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GENERALIZED CONTRACTION RESULTS ON PROBABILISTIC 2-METRIC SPACES USING A CONTROL FUNCTION

Samir Kumar Bhandari

Abstract

In the present work, we introduced a generalized contraction result on probabilistic 2-metric spaces. Some control functions are also used here. We get a unique fixed point, that is, Tx=x for such contraction. Fixed point has an important role in modern analysis. One corollary is also given here. An illustrative example is given to validate our results. Some recent references are also listed here which help us to establish this manuscript.

Keywords:
2-Menger space,
Cauchy sequence,
fixed point,
φ-function,
ψ-function.

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1. Introduction

In 1906, Frechet introduced a new concept which was known as metric spaces. Many authors have taken their interest on these spaces. This idea also opened many directions for researchers. In 1942, K. Menger established the idea of probabilistic metric spaces in his famous work [17]. Probabilistic metric spaces are probabilistic generalization of metric spaces. Distribution function plays the role of metric on these spaces. S. Banach established Banach contraction mapping principle in 1922 on metric space [1]. Particular type of probabilistic metric space is Menger space in which the triangle inequality is postulated with the help of t-norm. Sehgal and Bharucha-Reid generalized the Banach contraction mapping principle to probabilistic metric space in 1972 [22]. The theory of Menger spaces is an important part of stochastic analysis. Schweizer and Sklar have described several aspects of such spaces in their book [21].

The purpose of this paper is to introduced a generalized contraction results on probabilistic 2-metric space using some control functions. The space in which the results are deduced is a probabilistic 2-metric space which is a probabilistic extension of 2-metric space. 2-Menger space is a special case of probabilistic 2-metric space. The theory of Menger spaces is an important part of stochastic analysis. Some recent results on probabilistic 2-metric space may be noted as [2, 3, 7].

In 1984, Khan, Swaleh and Sessa introduced the concept of altering distance function in their research work on [16]. After that, this idea was generalized in various number of works. Some works may be referred as [18, 19, 20]. The main features of the paper are given below.

1. We introduce generalized probabilistic contraction results.
2. Here we use the continuous t-norm.
3. We use two control functions.
4. Finally we get a unique fixed point for the function f , that is, f x = x.

2. Definitions and Mathematical Preliminaries

The following definitions and mathematical preliminaries are required in our discussion.

Definition 2.1 2-metric space [11, 12]

Let X be a non empty set. A real valued function d on X × X × X is said to be a 2-metric on X if

(i) given distinct elements x, y ∈ X, there exists an element z of X such that d(x, y, z) ≠ 0,

(ii) d(x, y, z) = 0 when at least two of x, y, z are equal,

(iii) d(x, y, z) = d(x, z, y) = d(y, z, x) for all x, y, z ∈ X and

(iv) d(x, y, z) ≤ d(x, y, w) + d(x, w, z) + d(w, y, z) for all x, y, z, w ∈ X.

When d is a 2-metric on X, the ordered pair (X, d) is called a 2-metric space.

Definition 2.2 [15, 21] A mapping F: R → R^+ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t \in R} F(t) = 0$ and $\sup_{t \in R} F(t) = 1$, where R is the set of real numbers and R^+ denotes the set of non-negative real numbers.

Definition 2.3 Probabilistic metric space [15, 21]

A probabilistic metric space (briefly, PM-space) is an ordered pair (X, F), where X is a non empty set and F is a mapping from X × X into the set of all distribution functions. The function $F_{x,y}$ is assumed to satisfy the following conditions for all x, y, z ∈ X,

(i) $F_{x,y}(0) = 0$,

(ii) $F_{x,y}(t) = 1$ for all t > 0 if and only if x = y,

(iii) $F_{x,y}(t) = F_{y,x}(t)$ for all t > 0,

(iv) if $F_{x,y}(t_1) = 1$ and $F_{y,z}(t_2) = 1$ then $F_{x,z}(t_1+ t_2) = 1$ for all $t_1, t_2 > 0$.

A particular type of probabilistic metric space is Menger space in which the triangular inequality is proved with the help of a t-norm. Shi, Ren and Wang [23] introduced the following definition of n-th order t-norm.

Definition 2.4 n-th order t-norm [23]

A mapping T: $\prod_{i=1}^{n}[0,1] \to [0,1]$ is called a n-th order t-norm if the following conditions are satisfied :

(i) T (0, 0, ..., 0) = 0, T (a, 1, 1, ..., 1) = a for all a ∈ [0, 1],

(ii) T $(a_1, a_2,, a_3, ..., a_n)$ = T $(a_2, a_1, a_3, ..., a_n)$ = T $(a_2, a_3, a_1, ..., a_n)$

= .... = T $(a_2, a_3, a_4, ..., a_n, a_1)$,

(iii) $a_i \geq b_i$, i=1,2,3,....,n implies T $(a_1, a_2, a_3, ..., a_n) \geq$ T $(b_1, b_2, b_3, ..., b_n)$,

(iv) T (T $(a_1, a_2, a_3, ..., a_n)$, $b_2, b_3, ...b_n$)

= T $(a_1$, T $(a_2, a_3, ..., a_n, b_2), b_3, ..., b_n)$

= T $(a_1, a_2$, T $(a_3, a_4...., a_n, b_2, b_3), b_4, ..., b_n)$

=...........= T $(a_1, a_2, ..., a_{n-1}$, T$(a_n, b_2, b_3, ..., b_n))$. When n = 2, we have a binary t-norm, which is commonly known as t-norm.

Definition 2.5 Menger space [15, 21]

A Menger space is a triplet (X, F, Δ), where X is a non empty set, F is a function defined on X × X to the set of all distribution functions and Δ is a t-norm, such that the following are satisfied:

(i) $F_{x,y}(0) = 0$ for all x, y ∈ X,

(ii) $F_{x,y}(s) = 1$ for all s > 0 if and only if x = y,

(iii) $F_{x,y}(s) = F_{y,x}(s)$ for all x, y ∈ X, s > 0 and

(iv) $F_{x,y}(u + v) \geq \Delta (F_{x,z}(u), F_{z,y}(v))$ for all u, v ≥ 0 and x, y, z ∈ X.

The theory of these spaces is an important part of stochastic analysis. Schweizer and Sklar in their book [21] have given a comprehensive account of several aspects of such spaces. A probabilistic 2-metric space is a probabilistic generalization of 2-metric space. Wen-Zhi Zeng [25] introduced the concept of probabilistic 2-metric spaces.

Definition 2.6 probabilistic 2-metric space [25]

A probabilistic 2-metric space is an order pair (X, F) where X is an arbitrary set and F is a mapping from X × X × X into the set of all distribution functions such that the following conditions are satisfied:

(i) $F_{x,y,z}(t) = 0$ for $t \leq 0$ and for all x, y, z ∈ X,

(ii) $F_{x,y,z}(t) = 1$ for all $t > 0$ iff at least two of x, y, z are equal,

(iii) for distinct points x, y ∈ X there exists a point z∈ X such that $F_{x,y,z}(t) \neq 1$ for $t > 0$,

(iv) $F_{x,y,z}(t) = F_{x,z,y}(t) = F_{z,y,x}(t)$ for all x, y, z ∈ X and $t > 0$,

(v) $F_{x,y,w}(t_1) = 1$, $F_{x,w,z}(t_2) = 1$ and $F_{w,y,z}(t_3) = 1$ then $F_{x,y,z}(t_1 + t_2 + t_3) = 1$, for all x, y, z, w ∈ X and $t_1, t_2, t_3 > 0$.

The following is the special case of above definition.

Definition 2.7 2-Menger space [24]

Let X be a nonempty set. A triplet(X, F, Δ) is said to be a 2-Menger space if Fis a mapping from X × X × X into the set of all distribution functions satisfying the following conditions:

(i) $F_{x,y,z}(0) = 0$,

(ii) $F_{x,y,z}(t) = 1$ for all $t > 0$ if and only if at least two of x, y, z ∈ X are equal,

(iii) for distinct points x, y ∈ X there exists a point z ∈ Xsuch that $F_{x,y,z}(t) \neq 1$ for $t > 0$,

(iv) $F_{x,y,z}(t) = F_{x,z,y}(t) = F_{z,y,x}(t)$, for all x, y, z ∈ X and $t > 0$,

(v) $F_{x,y,z}(t) \geq \Delta(F_{x,y,w}(t_1), F_{x,w,z}(t_2), F_{w,y,z}(t_3))$

Where $t_1, t_2, t_3 > 0$, $t_1 + t_2 + t_3 = t$, x, y, z, w ∈ X and Δ is the 3rd order t norm.

Definition 2.8 [14] A sequence $\{x_n\}$ in a 2-Menger space(X, F, Δ) is said to be converge to a limit x if given $\epsilon > 0$, $0 < \lambda < 1$ there exists a positive integer $N_{\epsilon,\lambda}$ such that

$$F_{x_n,x,a}(\varepsilon) \geq 1 - \lambda \quad (1.1)$$

for all $n > N_{\epsilon,\lambda}$ and for every a ∈ X.

Definition 2.9 [14] A sequence $\{x_n\}$ in a 2-Menger space(X, F, Δ) is said to be a Cauchy sequence in X if given $\epsilon > 0$, $0 < \lambda < 1$ there exists a positive integer $N_{\epsilon,\lambda}$ such that

$$F_{x_n,x_m,a}(\varepsilon) \geq 1 - \lambda \quad (1.2)$$

for all $m, n > N_{\epsilon,\lambda}$ and for every a ∈ X.

Definition 2.10 [14] A 2-Menger space (X, F, Δ) is said to be complete if every Cauchy sequence is convergent in X.

In 2008, Choudhury and Das established φ-function on their works [4]. They actually extended the concept of ”altering distant function” in the context of Menger spaces. The important φ-function is given below.

Definition 2.11 Φ-function [4]

A function $\varphi : R \to R^+$ is said to be a Φ-function if it satisfies the following conditions:

(i) $\varphi(t) = 0$ if and only if $t = 0$,

(ii) $\varphi(t)$ is strictly monotone increasing and $\varphi(t) \to \infty$ as $t \to \infty$,

(iii) φ is left continuous in$(0, \infty)$,

(iv) φ is continuous at 0.

The idea of control function has opened new possibilities of proving more fixed point results in

Menger spaces. Many authors applied this concept to a coincidence point problems also. Some recent references using Φ-function may be noted in [2, 3, 5, 6, 8, 9] and [10].

Here we also use the ψ[13], the class of all continuous function satisfies the following conditions:

$\psi : R^+ \to R^+$ such that $\psi(0) = 0$ and $\psi^n(a_n) \to 0$ whenever $a_n \to 0$ as $n \to \infty$.

3. Main Results

In this section we have established one theorem, one corollary and one example. We are motivated by the recent results of [10, 13] to construct the present paper. Here we establish the fixed point results on probabilistic 2-metric spaces using some control functions.

Theorem 3.1 Let (X, F, Δ) be a complete 2-Menger space, Δ is a continuous t-norm and $f : X \to X$ be a mapping satisfying the following inequality for all x, y, a ∈ X,

$$\frac{1}{F_{fx,fy,a}(\phi(ct))} - 1 \le \psi\left(\frac{1}{F_{x,y,a}(\phi(t))} - 1\right), \tag{3.1}$$

where $t > 0$, $0 < c < 1$, φ is a Φ-function and ψ is a ψ -function. Then f has a unique fixed point in X.

Proof. Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X so that $x_n = f x_{n-1}$, $n \in N$ where N is the set of natural numbers. We suppose $x_{n+1} \neq x_n$ for all $n \in N$, otherwise f has trivially a fixed point.

Now, applying the inequality (3.1), we have

$$\frac{1}{F_{x_1,x_2,a}(\phi(ct))} - 1 = \frac{1}{F_{fx_0,fx_1,a}(\phi(ct))} - 1 \le \psi\left(\frac{1}{F_{x_0,x_1,a}(\phi(t))} - 1\right). \tag{3.2}$$

Obviously $F_{x_0,x_1,a}(\phi(t)) > 0$ implies $F_{x_0,x_1,a}(\phi(\frac{t}{c})) > 0$ for all a ∈X, t>0 and so, again by applying (3.1), we get

$$\frac{1}{F_{x_1,x_2,a}(\phi(t))} - 1 = \frac{1}{F_{fx_0,fx_1,a}(\phi(t))} - 1 \le \psi\left(\frac{1}{F_{x_0,x_1,a}(\phi(\frac{t}{c}))} - 1\right).$$

Repeating the above procedure successively n times, we obtain

$$\frac{1}{F_{x_n,x_{n+1},a}(\phi(t))} - 1 \le \psi^n\left(\frac{1}{F_{x_0,x_1,a}(\phi(\frac{t}{c^n}))} - 1\right).$$

Also , $F_{x_1,x_2,a}(\phi(ct)) > 0$ for all a ∈ X.

Then, following the above procedure, we have

$$\frac{1}{F_{x_n,x_{n+1},a}(\phi(ct))} - 1 \le \psi^{n-1}\left(\frac{1}{F_{x_1,x_2,a}(\phi(\frac{ct}{c^{n-1}}))} - 1\right).$$

In general, if we repeat the above step r times with r < n, we get

$$\frac{1}{F_{x_n,x_{n+1},a}(\phi(c^r t))} - 1 \le \psi^{n-r}\left(\frac{1}{F_{x_r,x_{r+1},a}(\phi(\frac{c^r t}{c^{n-r}}))} - 1\right). \tag{3.3}$$

Since $\psi^n(a_n) \to 0$ whenever $a_n \to 0$, then from (3.3) for all $r > 0$, we deduce that

$$F_{x_n,x_{n+1},a}(\phi(c^r t)) \to 1 \text{ as } n \to \infty. \tag{3.4}$$

Now, let $\varepsilon > 0$ be given, then by using the properties of function ϕ we can find $r > 0$, such that $\phi(c^r t) < \varepsilon$ .Therefore, from (3.4), we get

$$F_{x_n,x_{n+1},a}(\varepsilon) \to 1, \text{ as } n \to \infty \text{ for every } \varepsilon > 0. \tag{3.5}$$

We next prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then there exist $\varepsilon > 0$ and $0 < \lambda < 1$ for which we can find subsequences

$\{x_{m(k)}\}$and $\{x_{n(k)}\}$ of $\{x_n\}$ with n(k) > m(k) > k such that

$$F_{x_{m(k)},x_{n(k)},a}(\varepsilon) < 1-\lambda \quad . \tag{3.6}$$

We take n(k) corresponding to m(k) to be the smallest integer satisfying (3.6), so that

$$F_{x_{m(k)},x_{n(k)-1},a}(\varepsilon) \geq 1-\lambda \quad . \tag{3.7}$$

If $\varepsilon_1 < \varepsilon$, then we have

$$F_{x_{m(k)},x_{n(k)},a}(\varepsilon_1) \leq F_{x_{m(k)},x_{n(k)},a}(\varepsilon).$$

We conclude that it is possible to construct $\{x_{m(k)}\}$and $\{x_{n(k)}\}$with n(k) > m(k) > k and satisfying (3.6) and (3.7) whenever ε is replaced by a smaller positive value. As φ is continuous at 0 and strictly monotone increasing with φ(0) = 0, it is possible to obtain $\varepsilon_2 > 0$ such that $\phi(\varepsilon_2) < \varepsilon$.

Then, by the above argument, it is possible to obtain an increasing sequence of integers {m(k)} and {n(k)} with n(k) > m(k) > k such that

$$F_{x_{m(k)},x_{n(k)},a}(\phi(\varepsilon_2)) < 1-\lambda \tag{3.8}$$

and $F_{x_{m(k)},x_{n(k)-1},a}(\phi(\varepsilon_2)) \geq 1-\lambda$. (3.9)

By (3.8), we have

$$1-\lambda > F_{x_{m(k)},x_{n(k)},a}(\phi(\varepsilon_2)) \ ,$$

that is,

$$\frac{1}{1-\lambda} < \frac{1}{F_{x_{m(k)},x_{n(k)},a}(\phi(\varepsilon_2))},$$

that is,

$$\frac{1}{1-\lambda} - 1 < \frac{1}{F_{x_{m(k)},x_{n(k)},a}(\phi(\varepsilon_2))} - 1,$$

that is, $\frac{\lambda}{1-\lambda} < \frac{1}{F_{x_{m(k)},x_{n(k)},a}(\phi(\varepsilon_2))} - 1 \quad \leq \psi(\frac{1}{F_{x_{m(k)-1},x_{n(k)-1},a}(\frac{\phi(\varepsilon_2)}{c})} - 1)$. [using the inequality (3.1)]

Repeating the above procedure successively k times, we obtain

$$\frac{\lambda}{1-\lambda} \leq \psi^k(\frac{1}{F_{x_{m(k)-k},x_{n(k)-k},a}(\frac{\phi(\varepsilon_2)}{c^k})} - 1).$$

Now, for $k \to \infty$, $F_{x_{m(k)-k},x_{n(k)-k},a}(\frac{\phi(\varepsilon_2)}{c^k}) \to 1,$ (since $0 < c < 1$)

that is, $\psi^k(0) \to 0$,

that is, $\frac{\lambda}{1-\lambda} \leq 0$,

$\lambda \leq 0$,

since $\lambda \in (0,1)$, there is a contradiction.

Hence $\{x_n\}$ is a Cauchy sequence.

Since (X, F, Δ) be a complete 2-Menger space, therefore $x_n \to u$ as $n \to \infty$, for some u ∈ X.

Next, using the properties of function φ, we can find $t_2 > 0$ such that $\phi(t_2) < \frac{\varepsilon}{2}$. Again $x_n \to u$ as n ∞ and hence there exists $n_0 \in N$ such that, for all $n > n_0$ (sufficiently large), we have

$$\frac{1}{F_{x_{n+1},fu,a}(\frac{\varepsilon}{2})} - 1 \leq \frac{1}{F_{fx_n,fu,a}(\phi(t_2))} - 1 \leq \psi(\frac{1}{F_{x_n,u,a}(\phi(\frac{t_2}{c}))} - 1), \text{ for all } a \in X.$$

Now, letting n → ∞, since ψ(0) = 0 and the continuity of the function ψ, we obtain

$F_{u,fu,a}(\frac{\varepsilon}{2}) \geq 1$ as n → ∞,

that is,

function f has a fixed point, that is, f u = u.

Next, we establish the uniqueness of the fixed point.

Let x and y be two fixed point of f, that is, f x = x and f y = y.

By the properties of φ there exists s > 0 such that $F_{x,y,a}(\varphi(s)) > 0$ for all a ∈ X.

Then, by an application of (3.1), we have

$$\frac{1}{F_{fx,fy,a}(\phi(s))} - 1 = \frac{1}{F_{x,y,a}(\phi(s))} - 1 \leq \psi(\frac{1}{F_{x,y,a}(\phi(\frac{s}{c}))} - 1),$$

$$\leq \psi^2(\frac{1}{F_{x,y,a}(\phi(\frac{s}{c^2}))} - 1).$$

Repeating the above procedure successively n times, we obtain

$$\frac{1}{F_{fx,fy,a}(\phi(s))} - 1 = \frac{1}{F_{x,y,a}(\phi(s))} - 1 \leq \psi^n(\frac{1}{F_{x,y,a}(\phi(\frac{s}{c^n}))} - 1).$$

Taking limit as n → ∞ on both sides, we have

$\psi^n \to 0$, (since $F_{x,y,a}(\phi(\frac{s}{c^n})) \to 1$)

that is, $F_{x,y,a}(\phi(s)) = 1$, that is , $x = y$,

that is, the fixed point is unique.

Taking φ(t) = t in the above theorem we get the following Corollary.

Corollary 3.1 Let (X, F, Δ) be a complete 2-Menger space and f : X → X be a mapping satisfying the following inequality for all x, y, a ∈ X,

$$\frac{1}{F_{fx,fy,a}(t)} - 1 \leq \psi(\frac{1}{F_{x,y,a}(\frac{t}{c})} - 1),$$

where t > 0, 0 < c < 1. Then f has a unique fixed point in X.

The following example satisfied the above corollary.

Example 3.1 Let X = {α, β, γ, δ}, Δ is a continuous t-norm and F be defined as

$$F_{\alpha,\beta,\gamma}(t) = F_{\alpha,\beta,\delta}(t) = \begin{cases} 0, if t \leq 0, \\ 0.40, if\ 0 < t < 4, \\ 1, if t \geq 4, \end{cases}$$

$$F_{\alpha,\gamma,\delta}(t) = F_{\beta,\gamma,\delta}(t) = \begin{cases} 0, if t \leq 0, \\ 1, if t \geq 1, \end{cases}$$

Then (X, F, Δ) is a complete 2-Menger space. If we define f: X→X as follows:

f α=δ, f β=γ, f γ= δ, f δ=δ then the mappings f satisfies all the conditions of the Corollary 3.1. Here δ is the unique fixed point of f in X.

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