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# A GENERALIZED DIFFERENTIAL TRANSFORM METHOD FOR LINEAR AND NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

**Sheela Jacob**

**Abstract**

In this study a new generalization of the two-dimensional transform method is proposed for nonlinear partial differential equations with space- and time-fractional derivatives. This method is based on the two-dimensional differential transform method (DTM) and generalized Taylor's formula. The fractional derivatives are considered in the Caputo sense. Several illustrative examples are given to demonstrate the effectiveness of the present method. Results obtained using the scheme presented here agree well with the analytical solutions and the numerical results presented elsewhere. Results also show that the numerical scheme is very effective and convenient for solving nonlinear partial differential equations of fractional order.

***Keywords:***

*Generalized Taylor's formula,*
*Generalized differential transform method,*
*Non-linear fractionalpartial differential equations,*
*Caputo derivatives.*



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## 1. Introduction

Fractional differential equations are hot topics both in mathematics and physics. Recently, the fractional differential equations have been the subject of intensive research. In the last decades, it has been observed in many fields that any phenomena with strange kinetics cannot be described within the framework of classical theory using integer order derivatives. For high accuracy, fractional derivatives are then used to describe the dynamics of some structures. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Half order derivatives and integrals prove to be more useful for the formulation of certain electrochemical problems than the classical models.

Fractional calculus has found diverse applications in various scientific and technological fields, such as thermal engineering, acoustics, fluid mechanics, biology, chemistry, electromagnetism, control, robotics, viscoelasticity, diffusion, edge detection, turbulence, signal processing and many other physical processes. Mathematical modeling of many physical systems leads to linear and nonlinear fractional differential equations in various fields of physics and engineering.

A great deal of effort has been expended over the last 10 years or so in attempting to find robust and stable numerical and analytical methods for solving fractional partial differential equations of physical interest. Numerical and analytical methods have included finite difference method, Adomian decomposition method, variational iteration method,homotopy perturbation method and fractional difference method have been developed to obtain exact and approximate analytic solutions. Some of these methods use transformation in order to reduce equations into simpler equations or systems of equations and some other methods give the solution in a series form which converges to the exact solution. Among these solution techniques, the variational iteration method and the Adomian decomposition method are the clearest methods of solution of fractional differential and integral equations, because they provide immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations without linearization or discretization.

This work represents the application of the generalized differential transform method to provide approximate solutions for the system of fractional partial differential equations. Several numerical experiments of nonlinear systems of fractional partial differential equations shall be presented.

## FRACTIONAL CALCULUS

The theory of fractional calculus was first raised in the year 1695 by Marquis de L'Hopital. There are several definitions of a fractional derivative of order $\alpha > 0$. E.g. Riemann-Liouville, Grunwald-Letnikow, Caputo and Generalized Functions Approach. The most commonly used definitions are the Riemann-Liouville and Caputo. We give some basic definitions and properties of the fractional calculus theory which are used further in this work.

**Definition 1.1**

A real function $f(x), x > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$, if there exists a real number $p\ (> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$, and it is said to be in the space $C_\mu^m$ if and only if $f^m \in C_\mu, m \in N$.

**Definition 1.2**

The Riemann-Liouville fractional integral operator $(J^v)$of order $v \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as

$$J_a^v f(x) = \frac{1}{\Gamma(v)} \int_a^x (x-t)^{v-1} f(t)dt\,, \qquad v > 0,\ x > 0 \qquad (1.1) J^0 f(t) = f(t)$$

It has the following properties:

For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0, m-1 < \alpha < m, a \geq 0$ and $\gamma > 1$ :

1. $\left(J_a^\alpha J_a^\beta f\right)(x) = \left(J_a^\beta J_a^\alpha f\right)(x) = \left(J_a^{\alpha+\beta} f\right)(x),$ (1.2)

2. $J_a^\alpha (x-a)^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}(x-a)^{\gamma+\alpha}\,,$ (1.3)

3. $(J_a^\alpha D_a^\alpha f)(x) = J^m D^m f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(a)\dfrac{(x-a)^k}{k!}$ (1.4)

Here $D^m$ is the usual integer differential operator of order m.

The Riemann-Liouville fractional derivative is mostly used by mathematicians but this approach is not suitable for the physical problems of the real world since it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations.

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**Definition 1.3**
The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D_a^v f(x) = \frac{d^m}{dx^m}\left(J_a^{m-v} f(x)\right) \tag{1.5}$$

$$D_{*a}^v f(x) = J_a^{m-v}\left(\frac{d^m}{dx^m} f(x)\right) \tag{1.6}$$

where, $m-1 < v \leq m \; and \; m \in N.$
For now, the Caputo fractional derivative will be denoted by $D_*^v$ to maintain a clear distinction with the Riemann-Liouville fractional derivative.

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem. In this work, we consider the one-dimensional space- and time- fractional nonlinear partial differential equation, where the unknown function $u = u(x, t)$ is assumed to be a casual function of space and time respectively and the fractional derivatives are taken in Caputo sense as follows:

**Definition 1.4**
For m to be the smallest integer that exceeds μ, the Caputo time-fractional derivative operator of order $\mu > 0$ is defined as

$$D_{*t}^{\mu} u(x,t) = \frac{\partial^{\mu} u(x,t)}{\partial t^{\mu}}$$

$$= \begin{cases} \dfrac{1}{\Gamma(m-\mu)} \displaystyle\int_0^t (t-\tau)^{m-\mu-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau & for \quad m-1 < \mu < m, \\ \dfrac{\partial^m u(x,t)}{\partial t^m} & for \quad \mu = m \in N \end{cases} \tag{1.7}$$

And the space-fractional derivative operator of order v> 0 is defined as

$$D_{*x}^{v} u(x,t) = \frac{\partial^{v} u(x,t)}{\partial x^{v}}$$

$$= \begin{cases} \dfrac{1}{\Gamma(m-v)} \displaystyle\int_0^x (x-\theta)^{m-v-1} \frac{\partial^m u(\theta,t)}{\partial \theta^m} d\theta & for \quad m-1 < v < m, \\ \dfrac{\partial^m u(x,t)}{\partial x^m} & for \quad v = m \in N \end{cases} \tag{1.8}$$

In this study, we develop a semi numerical method based on the two dimensional differential transform method, generalized Taylor's formula and Caputo fractional derivative.
The organization of this work is as follows: In chapter 2, we give the generalized Taylor's formula. In chapter 3, the generalized differential transform, which is based on two dimensional differential transform and generalized Tailor's formula will be introduced. Chapter 4 consists of analysis of the method. Finally, in chapter 5, the mentioned scheme in chapter 4 is used to seek an approximate solution of some nonlinear fractional partial differential equations with the initial conditions. Also, the accuracy and efficiency of the scheme is investigated with three numerical illustrations for nonlinear fractional partial differential equation in this chapter.

## 2. GENERALIZED TAYLOR'S FORMULA

In this chapter, we present the generalized Taylor's formula that involves Caputo fractional derivatives. We begin by introducing the generalized mean value theorem.

**Theorem 2.1 (Generalized Mean Value Theorem)**

Suppose that $f(x) \in C[a,b]$ and $D_{*a}^{\alpha} f(x) \in C(a,b]$, for $0 < \alpha \leq 1$, then we have

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)} (D_{*a}^{\alpha} f)(\xi).(x-a)^{\alpha} \qquad (2.1)$$

with $a \leq \xi \leq x, \forall\, x \in (a,b]$ and $D^{\alpha}$ is the Caputo fractional derivative of order $\alpha > 0$

**Proof.**

From (1.1) and (1.4), we have,

$$(J_{*a}^{\alpha} D_{*a}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} (D_{*a}^{\alpha} f)(t) dt\,, \qquad (2.2)$$

Using the integral mean value theorem, we get

$$(J_{*a}^{\alpha} D_{*a}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} (D_{*a}^{\alpha} f)(\xi) \int_a^x (x-t)^{\alpha-1}\, dt$$

$$= \frac{1}{\Gamma(\alpha)} (D_{*a}^{\alpha} f)(\xi)(x-a)^{\alpha} \qquad (2.3)$$

for $0 \leq \xi \leq x$.

On the other hand, from (1.4), we have

$$(J_{*a}^{\alpha} D_{*a}^{\alpha} f)(x) = f(x) - f(a) \quad (2.4)$$

$$\Longrightarrow \quad f(x) = f(a) + (J_{*a}^{\alpha} D_{*a}^{\alpha} f)(x)$$

$$\therefore \quad f(x) = f(a) + \frac{1}{\Gamma(\alpha)} (D_{*a}^{\alpha} f)(\xi).(x-a)^{\alpha}$$

Hence, (2.1) is obtained. In case of $\alpha = 1$, the generalized mean value theorem reduces to the classical mean value theorem.

Before we present the generalized Taylor's formula in the Caputo sense, we need the following relation.

**Theorem 2.2.**

Suppose that $(D_{*a}^{\alpha})^n f(x), (D_{*a}^{\alpha})^{n+1} f(x) \in C(a,b]$, for $0 < \alpha \leq 1$, then we have

$$(J_{*a}^{n\alpha} (D_{*a}^{\alpha})^n f)(x) - \left(J_{*a}^{(n+1)\alpha} (D_{*a}^{\alpha})^{n+1} f\right)(x) = \frac{(x-a)^{n\alpha}}{\Gamma(n\alpha+1)} ((D_{*a}^{\alpha})^n f)(a), \quad (2.5)$$

Where,

$$(D_{*a}^{\alpha})^n = D_{*a}^{\alpha} . D_{*a}^{\alpha} \ . \ . \ . \ D_{*a}^{\alpha} (n - times).$$

**Proof.**

Using ( 1.2), we have

$$(J_{*a}^{n\alpha} (D_{*a}^{\alpha})^n f)(x) - \left(J_{*a}^{(n+1)\alpha} (D_{*a}^{\alpha})^{n+1} f\right)(x)$$

$$= J_{*a}^{n\alpha} (((D_{*a}^{\alpha})^n f)(x) - (J_{*a}^{\alpha} (D_{*a}^{\alpha})^{n+1} f)(x)),$$

$$= J_{*a}^{n\alpha} (((D_{*a}^{\alpha})^n f)(x) - (J_{*a}^{\alpha} D_{*a}^{\alpha})((D_{*a}^{\alpha})^n f)(x)),$$

$$= J_{*a}^{n\alpha} (((D_{*a}^{\alpha})^n f)(a)), \qquad (\text{using } (1.4))$$

$$= \frac{(x-a)^{n\alpha}}{\Gamma(n\alpha+1)} (((D_{*a}^{\alpha})^n f)(a)) (\text{using } (1.3))$$

$$\therefore \ (J_{*a}^{n\alpha} (D_{*a}^{\alpha})^n f)(x) - \left(J_{*a}^{(n+1)\alpha} (D_{*a}^{\alpha})^{n+1} f\right)(x) = \frac{(x-a)^{n\alpha}}{\Gamma(n\alpha+1)} ((D_{*a}^{\alpha})^n f)(a)$$

**Theorem 2.3.(Generalized Taylor's Formula)**

Suppose that $(D_{*a}^{\alpha})^k f(x) \epsilon\, C(a,b]$ for $k = 0,1,\ \ldots, n+1$, where $0 < \alpha \le 1$, then we have

$$f(x) = \sum\nolimits_{i=0}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} \big((D_{*a}^{\alpha})^i f\big)(a) + \frac{((D_{*a}^{\alpha})^{n+1} f)(\xi)}{\Gamma\big((n+1)\alpha+1\big)} \cdot (x-a)^{(n+1)\alpha} \quad (2.6)$$

With $a \le \xi \le x, \forall\, x \in (a,b]$.

**Proof.**

From (2.5), we have

$$\sum_{i=0}^{n} \big(J_{*a}^{i\alpha} (D_{*a}^{\alpha})^i f\big)(x) - \Big(J_{*a}^{(i+1)\alpha} (D_{*a}^{\alpha})^{i+1} f\Big)(x)$$

$$= \sum_{i=0}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} \big((D_{*a}^{\alpha})^i f\big)(a), \quad (2.7)$$

that is

$$f(x) - \Big(J_{*a}^{(i+1)\alpha} (D_{*a}^{\alpha})^{i+1} f\Big)(x) = \sum_{i=0}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} \big((D_{*a}^{\alpha})^i f\big)(a), \quad (2.8)$$

Applying the integral mean value theorem yields

$$\Big(J_{*a}^{(i+1)\alpha} (D_{*a}^{\alpha})^{i+1} f\Big)(x) = \frac{1}{\Gamma\big((n+1)\alpha+1\big)} \int_a^x (x-t)^{(n+1)\alpha} \,((D_{*a}^{\alpha})^{n+1} f)(t)dt,$$

$$= \frac{((D_{*a}^{\alpha})^{n+1} f)(\xi)}{\Gamma\big((n+1)\alpha+1\big)} \int_a^x (x-t)^{(n+1)\alpha}\ dt$$

$$= \frac{((D_{*a}^{\alpha})^{n+1} f)(\xi)}{\Gamma\big((n+1)\alpha+1\big)}\ .\ (x-a)^{(n+1)\alpha} \quad (2.9)$$

From (2.8) , we have

$$f(x) = \sum\nolimits_{i=0}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} \big((D_{*a}^{\alpha})^i f\big)(a) + \Big(J_{*a}^{(i+1)\alpha} (D_{*a}^{\alpha})^{i+1} f\Big)(x)$$

$$f(x) = \sum\nolimits_{i=0}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} \big((D_{*a}^{\alpha})^i f\big)(a) + \frac{((D_{*a}^{\alpha})^{n+1} f)(\xi)}{\Gamma\big((n+1)\alpha+1\big)} \cdot (x-a)^{(n+1)\alpha} \quad \text{(using (2.9))}$$

Hence, the generalized Taylor's formula is obtained. In case of $\alpha = 1$, the Caputo generalized Taylor's formula (2.6) reduces to the classical Taylor's formula.

The radius of convergence, R, for the generalized Taylor's series

$$\sum\nolimits_{i=0}^{\infty} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} \big((D_{*a}^{\alpha})^i f\big)(a) \quad (2.10)$$

depends on $f(x)$ and a, and is given by

$$R = |x-a|^{\alpha} \lim_{n\to\infty} \left| \frac{\Gamma(n\alpha+1)}{\Gamma\big((n+1)\alpha+1\big)} \cdot \frac{((D_{*a}^{\alpha})^{n+1} f)(a)}{((D_{*a}^{\alpha})^{n} f)(a)} \right| \quad (2.11)$$

**Theorem 2.4.**

Suppose that $(D_{*a}^{\alpha})^k f(x) \in C(a,b]$ for $k = 0,1,\ \ldots, n+1$, where $0 < \alpha \le 1$. If $x \in [a,b]$, then

$$f(x) \cong P^{\alpha}{}_{N}(x) = \sum\nolimits_{i=0}^{N} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} \big((D_{*a}^{\alpha})^i f\big)(a) (2.12)$$

Furthermore, there is a value ξ with $a \le \xi \le x$ so that the error term $R^{\alpha}{}_{N}(x)$ has the form:

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$$R^{\alpha}{}_N(x) = \frac{((D^{\alpha}_{*a})^{N+1} f)(\xi)}{\Gamma\big((N+1)\alpha+1\big)} \cdot (x-a)^{(N+1)\alpha} \qquad (2.13)$$

The proof follows directly from theorem 2.3. The accuracy of $P^{\alpha}{}_N(x)$ increases, when we choose large N and decrease as the value of $x$ moves away from the center $a$. Hence, we must choose N large enough so that the error does not exceed a specified bound.

In the following theorem, we find precise conditions under which the exponents hold for arbitrary fractional operators. This result is very useful on our approach for solving differential equations of fractional order.

**Theorem 2.5.**
Suppose that $f(x) = x^{\lambda} g(x)$, where $\lambda > -1$ and $g(x)$ has the generalized power series expansion $g(x) = \sum_{n=0}^{\infty} a_n \cdot (x-a)^{n\alpha}$ with radius of convergence $R > 0$, $0 < \alpha \le 1$. Then

$$D^{\gamma}_{*a} D^{\beta}_{*a} f(x) = D^{\gamma+\beta}_{*a} f(x), \qquad (2.14)$$

for all $x \in (0, R)$ if one of the following conditions is satisfied:

a) $\beta < \lambda + 1$ and α arbitrary ,
b) $\beta \ge \lambda + 1$ ,γ arbitrary, and $a_k = 0$ for $k = 0, 1, \ldots, m-1$, where $m-1 < \beta \le m$.

**Proof.**
In case of $\beta < \lambda + 1$, from the definition of Caputo fractional differential operator (1.6) and from the property (1.3), we have

$$D^{\beta}_{*a} f(x) = \sum_{n=0}^{\infty} a_n D^{\beta}_{*a} (x - x_0)^{n\alpha+\lambda}$$

$$= \sum_{n=0}^{\infty} a_n \cdot \frac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\beta+1)} (x-a)^{n\alpha+\lambda-\beta}, \qquad (2.15)$$

Since $\lambda - \beta > -1$, and

$$D^{\gamma}_{*a} D^{\beta}_{*a} f(x) = \sum_{n=0}^{\infty} a_n \cdot \frac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\beta+1)} D^{\gamma}_{*a} (x-a)^{n\alpha+\lambda-\beta}$$

$$= \sum_{n=0}^{\infty} a_n \cdot \frac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\beta+1)} \frac{\Gamma(n\alpha+\lambda-\beta+1)}{\Gamma(n\alpha+\lambda-\beta-\gamma+1)} (x-x_0)^{n\alpha+\lambda-\beta-\gamma}$$

$$= \sum_{n=0}^{\infty} a_n \cdot \frac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\beta-\gamma+1)} (x-a)^{n\alpha+\lambda-\beta-\gamma} \qquad (2.16)$$

which is precisely $D^{\gamma+\beta}_{*a} f(x)$.
For the other case, $\beta \ge \lambda + 1$, in a similar way we can prove (2.14).

## 3. GENERALIZED DIFFERENTIAL TRANSFORM METHOD

The differential transform method was first introduced by Zhou who solved linear and nonlinear initial value problems in electric circuit analysis. The differential transform method is a numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. The differential transform method is an iterative procedure for obtaining analytic Taylor series solution of ordinary or partial differential equations.

Recently, the application of differential transform method is successfully extended to obtain analytical approximate solutions to linear and nonlinear ordinary or partial differential

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equations of fractional order. The fact that the differential transform method solves nonlinear equations without using Adomian polynomials can be considered as an advantage of this method over the Adomian decomposition method.

In this section we shall derive the generalized two-dimensional differential transform method that we have developed for the numerical solution of nonlinear partial differential equations with space- and time- fractional derivatives. The proposed method is based on the combination of the classical two-dimensional differential transform method and generalized Taylor's formula.

Consider a function of two variables $u(x,y)$, and suppose that it can be represented as a product of two single-variable functions, $i.e., u(x,y) = f(x)g(y)$. Based on the properties of generalized two dimensional differential transform, the function $u(x,y)$ can be represented as

$$u(x,y) = \sum_{k=0}^{\infty} F_\alpha(k)(x-x_0)^{k\alpha} \sum_{h=0}^{\infty} G_\beta(h)(y-y_0)^{h\beta}$$

$$u(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U_{\alpha,\beta}(k,h)(x-x_0)^{k\alpha}(y-y_0)^{h\beta}, \qquad (3.1)$$

Where $0 < \alpha, \beta \leq 1$, $U_{\alpha,\beta}(k,h) = F_\alpha(k)G_\beta(h)$ is called the spectrum of $u(x,y)$.

The generalized two-dimensional differential transform of the function $u(x,y)$ is given by

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[\left(D^{\alpha}_{*x_0}\right)^k \left(D^{\beta}_{*y_0}\right)^h u(x,y)\right]_{(x_0,y_0)}, \qquad (3.2)$$

where, $\left(D^{\alpha}_{*x_0}\right)^k = D^{\alpha}_{*x_0} D^{\alpha}_{*x_0} \ . \ . \ . \ D^{\alpha}_{*x_0}, k$ −times.

In this work, the lowercase $u(x,y)$ represents the original function while the uppercase $U_{\alpha,\beta}(k,h)$ stands for the transformed function. In case of $\alpha = 1$ and β= 1 the generalized two-dimensional differential transform (3.1) reduces to the classical two-dimensional differential transform. Based on definitions (3.1)and (3.2), some basic properties of the generalized differential transform are introduced below.

**Theorem 3.1.**

Suppose that $U_{\alpha,\beta}(k,h)$, $V_{\alpha,\beta}(k,h)$and $W_{\alpha,\beta}(k,h)$ are the differential transforms of the functions $u(x,y), v(x,y)$ and $w(x,y)$, respectively. And if $u(x,y) = v(x,y) \pm w(x,y)$, then $U_{\alpha,\beta}(k,h) = V_{\alpha,\beta}(k,h) \pm W_{\alpha,\beta}(k,h)$.

**Proof**.

From the definition (3.2) we have

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[\left(D^{\alpha}_{*x_0}\right)^k \left(D^{\beta}_{*y_0}\right)^h u(x,y)\right]_{(x_0,y_0)}$$

Here, $u(x,y) = v(x,y) \pm w(x,y)$

$$\therefore \quad U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[\left(D^{\alpha}_{*x_0}\right)^k \left(D^{\beta}_{*y_0}\right)^h \left(v(x,y) \pm w(x,y)\right)\right]_{(x_0,y_0)}$$

$$= \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[\left(D^{\alpha}_{*x_0}\right)^k \left(D^{\beta}_{*y_0}\right)^h v(x,y)\right]_{(x_0,y_0)}$$

$$\pm \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[\left(D^{\alpha}_{*x_0}\right)^k \left(D^{\beta}_{*y_0}\right)^h w(x,y)\right]_{(x_0,y_0)}$$

$$= V_{\alpha,\beta}(k,h) \pm W_{\alpha,\beta}(k,h)$$

Hence, $U_{\alpha,\beta}(k,h) = V_{\alpha,\beta}(k,h) \pm W_{\alpha,\beta}(k,h)$.

**Theorem 3.2.**

Suppose that $U_{\alpha,\beta}(k,h)$, $V_{\alpha,\beta}(k,h)$and $W_{\alpha,\beta}(k,h)$ are the differential transforms of the functions $u(x,y), v(x,y)$ and $w(x,y)$, respectively. And if $u(x,y) = a\, v(x,y)$, $a \in \mathbb{R}$, then $U_{\alpha,\beta}(k,h) = a\, V_{\alpha,\beta}(k,h)$.

**Proof**.

From the definition (3.2) we have

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[\left(D_{*x_0}^{\alpha}\right)^k \left(D_{*y_0}^{\beta}\right)^h u(x,y)\right]_{(x_0,y_0)}$$

Here, $u(x,y) = a\, v(x,y)$

$$\therefore \quad U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[\left(D_{*x_0}^{\alpha}\right)^k \left(D_{*y_0}^{\beta}\right)^h a\, v(x,y)\right]_{(x_0,y_0)}$$

$$U_{\alpha,\beta}(k,h) = a\ \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[\left(D_{*x_0}^{\alpha}\right)^k \left(D_{*y_0}^{\beta}\right)^h v(x,y)\right]_{(x_0,y_0)}$$

$$= a\, V_{\alpha,\beta}(k,h)$$

Hence, $U_{\alpha,\beta}(k,h) = a\, V_{\alpha,\beta}(k,h)$.

**Theorem 3.3**

Suppose that $U_{\alpha,\beta}(k,h)$, $V_{\alpha,\beta}(k,h)$and $W_{\alpha,\beta}(k,h)$ are the differential transforms of the functions $u(x,y), v(x,y)$ and $w(x,y)$, respectively. And if $u(x,y) = v(x,y)w(x,y)$,then

$$U_{\alpha,\beta}(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} V_{\alpha,\beta}\,(r, h-s) W_{\alpha,\beta}(k-r, s)$$

**Proof**.

From the definition (3.2) we have

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[\left(D_{*x_0}^{\alpha}\right)^k \left(D_{*y_0}^{\beta}\right)^h u(x,y)\right]_{(x_0,y_0)}$$

Here,

$$u(x,y) = v(x,y)w(x,y)$$

$$\therefore \quad U_{\alpha,\beta}(k,h)$$
$$= \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[\left(D_{*x_0}^{\alpha}\right)^k \left(D_{*y_0}^{\beta}\right)^h \left(v(x,y)w(x,y)\right)\right]_{(x_0,y_0)}$$

From (3.1), we have

$$u(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U_{\alpha,\beta}\,(k,h)(x-x_0)^{k\alpha}(y-y_0)^{h\beta}$$

$$= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} V_{\alpha,\beta}\,(k,h)(x-x_0)^{k\alpha}(y-y_0)^{h\beta} \times \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W_{\alpha,\beta}\,(k,h)(x-x_0)^{k\alpha}(y-y_0)^{h\beta}$$

$$= \left(V_{\alpha,\beta}(0{,}0) + V_{\alpha,\beta}(1{,}1)(x-x_0)^{\alpha}(y-y_0)^{\beta} + V_{\alpha,\beta}(2{,}2)(x-x_0)^{2\alpha}(y-y_0)^{2\beta} + ...\right)$$
$$\times \left(W_{\alpha,\beta}(0{,}0) + W_{\alpha,\beta}(1{,}1)(x-x_0)^{\alpha}(y-y_0)^{\beta}\right.$$
$$\left. + W_{\alpha,\beta}(2{,}2)(x-x_0)^{2\alpha}(y-y_0)^{2\beta} + ...\right)$$

$$
\begin{aligned}
= V_{\alpha,\beta}(0,0)W_{\alpha,\beta}(0,0) + V_{\alpha,\beta}(0,0)W_{\alpha,\beta}(1,1)(x-x_0)^{\alpha}(y-y_0)^{\beta} \\
+ V_{\alpha,\beta}(0,0)\, W_{\alpha,\beta}(2,2)(x-x_0)^{2\alpha}(y-y_0)^{2\beta} \\
+ V_{\alpha,\beta}(1,1)W_{\alpha,\beta}(0,0)(x-x_0)^{\alpha}(y-y_0)^{\beta} \\
+ V_{\alpha,\beta}(1,1)W_{\alpha,\beta}(1,1)(x-x_0)^{2\alpha}(y-y_0)^{2\beta} \\
+ V_{\alpha,\beta}(1,1)W_{\alpha,\beta}(2,2)(x-x_0)^{3\alpha}(y-y_0)^{3\beta} \\
+ V_{\alpha,\beta}(2,2)W_{\alpha,\beta}(0,0)(x-x_0)^{2\alpha}(y-y_0)^{2\beta} \\
+ V_{\alpha,\beta}(2,2)W_{\alpha,\beta}(1,1)(x-x_0)^{3\alpha}(y-y_0)^{3\beta} \\
+ V_{\alpha,\beta}(2,2)W_{\alpha,\beta}(2,2)(x-x_0)^{4\alpha}(y-y_0)^{4\beta} + \ldots
\end{aligned}
$$

$$
\begin{aligned}
= V_{\alpha,\beta}(0,0)W_{\alpha,\beta}(0,0) + \left[V_{\alpha,\beta}(0,0)W_{\alpha,\beta}(1,1) + V_{\alpha,\beta}(1,1)W_{\alpha,\beta}(0,0)\right](x-x_0)^{\alpha}(y-y_0)^{\beta} \\
+ \left[V_{\alpha,\beta}(0,0)\, W_{\alpha,\beta}(2,2) + V_{\alpha,\beta}(1,1)W_{\alpha,\beta}(1,1)\right. \\
\left. + V_{\alpha,\beta}(2,2)W_{\alpha,\beta}(0,0)\right](x-x_0)^{2\alpha}(y-y_0)^{2\beta} + \ldots
\end{aligned}
$$

$$u(x,y) = \sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{h=0}^{\infty}\sum_{s=0}^{h} V_{\alpha,\beta}(r,h-s)W_{\alpha,\beta}(k-r,s)\,(x-x_0)^{k\alpha}(y-y_0)^{h\beta}$$

$$\Rightarrow \frac{u(x,y)}{\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}(x-x_0)^{k\alpha}(y-y_0)^{h\beta}} = \sum_{r=0}^{k}\sum_{s=0}^{h} V_{\alpha,\beta}\,(r,h-s)W_{\alpha,\beta}(k-r,s)$$

From the equation (3.1), we have

$$\frac{u(x,y)}{\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}(x-x_0)^{k\alpha}(y-y_0)^{h\beta}} = U_{\alpha,\beta}(k,h)$$

Hence, $$U_{\alpha,\beta}(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} V_{\alpha,\beta}\,(r,h-s)W_{\alpha,\beta}(k-r,s).$$

**Theorem 3.4.**

Suppose that $U_{\alpha,\beta}(k,h)$ is the differential transform of the function $u(x,y)$ and if $(x,y) = (x-x_0)^{n\alpha}(y-y_0)^{m\beta}$, then $U_{\alpha,\beta}(k,h) = \delta(k-n)\delta(h-m)$

**Proof**.

From the definition (3.2), $u(x,y) = (x-x_0)^{n\alpha}(y-y_0)^{m\beta}$ can be written as

$$u(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \delta(k-n)\delta(h-m)\,(x-x_0)^{k\alpha}(y-y_0)^{h\beta}$$

$$\Rightarrow \frac{u(x,y)}{\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}(x-x_0)^{k\alpha}(y-y_0)^{h\beta}} = \delta(k-n)\delta(h-m)$$

So, from the inverse differential transform (3.1), we get

$$\frac{u(x,y)}{\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}(x-x_0)^{k\alpha}(y-y_0)^{h\beta}} = U_{\alpha,\beta}(k,h)$$

$$\therefore\ U_{\alpha,\beta}(k,h) = \delta(k-n)\delta(h-m)$$

Hence, the result is obtained.

**Theorem 3.5.**

If $u(x,y) = D_{*x_0}^{\alpha} v(x,y),\ 0 < \alpha \le 1$, then

$$U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}\, V_{\alpha,\beta}(k+1,h) \quad (3.3)$$

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**Proof.**

From the definition (3.2) we have

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[\left(D^{\alpha}_{*x_0}\right)^k \left(D^{\beta}_{*y_0}\right)^h u(x,y)\right]_{(x_0,y_0)}$$

Here, $u(x,y) = D^{\alpha}_{*x_0} v(x,y)$

$$\therefore \quad U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[\left(D^{\alpha}_{*x_0}\right)^k \left(D^{\beta}_{*y_0}\right)^h D^{\alpha}_{*x_0} v(x,y)\right]_{(x_0,y_0)}$$

$$= \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[\left(D^{\alpha}_{*x_0}\right)^{k+1} \left(D^{\beta}_{*y_0}\right)^h v(x,y)\right]_{(x_0,y_0)}$$

$$= \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)\Gamma(\beta h+1)\Gamma(\alpha(k+1)+1)}\left[\left(D^{\alpha}_{*x_0}\right)^{k+1} \left(D^{\beta}_{*y_0}\right)^h v(x,y)\right]_{(x_0,y_0)}$$

$$= \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} V_{\alpha,\beta}(k+1,h)$$

$$\therefore \quad U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} V_{\alpha,\beta}(k+1,h)$$

**Theorem 3.6.**

If $u(x,y) = f(x)g(y)$ and the function $f(x) = x^{\lambda}h(x)$, where $\lambda > -1$ and $h(x)$ has the generalized Taylor series expansion

$$h(x) = \sum_{n=0}^{\infty} a_n \,.\, (x-x_0)^{\alpha k},$$

and if one of the following conditions is satisfied:

a) $\beta < \lambda + 1$ and α arbitrary ,
b) $\beta \geq \lambda + 1$ , α arbitrary, and $a_n = 0$ for n= $0, 1, \ldots, m-1$, where $m-1 < \beta \leq m$.

Then the generalized differential transform (3.2) becomes

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[D^{\alpha k}_{*x_0} \left(D^{\beta}_{*y_0}\right)^h u(x,y)\right]_{(x_0,y_0)} \quad (3.4)$$

**Proof**.

From the definition (3.2) we have

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[\left(D^{\alpha}_{*x_0}\right)^k \left(D^{\beta}_{*y_0}\right)^h u(x,y)\right]_{(x_0,y_0)}$$

where, $\left(D^{\alpha}_{*x_0}\right)^k = D^{\alpha}_{*x_0} D^{\alpha}_{*x_0} \ldots D^{\alpha}_{*x_0}, k$ −times.

Now, if the function $u(x,y)$ satisfies the conditions given in theorem 2.5, that are $f(x) = x^{\lambda}h(x)$, where $\lambda > -1$ and $h(x)$ has the generalized Taylor series expansion $h(x) = \sum_{n=0}^{\infty} a_n \,.\, (x-x_0)^{\alpha k}, 0 < \alpha \leq 1$, and $(a)$ $\beta < \lambda + 1$ and α arbitrary , or $(b)$ $\beta \geq \lambda + 1$ , α arbitrary, and $a_n = 0$ for n= $0, 1, \ldots, m-1$, where $m-1 < \beta \leq m$.

Then $\quad D^{\gamma}_{*a} D^{\beta}_{*a} f(x) = D^{\gamma+\beta}_{*a} f(x)$

So we get, $\left(D^{\alpha}_{*x_0}\right)^k = D^{\alpha}_{*x_0} D^{\alpha}_{*x_0} \ldots D^{\alpha}_{*x_0}, k$ − times $= \left(D^{k\alpha}_{*x_0}\right)$.

∴ the generalized differential transform (3.2) becomes

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[D^{\alpha k}_{*x_0} \left(D^{\beta}_{*y_0}\right)^h u(x,y)\right]_{(x_0,y_0)}$$

Hence, the equation (3.4) is obtained.

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**theorem 3.7.**

If $u(x,y) = D_{*x_0}^{\gamma} v(x,y)$, $m-1<\gamma\le m$, and $u(x,y) = f(x)g(y)$, the function $f(x)$ satisfies the conditions given in Theorem (2.5), then

$$U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha k+\gamma+1)}{\Gamma(\alpha k+1)} V_{\alpha,\beta}(k+\gamma/\alpha, h) \quad (3.5)$$

**Proof.**

Using Theorem 3.6, we have,

$$\begin{aligned} U_{\alpha,\beta}(k,h) &= \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[\left(D_{*x_0}^{\alpha k}\right)\left(D_{*y_0}^{\beta}\right)^h D_{*x_0}^{\gamma} v(x,y)\right]_{(x_0,y_0)} \\ &= \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[\left(D_{*x_0}^{\alpha k+\gamma}\right)\left(D_{*y_0}^{\beta}\right)^h v(x,y)\right]_{(x_0,y_0)} \\ &= \frac{\Gamma(\alpha k+\gamma+1)}{\Gamma(\alpha k+1)\Gamma(\beta h+1)\Gamma(\alpha k+\gamma+1)}\left[\left(D_{*x_0}^{\alpha k+\gamma}\right)\left(D_{*y_0}^{\beta}\right)^h v(x,y)\right]_{(x_0,y_0)} \\ &= \frac{\Gamma(\alpha k+\gamma+1)}{\Gamma(\alpha k+1)} V_{\alpha,\beta}(k+\gamma/\alpha, h) \end{aligned}$$

$$\therefore\ U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha k+\gamma+1)}{\Gamma(\alpha k+1)} V_{\alpha,\beta}(k+\gamma/\alpha, h)$$

**Remark .**

Now, if the function $u(x,y) = f(x)g(y)$, $f(x)$ and $g(y)$ satisfies the condition given in Theorem (2.5), then the generalized differential transform (3.2) becomes

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[\left(D_{*x_0}^{\alpha k}\right)\left(D_{*y_0}^{\beta h}\right) u(x,y)\right]_{(x_0,y_0)} \quad (3.6)$$

Therefore, in this case, if $u(x,y) = D_{*x_0}^{\gamma} D_{*y_0}^{\mu} v(x,y)$, where $m-1<\gamma\le m$, $n-1<\mu\le n$ and the functions $f(x)$ and $g(y)$ satisfies the condition given in Theorem (2.5), then we have the following result:

$$U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha k+\gamma+1)}{\Gamma(\alpha k+1)}\frac{\Gamma(\beta h+\mu+1)}{\Gamma(\alpha k+1)} V_{\alpha,\beta}(k+\gamma/\alpha, h+\mu/\beta) \quad (3.7)$$

## 4. ANALYSIS OF THE METHOD

In this chapter, we shall use the analysis presented in the previous section to construct our numerical method for solving the following nonlinear partial differential equation with space- and time-fractional derivatives

$$\frac{\partial^{\mu} u}{\partial t^{\mu}} = \frac{\partial^{\nu} u}{\partial x^{\nu}} + N_f\big(u(x,t)\big), \quad m-1<\nu\le m,\ n-1<\mu\le n,\ n,m\in N, \quad (4.1)$$

where, μ and ν are parameters describing the order of the fractional time- and space-derivatives in the Caputo sense, respectively, and $N_f$ is a nonlinear operator which might include other fractional derivatives with respect to the variables $x$ and $t$. The function $u(x,t)$ is assumed to be a casual function of time and space, i.e., vanishing for $t<0$ and $x<0$. The general response expression contains parameters describing the order of the fractional derivatives that can be varied to obtain various responses.

First, if $0<\mu\le 1$ and $0<\nu\le 1$, we suppose that the solution of the nonlinear equation (4.1) can be written as a product of single-valued functions. In this case, selecting $\alpha=\mu$, $\beta=\nu$ and applying theorem 2.4 to both sides of equation(4.1), it transform to

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} U_{\alpha,\beta}(k,h+1) = \frac{\Gamma(\beta(k+1)+1)}{\Gamma(\beta k+1)} U_{\alpha,\beta}(k+1,h) + F_{\alpha,\beta}(k,h), \quad (4.2)$$

where, $F_{\alpha,\beta}(k,h)$ is the generalized differential transformation of $N_f\big(u(x,t)\big)$.

Second, if , $m-1<\mu=m_1/m_2\le m$ and $0<\nu\le 1$, we suppose that the solution of the nonlinear equation (4.1) can be written as a product of single-valued functions

$u(x,t) = v(x)w(t)$, where the function $w(t)$ satisfies the condition given in theorem 2.5. In this case, selecting $\alpha = 1/m_2$, $\beta = v$, and applying theorem 2.4 to both sides of equation(4.1), it transform to

$$\frac{\Gamma(\alpha(h+1)+m_1)}{\Gamma(\alpha h+1)} U_{\alpha,\beta}(k,h+m_1) = \frac{\Gamma(\beta(k+1)+1)}{\Gamma(\beta k+1)} U_{\alpha,\beta}(k+1,h) + F_{\alpha,\beta}(k,h), \quad (4.3)$$

Finally, if $m-1 < \mu = m_1/m_2 \leq m$and $n-1 < v = n_1/n_2 \leq n$, we suppose that the solution of the nonlinear equation (4.1) can be written as a product of single-valued functions $u(x,t) = v(x)w(t)$, where the function $v(x)$ and $w(t)$ satisfy the condition given in theorem 2.5.

In this case, selecting $\alpha = 1/m_2$, $\beta = 1/m_2$, and applying theorem 2.4 to both sides of equation(4.1), it transform to

$$\frac{\Gamma(\alpha(h+1)+m_1)}{\Gamma(\alpha h+1)} U_{\alpha,\beta}(k,h+m_1) = \frac{\Gamma(\beta(k+1)+n_1)}{\Gamma(\beta k+1)} U_{\alpha,\beta}(k+n_1,h) + F_{\alpha,\beta}(k,h), \quad (4.4)$$

In all the above cases, the solution of the nonlinear space and time-fractional equation (4.1), using the definition (3.1) can be written as

$$u(x,y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U_{\alpha,\beta}(k,h)(x)^{k\alpha}(y)^{h\beta}. \quad (4.5)$$

## 5. APPLICATIONS AND RESULTS

In this chapter we consider a few examples that demonstrate the performance and efficiency of the generalized differential transform method for solving nonlinear partial differential equations with time- or space-fractional derivatives.

**Example 5. 1.**

Consider the following nonlinear time-fractional equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + u(x,t)u_x(x,t) = x + xt^2, \qquad t > 0, \quad 0 < \alpha \leq 1, \quad (5.1)$$

subject to the initial condition

$$u(x,0) = 0. \quad (5.2)$$

Suppose that the solution $u(x,t)$ can be represented as a product of single-valued functions, $u(x,t) = v(x)w(t)$. Selecting $\beta = 1$ and applying equation(4.2), the recurrence relation for the time-fractional equation (5.1) is given by

$$U_{\alpha,1}(k,h+1) = \frac{\Gamma(\alpha h+1)}{\Gamma(\alpha(h+1)+1)} \left\{ \delta(k-1)\delta(h) + \delta(k-1)\delta(h-2) - \sum_{r=0}^{k} \sum_{s=0}^{h} U_{\alpha,1}(r,h-s)(k-r+1)U_{\alpha,1}(k-r+1,s) \right\} \quad (5.3)$$

The generalized two-dimensional differential transform of the initial condition (5.2) is

$$U_{\alpha,1}(k,0) = 0. \quad (5.4)$$

Utilizing the recurrence relation (5.3) and the transformed initial condition(5.4), we get

$$U_{\alpha,1}(1,1) = \frac{1}{\Gamma(\alpha+1)}, \qquad U_{\alpha,1}(1,3) = \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} \left(1 - \frac{1}{(\Gamma(\alpha+1))^2}\right),$$

$$U_{\alpha,1}(1,5) = -\frac{2}{\Gamma(\alpha+1)} \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} \frac{\Gamma(4\alpha+1)}{\Gamma(4\alpha+1)} \left(1 - \frac{1}{(\Gamma(\alpha+1))^2}\right),$$

and the other coefficients equal zero for $k, h \leq 5$. Therefore, from (3.1), the approximate solution of the nonlinear equation (5.1) can be derived as

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$$u(x,t) = x\left[\frac{1}{\Gamma(\alpha+1)}t^{\alpha} + \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left(1-\frac{1}{\left(\Gamma(\alpha+1)\right)^2}\right)t^{3\alpha} - \frac{2}{\Gamma(\alpha+1)}\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\frac{\Gamma(4\alpha+1)}{\Gamma(4\alpha+1)}\left(1-\frac{1}{\left(\Gamma(\alpha+1)\right)^2}\right)t^{5\alpha}\right]. \quad (5.5)$$

Table 1 shows the approximate solutions for equation (5.1) obtained for different values of α using the generalized differential transform method, for $k, h \leq 5$. The values of $\alpha = 1$ is the only case for which we know the exact solution $u(x,t) = xt$ and our approximate solution using the generalized differential transform method is more accurate than the approximate solution obtained [19] using Homotopy perturbation method.

**Table 1:**Numerical values when $\alpha = 0.5,\ 0.75,\ 1.0$ for equation (5.1)

| t | x | $\alpha = 0.5$ | | $\alpha = 0.75$ | | $\alpha = 1.0$ | | $u_{\text{Exact}}$ |
|---|---|---|---|---|---|---|---|---|
| | | $u_{\text{GDTM}}$ | $u_{\text{HPM}}$ | $u_{\text{GDTM}}$ | $u_{\text{HPM}}$ | $u_{\text{GDTM}}$ | $u_{\text{HPM}}$ | |
| 0.2 | 0.25 | 0.1236 | 0.1046 | 0.0808 | 0.0783 | 0.0500 | 0.0500 | 0.0500 |
| | 0.50 | 0.2473 | 0.2091 | 0.1617 | 0.1566 | 0.1000 | 0.0100 | 0.1000 |
| | 0.75 | 0.3709 | 0.3137 | 0.2426 | 0.2349 | 0.1500 | 0.1500 | 0.1500 |
| | 1.00 | 0.4945 | 0.4183 | 0.3233 | 0.3132 | 0.2000 | 0.1999 | 0.2000 |
| 0.4 | 0.25 | 0.1771 | 0.1772 | 0.1354 | 0.1368 | 0.1000 | 0.0996 | 0.1000 |
| | 0.50 | 0.3543 | 0.3545 | 0.2709 | 0.2736 | 0.2000 | 0.1993 | 0.2000 |
| | 0.75 | 0.5314 | 0.5317 | 0.4063 | 0.4104 | 0.3000 | 0.2989 | 0.3000 |
| | 1.00 | 0.7086 | 0.7089 | 0.5418 | 0.5472 | 0.4000 | 0.3986 | 0.4000 |
| 0.6 | 0.25 | 0.2270 | 0.2305 | 0.1856 | 0.1851 | 0.1500 | 0.1471 | 0.1500 |
| | 0.50 | 0.4539 | 0.4610 | 0.3711 | 0.3703 | 0.3008 | 0.2943 | 0.3000 |
| | 0.75 | 0.6809 | 0.6915 | 0.5566 | 0.5554 | 0.4500 | 0.4414 | 0.4500 |
| | 1.00 | 0.9079 | 0.9220 | 0.7421 | 0.7406 | 0.6000 | 0.5886 | 0.6000 |

**Example 5.2**

Consider the following nonlinear space-fractional Fisher's equation

$$\frac{\partial u}{\partial t} - \frac{\partial^{1.5} u}{\partial x^{1.5}} - u(x,t)\left(1-u(x,t)\right) = x^2, \qquad x > 0, \qquad (5.6)$$

subject to the initial condition

$$u(x,0) = x. \qquad (5.7)$$

The fractional space derivatives are used to model anomalous diffusion or dispersion, where a particle plume spreads at a rate inconsistent with the classical Brownian motion model, and the plume may be asymmetric. When a fractional derivative replaces the second derivative in a diffusion model or dispersion model, it leads to enhanced diffusion (also called super diffusion).

Suppose that the solution $u(x,t)$ can be represented as a product of single-valued functions, $u(x,t) = v(x)w(t)$ where the function $v(x)$ satisfies the conditions given in theorem 2.5. Selecting $\alpha = 1,\ \beta = 0.5$, and applying equation(4.3), the recurrence relation for the space-fractional Fisher's equation (5.6) is given by

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$$U_{1,1/2}(k,h+1) = \frac{1}{h+1}\left\{\frac{\Gamma(k/2+2)}{\Gamma(k/2+1)}U_{1,1/2}(k+3,h) + \sum_{r=0}^{k}\sum_{s=0}^{h}U_{1,1/2}(r,h-s)\left(\delta(k-r)\delta(s)-U_{1,1/2}(k-r,s)\right)\right\} + \delta(h)\delta(k-4). \quad (5.8)$$

The generalized two-dimensional differential transform of the initial condition (5.7) is

$$U_{1,1/2}(k,0) = \delta(k-2)(5.9)$$

**Table 2**: The first components of $U_{1,1/2}(k,h)$ for equation (5.6)

| | $\mathbf{U_{1,1/2}(k,0)}$ | $\mathbf{U_{1,1/2}(k,1)}$ | $\mathbf{U_{1,1/2}(k,2)}$ | $\mathbf{U_{1,1/2}(k,3)}$ | $\mathbf{U_{1,1/2}(k,4)}$ |
|---|---|---|---|---|---|
| $\mathbf{U_{1,1/2}(0,h)}$ | 0 | 0 | 0 | 0 | 0 |
| $\mathbf{U_{1,1/2}(1,h)}$ | 0 | 0 | 0 | $-\frac{\Gamma(5/2)}{3\Gamma(3/2)}$ | $-\frac{\Gamma(5/2)}{3\Gamma(3/2)}$ |
| $\mathbf{U_{1,1/2}(2,h)}$ | 1 | 1 | $\frac{1}{2}$ | $\frac{1}{6}$ | $\frac{1}{24}$ |
| $\mathbf{U_{1,1/2}(3,h)}$ | 0 | 0 | 0 | 0 | $\frac{\Gamma(5/2)}{6\Gamma(3/2)}$ |
| $\mathbf{U_{1,1/2}(4,h)}$ | 0 | 0 | $-1$ | $-1$ | $-\frac{7}{12}$ |

Utilizing the recurrence relation (5.8) and the transformed initial condition(5.9), the first few components of $U_{1,1/2}(k,h)$ are calculated and given in Table 2.

Therefore, the approximate solution, for $k,h \le 4$ of the nonlinear space-fractional Fisher's equation (5.6) can be derived as

$$u(x,t) = -\frac{\Gamma(5/2)}{3\Gamma(3/2)}(t^3+t^4)x^{1/2} + \left(1+t+\frac{1}{2}t^2+\frac{1}{6}t^3+\frac{1}{24}t^4\right)x + \frac{\Gamma(5/2)}{6\Gamma(3/2)}t^4x^{3/2} - \left(t^2+t^3+\frac{7}{12}t^4\right)x^2 . \quad (5.10)$$

In order to make comparison with Variational Iteration Method, we follow the analysis given in [23] and solve the nonlinear space-fractional equation (5.6) by using Variational Iteration Method to obtain the following fourth-order approximate solution.

$$u(x,t) = x + xt + (x-2x^2)\frac{t^2}{2} + \left(\frac{1}{2}x - 3x^2 + 2x^3 - \frac{2}{\Gamma(3/2)}x^{1/2}\right)\frac{t^3}{3} - \left(\frac{4}{3}x^2 - \frac{8}{3}x^3 + \frac{2}{3\Gamma(3/2)}x^{1/2}\right)\frac{t^4}{4} - \left(\frac{1}{4}x^2 - \frac{5}{3}x^3 + x^4\right)\frac{t^5}{5} + \left(\frac{1}{3}x^3 - \frac{2}{3}x^4\right)\frac{t^6}{6} - \frac{x^4}{9}\frac{t^7}{7} . \quad (5.11)$$

**Table 3**: Numerical values for equation (5.6)

| x | $t = 0.1$ | | $t = 0.2$ | | $t = 0.3$ | | $t = 0.4$ | |
|---|---|---|---|---|---|---|---|---|
| | $u_{GDTM}$ | $u_{VIM}$ | $u_{GDTM}$ | $u_{VIM}$ | $u_{GDTM}$ | $u_{VIM}$ | $u_{GDTM}$ | $u_{VIM}$ |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.1 | 0.1102 | 0.1104 | 0.1201 | 0.1216 | 0.1283 | 0.1333 | 0.1328 | 0.1453 |
| 0.2 | 0.2203 | 0.2206 | 0.2402 | 0.2422 | 0.2574 | 0.2646 | 0.2693 | 0.2871 |
| 0.3 | 0.3303 | 0.3306 | 0.3595 | 0.3621 | 0.3847 | 0.3938 | 0.4025 | 0.4252 |
| 0.4 | 0.4400 | 0.4403 | 0.4778 | 0.4810 | 0.5099 | 0.5210 | 0.5317 | 0.5594 |
| 0.5 | 0.5494 | 0.5499 | 0.5952 | 0.5989 | 0.6328 | 0.6461 | 0.6567 | 0.6899 |
| 0.6 | 0.6587 | 0.6592 | 0.7117 | 0.7160 | 0.7534 | 0.7690 | 0.7773 | 0.8164 |
| 0.7 | 0.7678 | 0.7683 | 0.8272 | 0.8321 | 0.8717 | 0.8894 | 0.8934 | 0.9383 |
| 0.8 | 0.8766 | 0.8772 | 0.9418 | 0.9471 | 0.9877 | 1.0071 | 1.0050 | 1.0547 |
| 0.9 | 0.9852 | 0.9858 | 1.0554 | 1.0607 | 1.1013 | 1.1214 | 1.1120 | 1.1644 |
| 1.0 | 1.0936 | 1.0941 | 1.1681 | 1.1729 | 1.2125 | 1.2315 | 1.2144 | 1.2655 |

Table 3 shows the approximate solutions for equation (5.6) using the Generalized Differential Transform Method, for $k, h \leq 4$, and the Variational Iteration Method. From the numerical results in table 3, it is conclude that the approximate solution obtained using the Generalized Differential Transform Method is good agreement with the approximate solution obtained using the Variational Iteration Method for all values of *x* and *t*.

**Example 5.3**

Consider the following nonlinear space-and time-fractional hyperbolic equation

$$\frac{\partial^{\gamma} u}{\partial t^{\gamma}} = \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial^{\beta} u}{\partial x^{\beta}}\right), \quad t > 0, \quad x > 0, \qquad (5.12)$$

Where $0 < \beta \leq 1$ and $1 < \gamma = m_1/m_2 \leq 2$, subject to the initial conditions

$$u(x,0) = x^{2\beta}, \qquad u_t(x,0) = -2\,x^{2\beta}\,. \qquad (5.13)$$

The space-time-fractional diffusion equation (5.12), in which the dispersive flux is described by a fractional space derivative, has been applied to modeling the anomalous or super diffusion of solutes observed in heterogeneous porous media.

Equation (5.12) is a time-fractional version of the advection dispersion equation solved in [28] via particle tracking methods.

Suppose that the solution $u(x,t)$ can be represented as a product of single-valued functions, $u(x,t) = v(x)w(t)$, where the function $w(t)$ satisfies the conditions given in theorem 2.5 . According to the equation(4.3), selecting $\alpha = 1/m_2$, the recurrence relation for the space-time and fractional hyperbolic equation (5.12) is given by

$$\frac{\Gamma(\alpha h + \gamma + 1)}{\Gamma(\alpha h + 1)} U_{1/2,\beta}(k, h + m_1)$$
$$= (k+1)\sum_{r=0}^{k+1}\sum_{s=0}^{h}\frac{\Gamma(\beta(k-r+2)+1)}{\Gamma(\beta(k-r+1)+1)} U_{1/2,\beta}(r, h-s) U_{1/2,\beta}(k-r+2, s) \quad (5.14)$$

In case of $\gamma = 2$and $m_1 = 2$, the generalized two-dimensional differential transform method of the initial conditions (5.13) are

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$U_{1/2,\beta}(k,0) = \delta(k-2),$ (5.15) $U_{1/2,\beta}(k,1) = -2\delta(k-2),$ (5.16)

Utilizing the recurrence relation (5.14) and the transformed initial condition (5.15)and (5.16), we get

$$\begin{aligned}
U_{1/2,\beta}(2,0) &= 1,\\
U_{1/2,\beta}(2,1) &= -2,\\
U_{1/2,\beta}(2,2) &= \frac{3}{2}\frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)},\\
U_{1/2,\beta}(2,3) &= -2\frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)},\\
U_{1/2,\beta}(2,4) &= \frac{\Gamma(2\beta+1)}{4\Gamma(\beta+1)}\left(\frac{3\Gamma(2\beta+1)}{\Gamma(\beta+1)}+4\right),\\
U_{1/2,\beta}(2,5) &= -\frac{3}{2}\frac{\Gamma(2\beta+1)^2}{\Gamma(\beta+1)^2}
\end{aligned}$$

and the other coefficients equal zero for $k,h \le 5$. Therefore, from(3.1), the approximate solution of the nonlinear space and time fractional hyperbolic equation (5.12) can be derived as

$$u(x,t) = \left[1 - 2t + \frac{3\Gamma(2\beta+1)}{2\Gamma(\beta+1)}t^2 - 2\frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)}t^3 + \frac{\Gamma(2\beta+1)}{4\Gamma(\beta+1)}\left(\frac{3\Gamma(2\beta+1)}{\Gamma(\beta+1)}+4\right)t^4 - \frac{3\Gamma(2\beta+1)^2}{2\Gamma(\beta+1)^2}t^5\right]x^{2\beta}. \quad (5.17)$$

**Table 4:** Numerical values when $\alpha = 2$ and $\beta = 1$for equation(5.12)

| $t$ | $x$ | $u_{GDTM}$ | $u_{ADM}$ | $u_{VIM}$ | $u_{Exact}$ |
|---|---|---|---|---|---|
| 0.2 | 0.25 | 0.0434 | 0.04340 | 0.0434 | 0.0434 |
| | 0.50 | 0.1736 | 0.1736 | 0.1736 | 0.1736 |
| | 0.75 | 0.3906 | 0.3906 | 0.3906 | 0.3906 |
| | 1.00 | 0.6944 | 0.6943 | 0.6945 | 0.6944 |
| 0.4 | 0.25 | 0.0319 | 0.0316 | 0.0318 | 0.0319 |
| | 0.50 | 0.1276 | 0.1263 | 0.1271 | 0.1275 |
| | 0.75 | 0.2870 | 0.2841 | 0.2860 | 0.2870 |
| | 1.00 | 0.5102 | 0.5051 | 0.5085 | 0.5085 |
| 0.6 | 0.25 | 0.0244 | 0.0220 | 0.0237 | 0.0244 |
| | 0.50 | 0.0977 | 0.0880 | 0.0947 | 0.0977 |
| | 0.75 | 0.2199 | 0.1980 | 0.2130 | 0.2198 |
| | 1.00 | 0.3909 | 0.3521 | 0.3786 | 0.3906 |

Table 4 shows the approximate solutions for equation (5.12) obtained for $\alpha = 2$ and $\beta = 1$ using the generalized differential transform method, for $k,h \le 20$.

Equation(5.12), for $\beta = 1$, is solved in [22] using the variational iteration method and Adomian decomposition method. The values of $\alpha = 2$ and $\beta = 1$ is the only case for which we know the exact solution $u(x,t) = (x/(t+1))^2$ and our approximate solution using the generalized differential transform method is high agreement with the approximate solution obtained in [22] using the variational iteration method and Adomian decomposition method.

**6. CONCLUSIONS**

This present analysis exhibits the applicability of the differential transform method to solve the nonlinear partial differential equations of fractional order with space- and time-fractional derivatives. The algorithm is based on the two-dimensional differential transform method, generalized Taylor's formula and Caputo fractional derivative. For illustration purpose, we considered three examples. The work emphasized our belief that the method is a reliable technique to handle nonlinear partial fractional differential equations. It provides the solutions with easily computable components in a direct way without using linearization, perturbation or restrictive assumptions. The results of this method are in good agreement with those obtained by using the Homotopy perturbation method, variational iteration method and Adomian decomposition method. As an advantage of this method over the Adomian decomposition method, in this method we do not need to do the difficult computation for finding the Adomian polynomials. A clear conclusion can be drawn from the results that the generalized differential transform method is promising and applicable to a broad class of nonlinear problems in the theory of fractional calculus.

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