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Quantum Mechanical Behaviour of Fractional Harmonic Oscillator in the Time Independent Domain

Mintu Debnath *

Abstract

Quantum mechanical harmonic oscillators have been used to explain different complex phenomenon in physics. In this paper, fractional dimensional harmonic oscillator has been studied. We have considered nonlinear variation of potential with space. The time independent fractional Schrodinger equation has been investigated. Solutions of the fractional Schrodinger equation for different plausible fractional parameter are also critically examined.

Keywords:

Harmonic oscillator;
Schrodinger equation;
Fractional potential;
Eigenvalue;
Normalised wavefunction.

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1. Introduction

Harmonic oscillator in quantum mechanics is a very important topic for researcher in recent years. It has very wide applications like molecular spectroscopy, radiation phenomena, structure of solids and so on. But in many instances force law involves nonlinear function of space coordinates in physical realm. Hence the potential varies as fractional power of space coordinates. So, conventional methods of solution for harmonic oscillator problem are not appropriate in such cases. Regarding this problem, fractional contribution of space coordinate has satisfactorily been applied as an effective description of restraint in some low-dimensional problem [1]. Laskin has developed fractional Schrödinger equation originated from fractional path integral [2], [3]. Xu, Dong and Guo discussed the method of solution of Schrodinger equation for different cases, such as barrier penetration problems, infinite potential well and other physical applications [4], [5]. Laskin solved the fractional Schrodinger equation for single dimensional oscillator using semi-classical approaches [6]. Herrman studied fractional form of Schrodinger equation with potential well of infinite depth [7]. Analytical and numerical method is used to solve fractional Schrodinger equation with Riemann-Liouville operator by Ibrahim and Jalab [8]. In all above mentioned papers the authors concerned with second order derivative dealing the kinetic energy. They replaced the second order derivative to a fractional order derivative and derive the effect on the eigenfunction and energy eigenvalues [9].

The purpose of this paper is to find out the solutions of Schrödinger equation for one dimensional fractional harmonic oscillator. Here the order of the derivative representing the kinetic energy kept identical. The nonlinearity of force and corresponding fractional potential are taken into account as the evolution of the problem. The suitable stable wave functions have been found out for fractional potential by numerical solution of time independent Schrödinger equation.

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2. Method and analysis

Time independent Schrödinger equation in one dimension is given by:

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2}+V(x)\Psi(x)=E\Psi(x) \tag{1}$$

Where m is the mass of the particle, $V(x)$ is the potential function, $\Psi(x)$ is the wavefunction, E is the energy of the oscillator system. For physical oscillator, we take restoring force, $F=-sx^{\alpha}$ and potential, $V(x)=kx^{\beta}$, where s, k, α and β(= α + 1) are constants. Then equation (1) reduces to

$$\frac{d^2\Psi(x)}{dx^2}=-\frac{2m}{\hbar^2}\left(E-\frac{1}{2}kx^{\beta}\right)\Psi(x) \tag{2}$$

If β = 2, the quantized energy of the oscillator is given by

$$E_n=\hbar\omega\left(\mathrm{n}+\frac{1}{2}\right), \qquad \mathrm{n}=0,1,2,\ldots \tag{3}$$

But we take the value of α and β equal to a fraction. The corresponding eigenfunctions (and quantized eigenvaules) have been derived by the method of numerical solution of differential equation to a desired precision by *Mathematica* software (and graphical analysis).

To solve Schrödinger equation, the values of constants throughout the whole work will be taken as follows:

$$m=1,\hbar=1,k=1 \tag{4}$$

We consider the ground state (n=0) and first excited state (n=1) of the fractional harmonic oscillator for numerical analysis. So the Schrödinger equations take the form respectively,

$$\frac{d^2\Psi(x)}{dx^2}=-2\pi\left((1+a)-2\pi x^{\beta}\right)\Psi(x) \tag{5}$$

$$\frac{d^2\Psi(x)}{dx^2}=-2\pi\left((3+a)-2\pi x^{\beta}\right)\Psi(x) \tag{6}$$

Where a stands for the fractional perturbation term over the energy eigenvalue of linear harmonic oscillator.

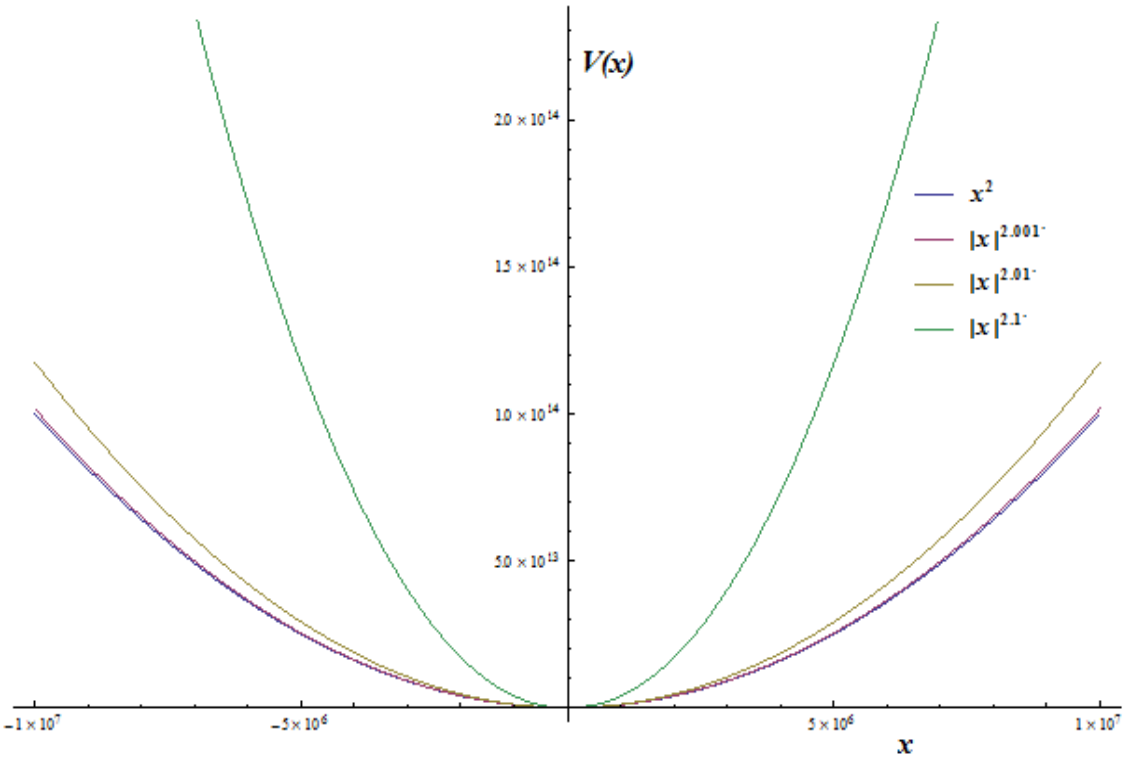

Figure 1. *Variation of linear harmonic and fractional potential with respect to position coordinates.*

In the figure 1, we have plotted variation of potential with position for different value of β. This figure clearly shows that shape of potential well has a large difference when we increase β from 2.01 to 2.1. But potential well are almost identical for $\beta = 2$ and $\beta = 2.001$. We have plotted the wavefunctions with respect to some arbitrary scale.

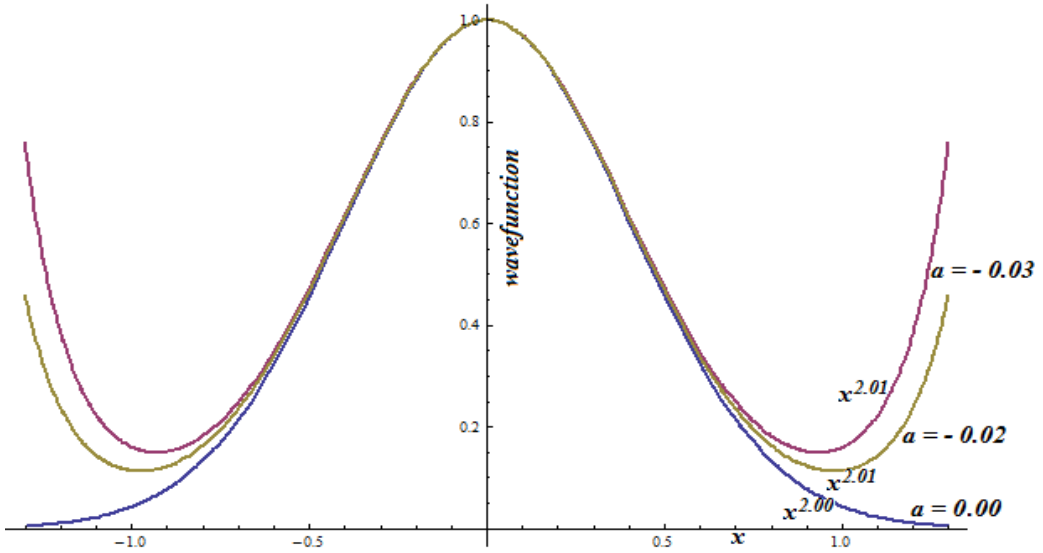

Figure 2. *Solutions of Schrödinger equation (n=0) for different values of energy where* $\beta = 2$ and $\beta = 2.01$.

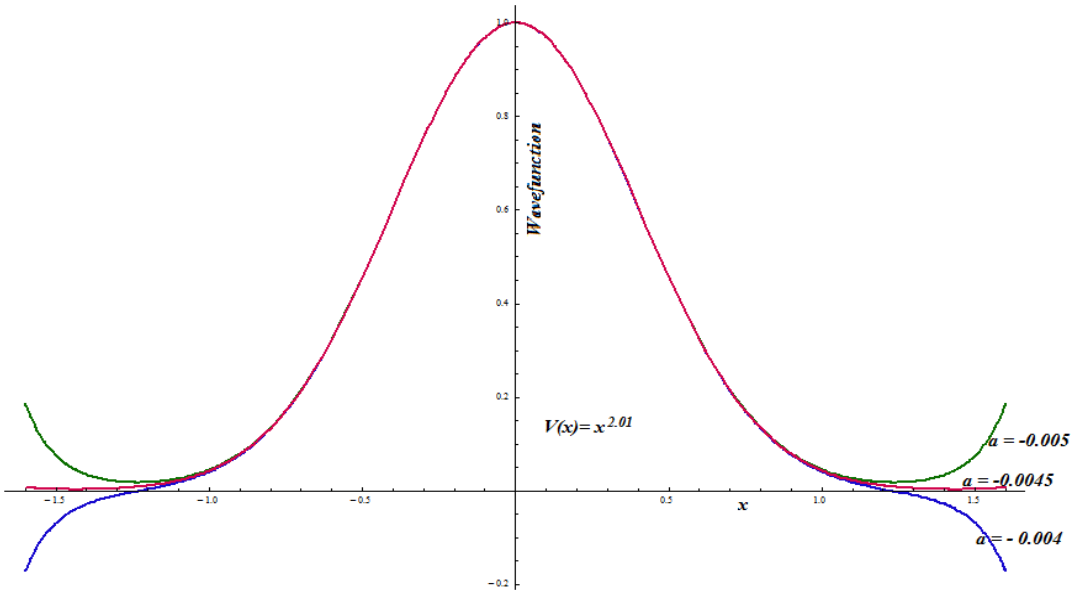

Figure 3. *Solutions of Schrödinger equation (n=0) for different values of energy where* $\beta = 2.01$.

Figure 2 depicts the ground state wave functions for $V(x) = x^{2.00}$ and $V(x) = x^{2.01}$ for different values of the parameter 'a'. We get a usual normalized wavefunction when $a = 0.00$ which corresponds to the energy $E_0 = 0.5\hbar\omega$. But when we change the energy by changing the parameter 'a' we have got the wavefunctions that blow exponentially towards the infinity. So to find the normalizable wavefunction, we have varied the parameter 'a' and solutions are plotted in the figure 3. It is clearly evident that $a = -0.0045$ is the nearest approach towards the normalized wave solution among the three solutions of this figure for $V(x) = x^{2.01}$. Finally we have got a normalized ground state wavefunction for $a = -0.00448$ which corresponds to the energy $E_0 = 0.49776\hbar\omega$. Which is 0.448% lower than the usual ground state energy of linear harmonic oscillator. The precision of results are correct up to the value $\pm 10^{-5}$.

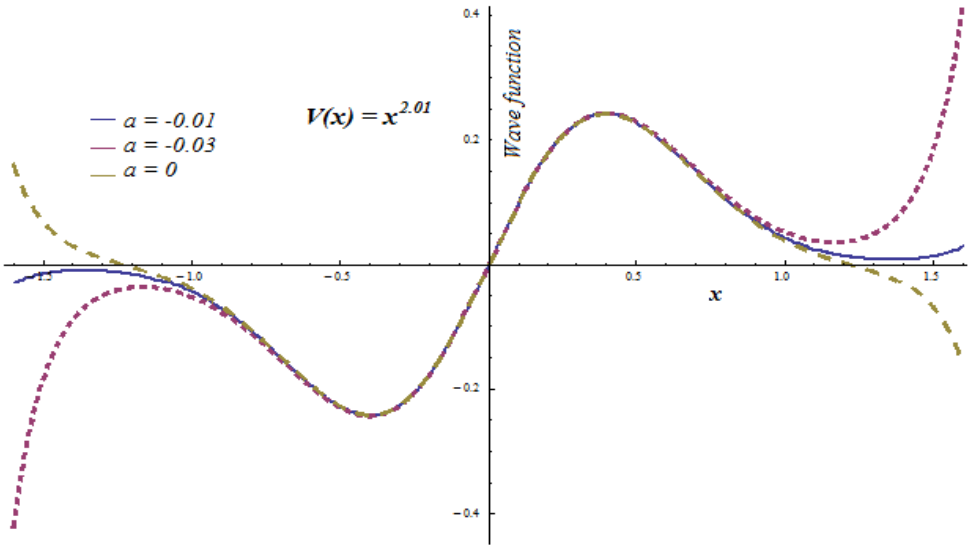

Figure 4. *Solutions of Schrödinger equation (n=1) for different values of energy when* $\beta = 2.01$.

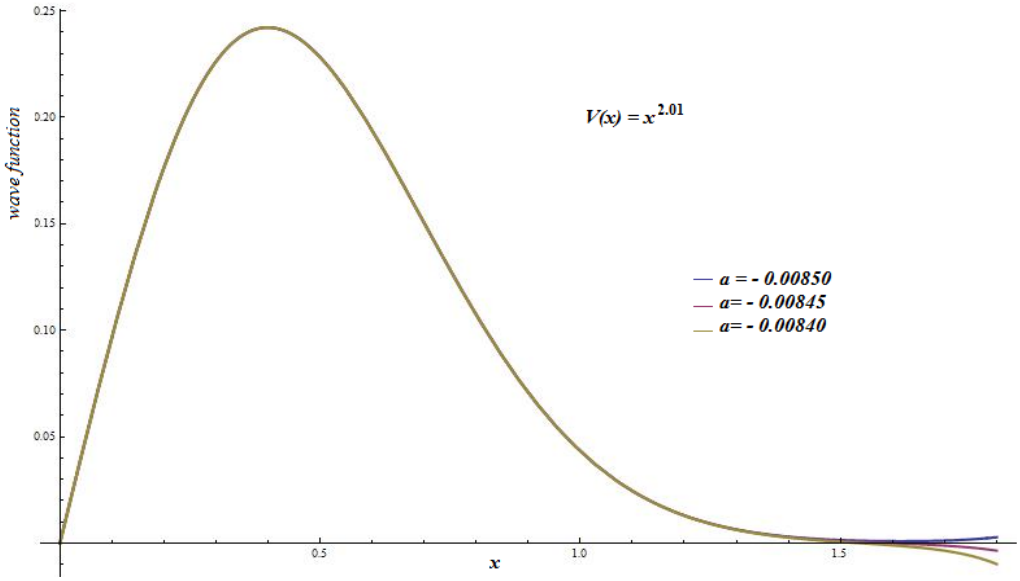

Figure 5. *Solutions of Schrödinger equation (n=1) for different values of energy* ($\beta = 2.01$) *in the first quadrant.*

We have found three distinct wavefunctions for three respective energies specified by the parameter $a = -0.01, -0.03, 0.00$ for first excited state of the oscillator. These three solutions are plotted in figure 4. Among the above three, last value of a is taken from conventional linear harmonic energy for n=1. But it is evident from the figure that these solutions are not normalizable, because these solutions blow up exponentially at large x. Figure 4 suggests that normalizable solution should be obtained between $a = 0$ and $a = -0.01$. For higher precision we have plotted the solutions for $a = -0.00850, a = -0.00845$ and $a = -0.00840$ in the figure 5. This plot shows that $a = -0.00850$ is the better approach to the normalised wavefunction for first excited state with the said potential well. The corresponding energy for the first excited state we have calculated is $E_1 = 1.49575\hbar\omega$.Which is 0.283% lower than the usual first excited state energy of linear harmonic oscillator.

3. Conclusion and Summary

Nature follows nonlinearity of space, so the restoring force can have nonlinear part in any circumstances while dealing with the real world. So we introduce fractional potential and deviations from linear harmonic oscillators are analysed graphically. We have solved time independent Schrödinger equations numerically and derive the normalised wavefunctions for ground state and first excited state with high degree of accuracy. The calculation is mainly focussed on $\beta = 2.01$. Wavefunction blows exponentially towards infinity at the classical forbidden regions with a slight departure from proper energy eigenvalue.

Energy eigenvalues for ground state and first excited state are $E_0 = 0.49776\hbar\omega$ and $E_1 = 1.49575\hbar\omega$ respectively. Energy gap between these adjacent energy states is $0.99799\hbar\omega$ which 0.201% lower than that for one dimensional linear harmonic oscillator. The present work could be extended for higher excited states and dimensions.

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