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ALTERNATEPROOFS FOR THE INFINITE NUMBER OF SOLUTIONS OF PELL'S EQUATION

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Abstract

In this paper we present a brief history and then derive the formula for infinite number of solutions of Pell's equation $x^2 - Dy^2 = N$, by two differenttechniques, when it is solvable. Here$D$ and N are fixed non-zero integers and$D$ is not a perfect square.

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1. Introduction :

In number theory, Pell's equation fall in the category of Diophantine equations and it is considered to be one of the oldest Diophantine equations. Named after the Greek mathematician Diophantus, Diophantine equations are equations for which integer solutions are desired. Specifically, the term *Pell's equation* is used to refer any Diophantine equation of the form

$$x^2 - Dy^2 = N; \tag{1.1}$$

where D and N are fixed non-zero integers and we wish to find integers x and y that satisfy the equation.Throughout, we consider the integer D to be positive and non-square.This condition is helpful because it leaves open the possibility of infinitely many solutions in positive integers x, y.

Pell's equations have been of interest to mathematicians for centuries. There is perhaps no other equation that has influenced the development of number theory as much as Pell's equation.Its main importance lies in the role it plays in the arithmetic of quadratic number fields. The special case of (1.1), viz.

$$x^2 - Dy^2 = 1; \tag{1.2}$$

where D is fixed positive non-square integer attracted attention of early mathematicians.

Mathematician Brahmagupta in the sixth century had methods to find solutions to (1.2), and he was able to generate infinitely many solutions from an initial solution. In Europe, Pierre de Fermat studied the equation and inspired some of his contemporaries to do the same. Leonhard Euler, after a brief reading of Wallis's *Opera Mathematica*, mistakenly credited the first serious study of nontrivial solutions of (1.2) to John Pell in a letter to Goldbach in 1730, and the name stuck (See [1, 2, 4, 5, 6]).John Pell was an English mathematician who taught at the University of Amsterdam and had a reputation as an algebraist in the middle of the 17th century. However, there is no evidence that Pell had ever considered solving such equations. They should be thus probably called Fermat – Brahmagupta equations, since Fermat and Brahmagupta were the first who investigated the properties of nontrivial solutions of many important such equations.

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Nevertheless, Pell-type equations have a long history and can be traced back at least to the ancient Greeks. Theon of Smyrna used x/y to approximate $\sqrt{2}$, which led to Pell's equation $x^2 - 2y^2 = 1$. In a letter to Eratosthenes (267 – 197 B.C.), Archimedes posed the so-called *cattle problem* in which he used the theory Pell's equation to solve the problem. We refer to Vardi [7] and Lenstra [3] for nicely written surveys on the cattle problem and its history. In *Arithmetica*, Diophantus asks for rational solutions to equation $x^2 - dy^2 = 1$ and he offered the integral solutions for the particular values of d. Pell-type equations are also found in Hindu Mathematics. In the fourth century, Baudhayana noted that $x = 577, y = 408$ is a solution of $x^2 - 2y^2 = 1$ and used the fraction $577/408$ to approximate the value of $\sqrt{2}$. In the seventh century, Brahmagupta obtained the smallest solution of Pell's equation$x^2 - 92y^2 = 1$as$x = 1151, y = 120$. In the twelfth century, Bhaskara found the smallest solution of $x^2 - 61y^2 = 1$ to be $x = 1766319049$ and$y = 226153980$.

In 1657, Fermat stated without proof that (1.2) has an infinite number of solutions, when D is a positive non-square integer. Fermat also challenged William Brouncker and John Wallis to find integral solutions to the equations $x^2 - 151y^2 = 1$ and $x^2 - 313y^2 = -1$. Wallis gave solution to first equation and Brouncker supplied the solution to second equation. In 1770 Euler devised a method, involving solutions to Pell's equations, to determine natural numbers that are both triangular numbers and perfect square. In 1766, Lagrange proved that (1.2) has an infinite number of solutions, when D is a positive non-square integer.

The Pell's equation is very important with respect to its role in quadratic number fields. For instance, an integer $\alpha = x + y\sqrt{D}$ in$\mathbb{Q}(\sqrt{D})$, where D is a square free integer with $D \equiv 3\ (mod\ 4)$, is a unit if and only if its norm is equal to 1, that is, $N(x + y\sqrt{D}) = (x - y\sqrt{D})(x + y\sqrt{D}) = x^2 - Dy^2 = 1$. Hence, solutions to the Pell's equation correspond one-to-one to units in the ring of integers $\mathbb{Q}(\sqrt{D})$.

Solving Pell's equation is of special interest to mathematicians because it may have infinitely many solutions or even no solution at all.If N is a perfect square, then (1.1) is solvable in integers for all positive, non-square integers D. If N is a quadratic nonresidue modulo D, then the Pell's equation (1.1) has no integer solution.

Let us right now assume that (1.1) is solvable and$x, y$are the integers satisfying it. Then $x + y\sqrt{D}$ is called a *solution* of (1.1). It will be called positive solution if both$x, y > 0$. We may denote this solution by (x, y).Let $u + v\sqrt{D}$ be the smallest positive solution of Pell's equation (1.2). We call $u + v\sqrt{D}$ to be the *basic solution* of (1.2).There are methods to generate other solutions from a single solution of Pell's equation. It is known that (See [1, 2, 4, 5, 6])if (u, v) is the basic solution of (1.2) then all its solutions (u_n, v_n) can be obtained by

$$u_n + v_n\sqrt{D} = \left(u + v\sqrt{D}\right)^n \text{ ; for } n = 0, 1, 2, 3, \ldots .$$

We note that the numbers defined by $(x + y\sqrt{D})(u + v\sqrt{D})^n$ for $n = 0, 1, 2, 3, \ldots$ are also positive solutions of (1.1). If $x + y\sqrt{D}$ is a positive solution of (1.1) then it is called a *fundamental solution* if the product $(x + y\sqrt{D})(u - v\sqrt{D})$ is not a positive solution of (1.1).Note that any fundamental solutionof (1.1)will never exceed the value$(x + y\sqrt{D})(u + v\sqrt{D})$.

It can also happen that the integers needed to solve $x^2 - Dy^2 = 1$ are small for a given value of D and very large for the succeeding value. A striking illustration of this variation is provided by the equation $x^2 - 61y^2 = 1$, whose fundamental solution is given by $x = 1766319049$, $y = 226153980$. These numbers are enormous when compared with the case $D = 60$, where the solution is$x = 31, y = 4$ or with $D = 62$, where the solution is$x = 63, y = 8$.The innocent-looking equation $x^2 - 991y^2 = 1$ has the following smallest positive solution:

$$x = 379516400906811930638014896080, y = 12055735790331359447442538767.$$

The situation is even worse with $x^2 - 1000099y^2 = 1$, where the smallest positive integer x satisfying this equation has 1118 digits.

With the help of the fundamental solution which can be found by means of continued fractions or by successively substituting $y = 1, 2, 3, \ldots$ into the expression $N + Dy^2$until it becomes a perfect square, we are able to construct all the remaining positive solutions. There are known algorithms which find the fundamental solution relatively efficiently. The most famous and elementary of them is as follows:

One can find the fundamental solution as a convergent in the continued fraction expansion of $\sqrt{D}$ and this is relatively fast – it depends upon the period length. Needless to say, everything depends upon the continued fraction expansion of $\sqrt{D}$.

We call two positive solutions $x_\alpha + y_\alpha\sqrt{D}$ and $x_\beta + y_\beta\sqrt{D}$to be*associated* if there exists an integer t such that

$$x_\beta + y_\beta\sqrt{D} = \left(x_\alpha + y_\alpha\sqrt{D}\right)\left(u + v\sqrt{D}\right)^t; \quad (1.3)$$

for $t = 0, \pm 1, \pm 2, \ldots$. If $x_\alpha + y_\alpha\sqrt{D}$is any fixed fundamental solution, then all thepositive solutions given in (1.3) are said to be associated with each other. The set of all solutions associated with each other forms a *class* of solutions of (1.1). And so clearly, we can consider fundamental solution to be the smallest positive solution belonging to the class. If solvable, the equation (1.1) has only finite number of classes of solutions (and so finite number of fundamental solutions). Throughout we assume that (1.1) has β classes of solutions.

Proof of the followingmain result related with the solutions of (1.1) can be found in Andreescu et el [1], Burton [2], LeVeque [4], Steuding [5], Telang [6]andmany other standard books of number theory.

Theorem: If D is a positive integer that is not a perfect square and if (1.1) is solvable then it has infinitely many solutions in nonnegative integers, and the general solution is given by $\left(x_{i,n}, y_{i,n}\right)_{n\geq 0}$, where

$$x_{i,n} + y_{i,n}\sqrt{D} = \left(x_i + y_i\sqrt{D}\right)\left(u + v\sqrt{D}\right)^n; \quad (1.4)$$

for $n = 0, 1, 2, \ldots$ and $x_i + y_i\sqrt{D}$ runs through all the fundamental solutions of (1.1).

In this paper we present the proof of above theorem by two different elementary approaches which are not available in the literature. Throughout we assume that$u + v\sqrt{D}$ is the smallest positive solution of (1.2),$x_1 + y_1\sqrt{D}$is the smallest positive solution of (1.1) and $x_\alpha + y_\alpha\sqrt{D}$ runs through all the fundamental solutions of (1.1).

2. Main Result:

Theorem: If D is a positive integer that is not a perfect square and if (1.1) is solvable then all the positive integral solution of (1.1) are given by

$$X_{\alpha,n} + Y_{\alpha,n}\sqrt{D} = \left(x_\alpha + y_\alpha\sqrt{D}\right)\left(u + v\sqrt{D}\right)^n; \quad (2.1)$$

for $n = 0, 1, 2, \ldots$.

Proof (First Method): We consider the conjugate surd of (2.1) as

$$X_{\alpha,n} - Y_{\alpha,n}\sqrt{D} = \left(x_\alpha - y_\alpha\sqrt{D}\right)\left(u - v\sqrt{D}\right)^n. \quad (2.2)$$

Multiplying relations (2.1) and (2.2) we get,

$$X_{\alpha,n}^2 - DY_{\alpha,n}^2 = (x_\alpha^2 - Dy_\alpha^2)(u^2 - Dv^2)^n = x_\alpha^2 - Dy_\alpha^2 = N.$$

Now we suppose that there exists some solution $x_{\alpha,t} + y_{\alpha,t}\sqrt{D}$ (for a fixedα) which is not covered by (2.1).But then for this solution of (1.1) and forfixed α, we have

$$x_{\alpha,t} + y_{\alpha,t}\sqrt{D} = \left(x_\alpha' + y_\alpha'\sqrt{D}\right)\left(u + v\sqrt{D}\right)^t; \quad (2.3)$$

where $x_\alpha' + y_\alpha'\sqrt{D}$ is one of the fundamental solution(of(1.1)) among all the$x_\alpha + y_\alpha\sqrt{D}$. Then we have

$$x_\alpha' + y_\alpha'\sqrt{D} > \left(x_1 + y_1\sqrt{D}\right)\left(u + v\sqrt{D}\right). \quad (2.4)$$

For, if $x_\alpha' + y_\alpha'\sqrt{D} < \left(x_1 + y_1\sqrt{D}\right)\left(u + v\sqrt{D}\right)$, then$x_\alpha' + y_\alpha'\sqrt{D}$ will be some fundamental solution (of(1.1)) amongst$x_\alpha + y_\alpha\sqrt{D}$ and so by (2.3),will be covered in(2.1).

Again by (2.3),$x_\alpha' + y_\alpha'\sqrt{D}$ is the positive solution of (1.1) and smallest in his class(for a fixed α).Then $\left(x_\alpha' + y_\alpha'\sqrt{D}\right)\left(u - v\sqrt{D}\right)$ is not a positive solution of (1.1). This gives $\left(x_\alpha' + y_\alpha'\sqrt{D}\right)\left(u - v\sqrt{D}\right) < x_1 + y_1\sqrt{D}$, which results in to the inequality

$$x_\alpha' + y_\alpha'\sqrt{D} < \left(x_1 + y_1\sqrt{D}\right)\left(u + v\sqrt{D}\right). \quad (2.5)$$

We finally arrive at a contradiction from (2.4) and (2.5),and thus all the positive solutions of (1.1) are given by (2.1). □

Wenext prove the same result by alternate method.

Proof (Second Method): We prove that all the positive integral solutions of (1.1) are given by (2.1).On the contrary assume that there exists some solution,say $X+Y\sqrt{D}$, of (1.1) which is not covered by (2.1).Then this solution, as we know,will lie between any two successive solutions(of (1.1))of some class generated by$x_\alpha+y_\alpha\sqrt{D}$ (for a fixed α). This means for some fixed α and for some fixed m, we have

$$\left(x_\alpha+y_\alpha\sqrt{D}\right)\left(u+v\sqrt{D}\right)^m \le X+Y\sqrt{D} < \left(x_\alpha+y_\alpha\sqrt{D}\right)\left(u+v\sqrt{D}\right)^{m+1}.$$

Then$x_\alpha+y_\alpha\sqrt{D} \le \left(X+Y\sqrt{D}\right)\left(u-v\sqrt{D}\right)^m < \left(x_\alpha+y_\alpha\sqrt{D}\right)\left(u+v\sqrt{D}\right)$.

We denote

$$\epsilon+\delta\sqrt{D} = \left(X+Y\sqrt{D}\right)\left(u-v\sqrt{D}\right)^m. \tag{2.6}$$

$$\therefore x_\alpha+y_\alpha\sqrt{D} \le \epsilon+\delta\sqrt{D} < \left(x_\alpha+y_\alpha\sqrt{D}\right)\left(u+v\sqrt{D}\right). \tag{2.7}$$

To prove the required result, it is sufficient to prove that

(i) $\epsilon+\delta\sqrt{D}$ is a solution of (1.1)

(ii) $\epsilon>0, \delta>0$.

and (iii) $\epsilon+\delta\sqrt{D}$ is the smallest solution of(1.1) for some fixed class α.

We first prove (i).Taking surd conjugate of (2.6) we get$\epsilon-\delta\sqrt{D} = \left(X-Y\sqrt{D}\right)\left(u+v\sqrt{D}\right)^m$.

Combining this result with (2.6) gives $\epsilon^2-D\delta^2 = (X^2-DY^2)(u^2-Dv^2)^m$which finally turn out to be$\epsilon^2-D\delta^2 = N$,as required.

Next we prove (ii):We know that$X+Y\sqrt{D}>1$ and $1<\left(u+v\sqrt{D}\right)^m<\infty$. So we have $0<\left(u-v\sqrt{D}\right)^m<1$.Hence,as $\epsilon+\delta\sqrt{D} = \left(X+Y\sqrt{D}\right)\left(u-v\sqrt{D}\right)^m$, we get $0<\epsilon+\delta\sqrt{D}<\infty$. This gives $0<\frac{\epsilon-\delta\sqrt{D}}{N}<\infty$, as$\epsilon^2-D\delta^2=N$. Thus $0<\epsilon-\delta\sqrt{D}<\infty$. Adding this result with $0<\epsilon+\delta\sqrt{D}<\infty$ proves that $\epsilon>0$.

We then observeby (2.7) that$1<x_\alpha+y_\alpha\sqrt{D} \le \epsilon+\delta\sqrt{D} < \left(x_\alpha+y_\alpha\sqrt{D}\right)\left(u+v\sqrt{D}\right)<\infty$.

$\therefore 0<\frac{1}{\epsilon+\delta\sqrt{D}} \le \frac{1}{x_\alpha+y_\alpha\sqrt{D}}<1$ implies $0<\epsilon-\delta\sqrt{D}<x_\alpha-y_\alpha\sqrt{D}$.

And hence

$$\begin{aligned}2\delta\sqrt{D} &= \left(\epsilon+\delta\sqrt{D}\right)-\left(\epsilon-\delta\sqrt{D}\right) \ge \left(x_\alpha+y_\alpha\sqrt{D}\right)-\left(\epsilon-\delta\sqrt{D}\right)\\ &= \left(x_\alpha-y_\alpha\sqrt{D}\right)-\left(\epsilon-\delta\sqrt{D}\right)+2y_\alpha\sqrt{D}\\ &> \left(\epsilon-\delta\sqrt{D}\right)-\left(\epsilon-\delta\sqrt{D}\right)+2y_\alpha\sqrt{D}\end{aligned}$$

$=2y_\alpha\sqrt{D}$.

Thus $\delta>y_\alpha>0$, which proves (ii).

Finally we prove (iii):

We have to prove that $\epsilon+\delta\sqrt{D}$ is the smallest solution of (1.1).On the contrary suppose $\epsilon+\delta\sqrt{D}$ is positive solution of (1.1) but not the smallest solution of any class α. By(2.7) we have

$$x_\alpha+y_\alpha\sqrt{D} \le \epsilon+\delta\sqrt{D} < \left(x_\alpha+y_\alpha\sqrt{D}\right)\left(u+v\sqrt{D}\right).$$

This gives

$$\left(x_\alpha+y_\alpha\sqrt{D}\right)\left(u-v\sqrt{D}\right) \le \left(\epsilon+\delta\sqrt{D}\right)\left(u-v\sqrt{D}\right) < x_\alpha+y_\alpha\sqrt{D}.$$

Thus $\left(\epsilon+\delta\sqrt{D}\right)\left(u-v\sqrt{D}\right)<x_\alpha+y_\alpha\sqrt{D}$and $\epsilon+\delta\sqrt{D}$ is not the smallest solution of some classα. We then say that $\left(\epsilon+\delta\sqrt{D}\right)\left(u-v\sqrt{D}\right)$ should be a positive solution.i.e.$\left(\epsilon+\delta\sqrt{D}\right)\left(u-v\sqrt{D}\right)<x_\alpha+y_\alpha\sqrt{D}$,the smallest positive solution for fixed class α.i.e.$\left(\epsilon+\delta\sqrt{D}\right)\left(u-v\sqrt{D}\right)$ is the positive solution and is smaller than fundamental solution (for a class α),which is a contradiction.This proves that $\epsilon+\delta\sqrt{D}$ is the smallest solution (for a fixed class α), as required.

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