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ON UNIFIED INTEGRAL ASSOCIATED WITH THE GENERALIZED FUNCTION $\boldsymbol{G_{\rho,\eta,r}}$[A, Z]

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Abstract:

The main object of the present is to provide an interesting double integral involving generalized function $G_{\rho,\eta,r}$ defined in [3], which is expressed in terms of generalized (Wright) hyper geometric function . A further extension of our main result and their associated special cases are also considered.

AMS subject classification: 33C45, 33C60, 33E12.

Key Words: Generalized function, generalized Wright hyper geometric function and integrals.

Introduction:

The well known generalized function $\boldsymbol{G_{\rho,\eta,r}}$**[a, z]** defined by [3,4,8]

$$G_{\rho,\eta,r}[a,z] = z^{r\rho-\eta-1}\sum_{n=0}^{\infty}\frac{(r)_n\,(a\,z^{\rho})^n}{\Gamma(n\rho+r\rho-\eta)n!}\,, \qquad Re(\rho r-\eta)>0 \tag{1.1}$$

The well known Mittag – Leffler function of the form

$$E_{\rho}(z) = \sum_{n=0}^{\infty}\frac{z^{\rho n}}{\Gamma(\rho n+1)} \tag{1.2}$$

Where$\rho\,\epsilon C$, $Re\,(\alpha)>0$, $z\,\epsilon C$, defines the Mittag -Leffler function [9]

A generalized function of (1.2) in the form

$$E_{\rho,\mu}(z) = \sum_{n=0}^{\infty}\frac{z^{\rho n}}{\Gamma(\rho n+\mu)} \tag{1.3}$$

Where$\rho,\mu\,\epsilon C$, $Re\,(\rho)>0$, $Re(\mu)>0$ $z\,\epsilon C$, defines the Mittag -Leffler function [2]

A Generalized function of (1.3) in the form

$$E^{r}_{\rho,\mu}(z) = \sum_{n=0}^{\infty} \frac{(r)_n z^{\rho n}}{\Gamma(\rho n+\mu)} \tag{1.4}$$

Where$\rho, \mu, r \epsilon C$, $Re(\rho) > 0$, $Re(\mu) > 0$, $Re(r) > 0$ $z \epsilon C$, defines the Mittag- Leffler function [11,1]

Where $(r)_n$ is the Pochhammer symbol (cf. [6, p.2 and p.5]):

$$(r)_n = \frac{\Gamma(r+n)}{\Gamma(r)} \tag{1.5}$$

$$(r)_0 = 1, (r)_n = (r)(r+1)\ldots.(r+n-1), (n = 1,2,3\ldots.); \tag{1.6}$$

The Generalized Wright Hypergeometric function ${}_p\Psi_q(z)$ (see, for details, Shrivastava and Karlsson [7]) for $z \in \mathbb{C}$ complex, $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}$

Where $(\alpha_i, \beta_j \neq 0; i = 1,2,3,..,p; j = 1,2,3,\ldots q)$ is defined as bellow :

$${}_p\Psi_q = \mathrm{p}\Psi\mathrm{q}\left[\begin{matrix}(a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q}\end{matrix}\middle| z = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i k)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)} \frac{z^k}{k!}\right] \tag{1.7}$$

Introduced by Wright [5], the generalized Wright function proved several theorems on the asymptotic expansion of ${}_p\Psi_q(z)$ for all values of the argument z, under the condition:

$$\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_j > -1 \tag{1.8}$$

Furthermore, we also recall here the following interesting and useful result due to Edward

[10, p.445]

$$\int_0^1 \int_0^1 y^{\alpha} (1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{1-\alpha-\beta}\, dx\, dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{1.9}$$

2. On unified integral associated with the generalized special function

Theorem 2.1

If $\rho, \eta, r \epsilon C \ \ Re(\rho), Re(\eta) > 0 \, , Re(\rho r - \eta) > 0 \ and \ n \epsilon N$ then hold the following the special function $G_{\rho,\eta,r}[a, z]$, then we have the following relation,

$$\int_0^1 \int_0^1 y^{\alpha} (1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{1-\alpha-\beta} G_{\rho,\eta,r}\left[c, \frac{ay(1-x)(1-y)}{(1-xy)^2}\right] dx\, dy$$

$$= a^{r\rho-\eta-1} \frac{1}{\Gamma(r)} \ {}_3\Psi_2 \left[\begin{matrix}(r,1), (\alpha + r\rho - \eta - 1, \rho), (\beta + r\rho - \eta - 1, \rho) \\ (\rho r - \eta, \rho)\ (\alpha + \beta + 2(\rho r - \eta - 1), 2\rho)\end{matrix}; a^{\rho}\right] \quad (2.1)$$

Where ${}_p\Psi_q$ is defined by (1.7)

Proof:

In order to establish our main result (2.1) , we denote the left –hand side of (2.1) by Δ

And then using (1.1), we get:

$$\Delta = \int_0^1 \int_0^1 y^{\alpha} (1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{1-\alpha-\beta}$$

$$\times \left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right]^{r\rho-\eta-1} \sum_0^{\infty} (r)_n \frac{\left[c\left\{\frac{ay(1-x)(1-y)}{(1-xy)^2}\right\}^{\rho}\right]^n}{\Gamma(\mathrm{n}\rho+\rho\mathrm{r}-\eta)\ \mathrm{n}!} dx\, dy \quad (2.2)$$

Now changing the order of integration and summation and then applying the result (1.9), we get

$$\Delta = [a]^{r\rho-\eta-1+n\rho} \ \frac{C^n}{\Gamma(\mathrm{n}\rho + \rho\mathrm{r} - \eta)\ \mathrm{n}!} \ \frac{\Gamma(\mathrm{r}+\mathrm{n})}{\Gamma(\mathrm{r})}$$

$$\times \frac{\Gamma(\alpha + r\rho - \eta - 1 + \rho n)\Gamma(\beta + r\rho - \eta - 1 + \rho n)}{\Gamma(\alpha + \beta + 2(r\rho - \eta - 1 + \rho n))} \quad (2.3)$$

Finally, summing up the above series with the help of (1.7), we easily arrive at the right hand side of (2.1). This completes the proof of our main result.

3. Special cases

(1) On setting η by $\rho r-\mu$ and $a=1$ in (2.1) and then by using (1.4), we get the following interesting integral:

$$\int_0^1\int_0^1 y^{\alpha}\,(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{1-\alpha-\beta}\,E_{\rho,\mu}^{r}\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right] dx\,dy$$

$$= a^{\mu-1}\,\frac{1}{\Gamma(r)}\;{}_3\Psi_2\left[\begin{matrix}(r,1),(\alpha+\mu-1,\rho),(\beta+\mu-1,\rho)\\(\mu,\rho)\,(\alpha+\beta+2(\mu-1),2\rho)\end{matrix};\,a^{\rho}\right] \tag{3.1}$$

(2) On setting η by $\rho r-\mu$, $r=1$ and $a=1$ in (2.1) and then by using (1.3), we get the following interesting integral:

$$\int_0^1\int_0^1 y^{\alpha}\,(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{1-\alpha-\beta}\,E_{\rho,\mu}\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right] dx\,dy$$

$$= a^{\mu-1}\;{}_3\Psi_2\left[\begin{matrix}(1,1),(\alpha+\mu-1,\rho),(\beta+\mu-1,\rho)\\(\mu,\rho)\,(\alpha+\beta+2(\mu-1),2\rho)\end{matrix};\,a^{\rho}\right] \tag{3.2}$$

(3) On setting η by $\rho r-\mu$, $r=1$, $\mu=1 and$ $a=1$ in (2.1) and then by using (1.2), we get the following interesting integral:

$$\int_0^1\int_0^1 y^{\alpha}\,(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{1-\alpha-\beta}\,E_{\rho}\left[\frac{ay(1-x)(1-y)}{(1-xy)^2}\right] dx\,dy$$

$$= \;{}_3\Psi_2\left[\begin{matrix}(1,1),(\alpha,\rho),(\beta,\rho)\\(1,\rho)\,(\alpha+\beta\,,2\rho)\end{matrix};\,a^{\rho}\right]$$

(3.3)

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