** | 89.57 | 2.021 | 96.06 | 1.960 | 96.30 | 1.958 | 96.35 | 1.957 | 96.352 | 1.957 |
| **0.1** | 92.06 | 2.048 | 100.27 | 1.991 | 100.69 | 1.988 | 100.75 | 1.987 | 100.75 | 1.987 |
| **1** | 115.28 | 2.258 | 142.22 | 2.202 | 142.54 | 2.194 | 142.63 | 2.193 | 142.63 | 2.193 |
| **10** | 303.92 | 2.962 | 518.48 | 2.688 | 524.06 | 2.652 | 524.65 | 2.649 | 524.65 | 2.649 |
| **100** | 1953.9 | 3.652 | 4170.5 | 2.925 | 4239.39 | 2.859 | 4245.41 | 2.854 | 4245.41 | 2.854 |
| **100** | 8 | 3.845 | 1 | 2.961 | 41348.9 | 2.889 | 41409.5 | 2.884 | 41409.5 | 2.884 |

**Table 4:** Critical values of $Ma_c$ and $a_c$ for different values of Biot number $Bi$ when $\eta = 0.5$, $\delta = 0.2$ and $R_t = 100$: lower boundary at fixed heat flux conditions (i.e., $D\Theta = 0$)

| $Bi$ | $i=j=1$ | | $i=j=3$ | | $i=j=5$ | | $i=j=7$ | | $i=j=8$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $Ma_c$ | $a_c$ | $Ma_c$ | $a_c$ | $Ma_c$ | $a_c$ | $Ma_c$ | $a_c$ | $Ma_c$ | $a_c$ |
| 0 | 49 | 0 | 51.10 | 0 | 51.11 | 0 | 51.12 | 0 | 51.12 | 0 |
| 0.01 | 52.81 | 0.553 | 54.55 | 0.582 | 54.84 | 0.582 | 54.88 | 0.582 | 54.88 | 0.582 |
| 0.1 | 61.66 | 0.971 | 64.81 | 1.014 | 64.82 | 1.013 | 64.82 | 1.013 | 64.82 | 1.013 |
| 1 | 95.25 | 1.657 | 113.77 | 1.680 | 113.92 | 1.676 | 113.93 | 1.675 | 113.93 | 1.675 |
| 10 | 264.53 | 2.628 | 473.83 | 2.365 | 478.42 | 2.333 | 478.66 | 2.331 | 478.66 | 2.331 |
| 100 | 1550.62 | 3.511 | 3912.5 | 2.630 | 3973.2 | 2.567 | 3976.2 | 2.563 | 3976.2 | 2.563 |
| 100 | 13020.0 | [illegible] | 7 | 2.669 | 8 | 2.599 | 3 | 2.595 | 3 | 2.595 |


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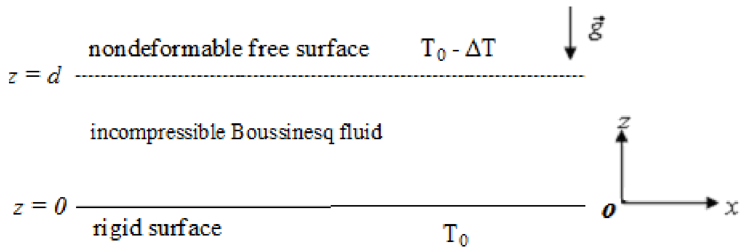


**Fig.1** Physical configuration

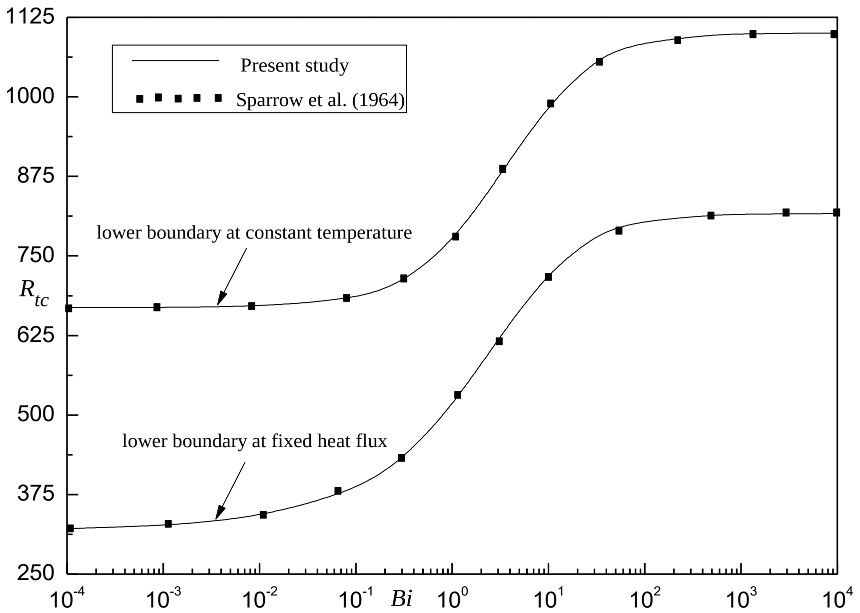


**Fig. 2** Comparison of critical thermal Rayleigh number $R_{tc}$ as a function of $Bi$ for two types of thermal boundary conditions when $\eta = 0$ and $\delta = 0$

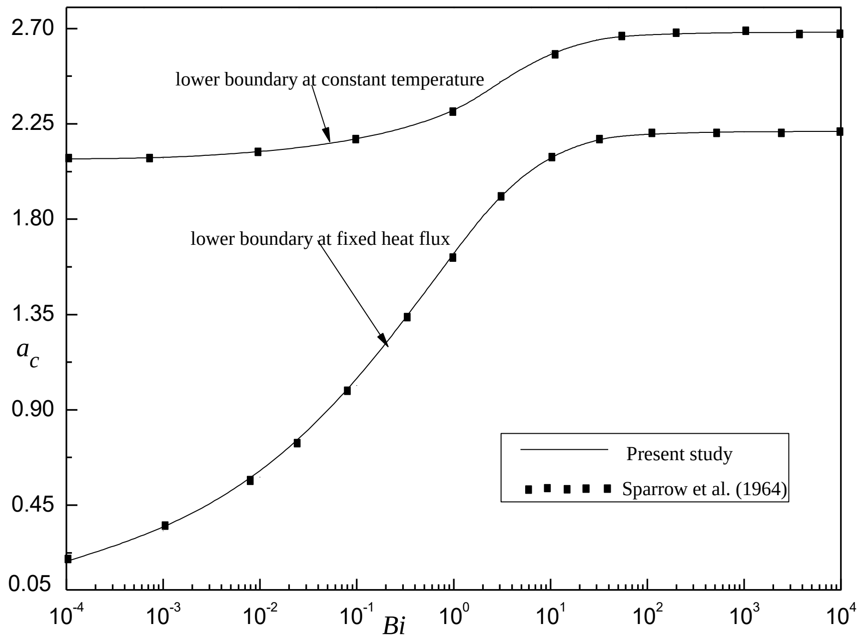


**Fig. 3** Comparison of critical wave number $a_c$ as a function of $Bi$ for two types of thermal boundary conditions when $\eta = 0$ and $\delta = 0$

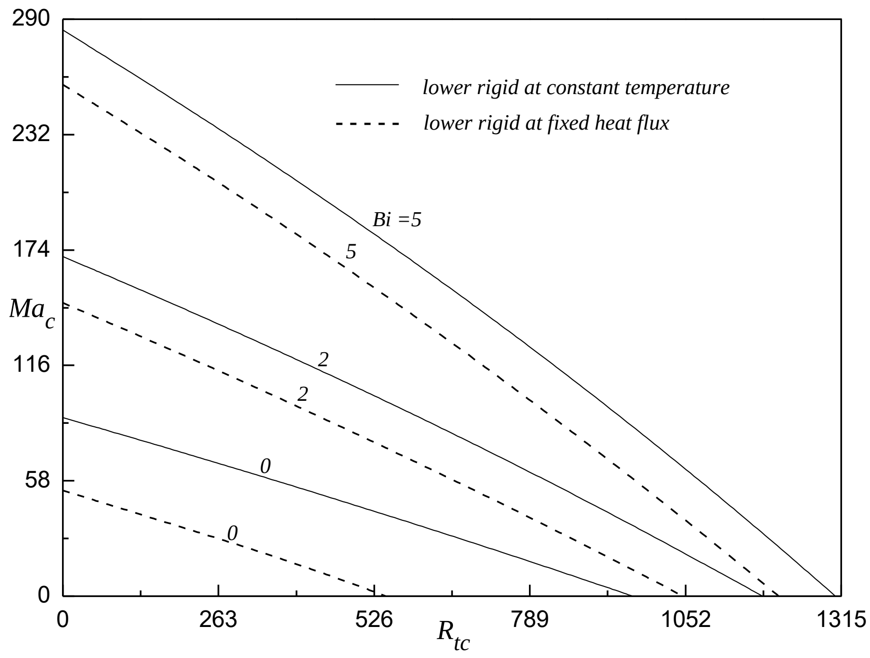


**Fig.4** Locus of critical thermal Rayleigh number $R_{tc}$ verses critical Marangoni number $Ma_c$ for different values of $Bi$ when $\eta = 0.2$ and $\delta = 0.2$


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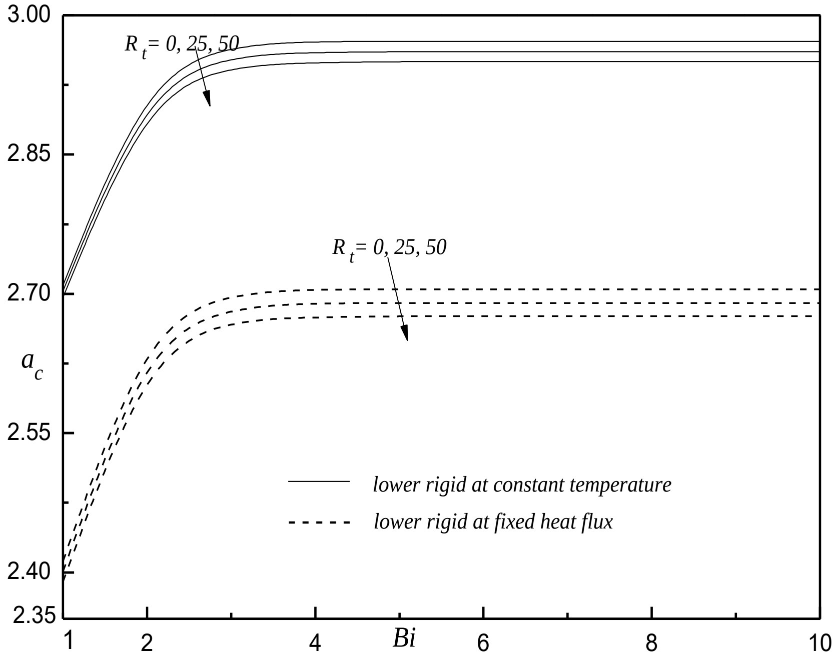


**Fig.5** Variation of critical wave number $a_c$ as a function of Biot number $Bi$ for different values of $R_t$ when $\delta = 0.2$ and $\eta = 0.2$

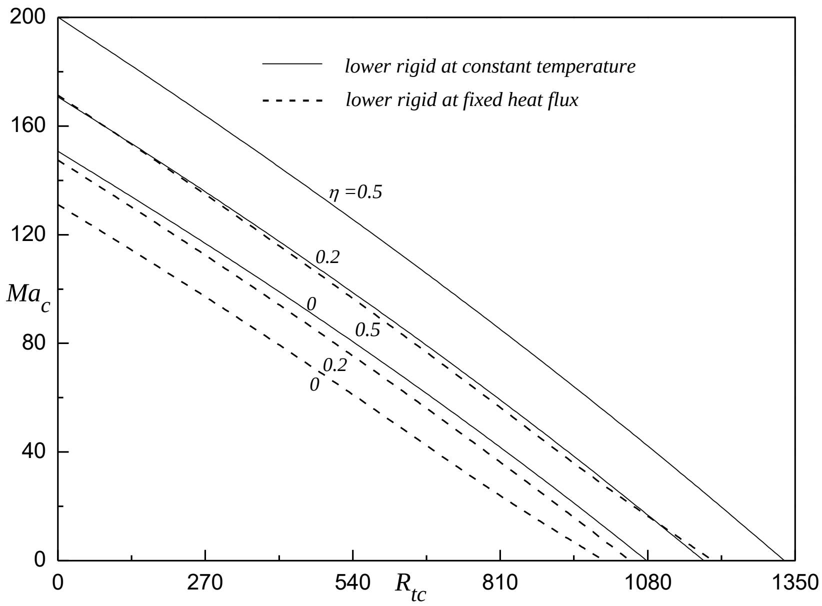


**Fig.6** Locus of critical thermal Rayleigh number $R_{tc}$ as a function of critical Marangoni number $Ma_c$ for different values of $\eta$ when $\delta = 0.2$ and $Bi = 2$

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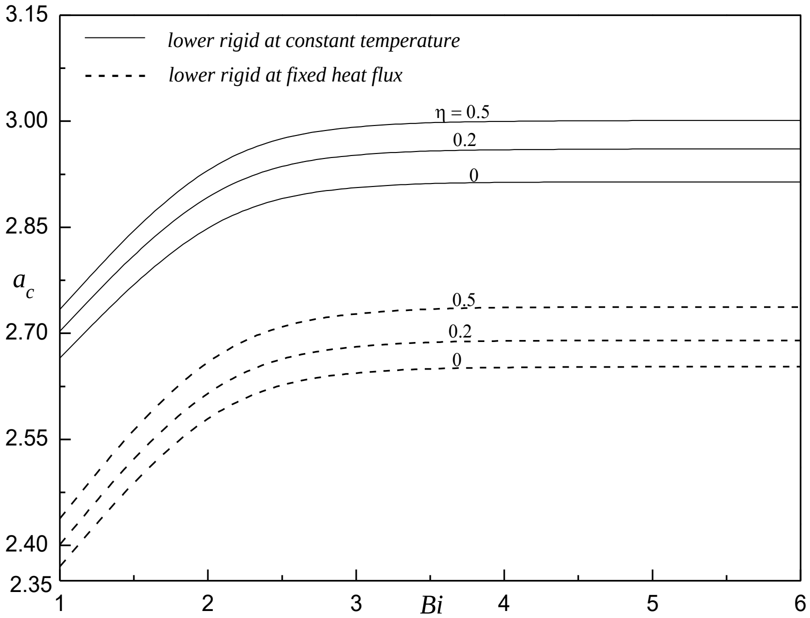


**Fig.7** Variation of critical wave number $a_c$ as a function of Biot number $Bi$ for different values of $\eta$ when $R_t = 25$ and $\delta = 0.2$

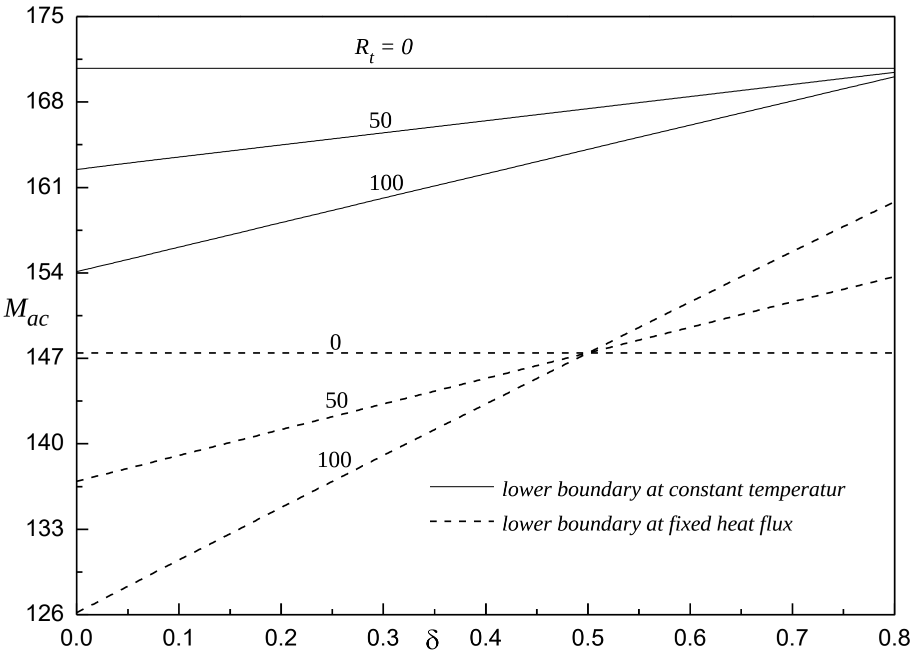


**Fig.8** Variation of critical Marangoni number $Ma_c$ as a function of thermal expansion coefficient $\delta$ when $\eta = 0.2$ and $Bi = 2$


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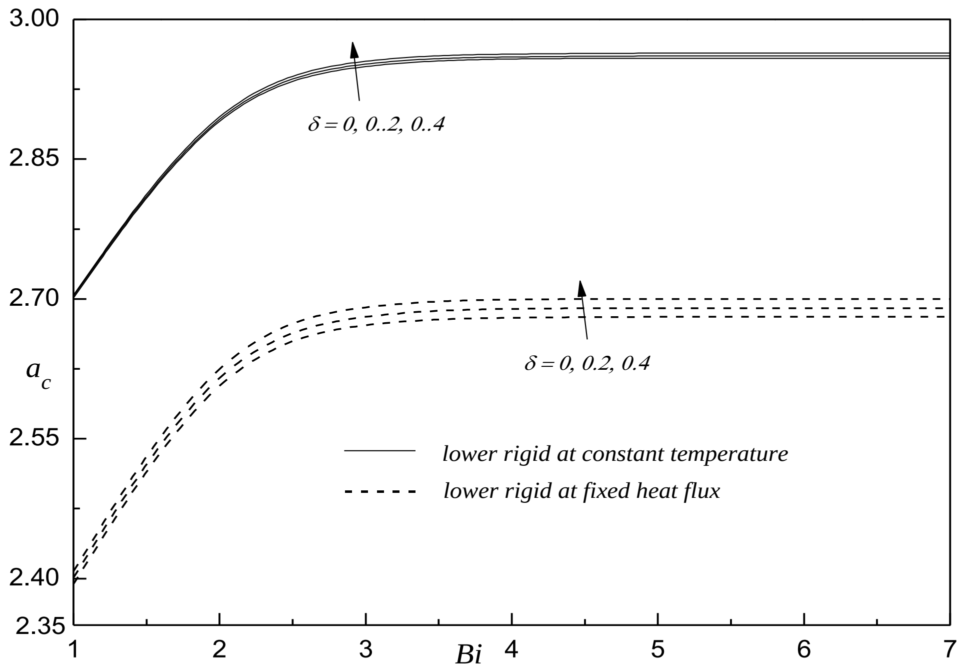


**Fig.9** Variation of critical wave number $a_c$ as a function of Biot number $Bi$ for different values of $\delta$ when $R_t = 25$ and $\eta = 0.2$


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