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On the growth analysis of complex linear differential equations with entire and meromorphic coefficients

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Abstract

The theory of complex differential equation has been developed since 1960's.Many researchers like IlpoLaine (1993) have investigated the system of complex differential equation of the following form $k \geq 2$,

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+.......+A_1(z)f'+A_0(z)f=0$$
$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+.......+A_1(z)f'+A_0(z)f=F(z)$$

where $A_i(z)$'s (i=0,1,2….k-1) and $F(z)\neq 0$ are entire or meromorphic functions.The prime concern of this paper is to investigate the comparative growth analysis of the solution as well as the coefficients of the above system of equations.

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1. Introduction

For any two transcendental entire functions f and g defined in the open complex plane C, Clunie[3] proved that

$\lim_{r\to\infty}\frac{T(r,fog)}{T(r,f)}=\infty$ **and** $\lim_{r\to\infty}\frac{T(r,fog)}{T(r,g)}=\infty.$

Singh [13] proved some comparative growth properties of $\log T(r,fog)$ and $T(r,f)$.He also raised the problem of investigating the comparative growth of $\log T(r,fog)$ and $T(r,g)$ which he was unable to solve.However some result on comparative growth of $\log T(r,fog)$ and $T(r,g)$ are proved later.

Let f be an entire function defined in the open complex plane C.Known[8] studied on the growth of an entire function f satisfying second order linear differential equation.Later Chen[4] proved some result on the growth of solutions of second order linear differential equations with meromorphiccoefficients.Chen and Yang[5]eshtablished a few theorems on the zeros and growths of entire functions of second order linear differential equations.The purpose of this paper is to study on the growth of the solution $f \neq 0$ of the nth order linear differential equation

$$f^{(n)} + A_1(z) f^{(n-1)} + A_2(z) f^{(n-2)} + ... + A_n(z) f = 0 ,$$

where A_i's $(\neq 0)$ are entire functions.In this paper we investigate the comparative growth of composite entire functions which satisfy nth order linear differential equations.

We do not explain the standard notations and definitions in the theory of entire and meromorphic functions as those are available in [13] and [7].

The following definitions are well known.

Definition 1 The order ρ_f and lower order λ_f of an entire function f is defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}$$

where $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$, $for\, k = 1,2,....,$ and $\log^{[0]} x = x.$

If f is meromorphic, one can easily verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

Definition 2 The hyper order $\overline{\rho}_f$ and hyper lower order $\overline{\lambda}_f$ of an entire function f is defined as follows

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r} \text{ and } \overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

If f is meromorphic then

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r} \text{ and } \overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r}.$$

Definition 3 The type σ_f of an entire function f is defined as

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

If f is meromorphic then

$$\sigma_f = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

Definition 4 Let f be an entire function order zero.Then the quantities $\rho_f^*, \lambda_f^*, \overline{\rho}_f^*, \overline{\lambda}_f^*$ are defined in the following way:

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r} \qquad \lambda_f^* = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r}.$$

$$\overline{\rho}_f^* = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log^{[2]} r} \qquad \overline{\lambda}_f^* = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log^{[2]} r}.$$

If f is meromorphic then clearly

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log^{[2]} r} \qquad \lambda_f^* = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log^{[2]} r}$$

$$\overline{\rho}_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log^{[2]} r} \qquad \overline{\lambda}_f^* = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log^{[2]} r}.$$

Definition 5 Let 'a' be a complex number, finite or infinite . The Nevanlinna deficiency and the Valiron deficiency of 'a' w.r.t. a meromorphic function f are defined as

$$\delta(a;f)=1-\limsup_{r\to\infty}\frac{N(r,a;f)}{T(r,f)}=\liminf_{r\to\infty}\frac{m(r,a;f)}{T(r,f)}$$
$$\Delta(a;f)=1-\liminf_{r\to\infty}\frac{N(r,a;f)}{T(r,f)}=\limsup_{r\to\infty}\frac{m(r,a;f)}{T(r,f)}.$$

Now let us define another function :

Let $\Psi:[0,\infty)\to(0,\infty)$ be a non-decreasing unbounded function, satisfying the following two conditions:

$$(i)\lim_{r\to\infty}\frac{\log^{[p]}r}{\log^{[q]}\Psi(r)}=0$$
$$(ii)\lim_{r\to\infty}\frac{\log^{[q]}\Psi(\alpha r)}{\log^{[q]}\Psi(r)}=1$$

for some $\alpha>1$.

With the help of the function Ψ ,the classical definitions can be written as,

Definition 6 The Ψ- order $\rho_{f,\Psi}$ and lower Ψ -order $\lambda_{f,\Psi}$ of an entire function f is defined as follows:

$$\rho_{f,\Psi}=\limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log\Psi(r)} \text{ and } \lambda_{f,\Psi}=\liminf_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log\Psi(r)}$$

where $\log^{[k]}x=\log\left(\log^{[k-1]}x\right), for k=1,2,....,and \log^{[0]}x=x.$

If $\rho_{f,\Psi}<\infty$ then f is of finite Ψ -order.Also $\rho_{f,\Psi}$ =0 means that f is of Ψ -order zero. In this connection following Liao and Yang [11] we may give the definition as below:

Definition 7{ *cf.* [11]} Let f be an entire function of Ψ order zero.Then the quantities $\rho^{*}_{f,\Psi},\lambda^{*}_{f,\Psi}$,are defined in the following way:

$$\rho^{*}_{f,\Psi}=\limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log^{[2]}\Psi(r)} \quad \text{and} \quad \lambda^{*}_{f,\Psi}=\liminf_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log^{[2]}\Psi(r)}\cdot$$

In the line of Datta and Biswas [6] an alternative definition of zero Ψ -order and zero Ψ -lower order of an entire function may be given as:

Definition 8{ *cf.* [6]} Let f be an entire function of Ψ order zero.Then the quantities $\rho^{**}_{f,\Psi},\lambda^{**}_{f,\Psi}$,are defined in the following way:

$$\rho^{**}_{f,\Psi}=\limsup_{r\to\infty}\frac{\log M(r,f)}{\log \Psi(r)} \quad \text{and} \quad \lambda^{**}_{f,\Psi}=\liminf_{r\to\infty}\frac{\log M(r,f)}{\log \Psi(r)}\cdot$$

Definition 9 The $\Psi-type\,\sigma_{f,\Psi}$ and $\Psi-lowertype\,\overline{\sigma}_{f,\Psi}$ of an entire function f are defined as:

$$\sigma_{f,\Psi}=\limsup_{r\to\infty}\frac{\log M(r,f)}{\Psi(r)^{\rho_{f,\Psi}}} \quad \text{and} \quad \overline{\sigma}_{f,\Psi}=\liminf_{r\to\infty}\frac{\log M(r,f)}{\Psi(r)^{\rho_{f,\Psi}}},\ 0<\rho_{f,\Psi}<\infty.$$

2. Research Method : Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 [1] If f is meromorphic and g is entire then for all sufficiently large values of r,

$$T(r, fog) \le \{1+o(1)\}\frac{T(r,g)}{\log M(r,g)} T(M(r,g), f).$$

Lemma 2 [2] If f is meromorphic and g is entire and suppose that $0<\mu\le\rho_g\le\infty$. Then for a sequence of values of r tending to infinity,

$$T(r, fog) \ge T(\exp(r^{\mu}), f).$$

Lemma 3 [12] If f , g be two transcendental entire functions with $\rho_g<\infty$, η be a constant satisfying 0< η <1 and α be a positive number.Then

$$T(r, fog)+O(1) \ge N(r,0; fog) \ge \log\left(\frac{1}{\eta}\right)\left[\frac{N(M((\eta r)^{\frac{1}{1+\alpha}}, g),0, f)}{\log M((\eta r)^{\frac{1}{1+\alpha}}, g)-O(1)} - O(1)\right]$$

as $r \to \infty$ through all values.

3. Results and Analysis.

In this section we present the main results of the paper.

Theorem 1 Let f be an entire function satisfying the n[th] order linear differential equation

$$f^{(n)} + A_1(z) f^{(n-1)} + A_2(z) f^{(n-2)} + \dots\dots + A_n(z) f = 0$$

where $A_i(z)'s$ are non zero entire functions.If

(i) $\rho_{A_1,\Psi}, \rho_{A_2,\Psi}, \dots\dots, \rho_{A_n,\Psi}$ all are finite,

(ii) $\lambda_{(A_1\circ A_2\circ\dots\circ A_{n-1}),\Psi}, \lambda_{f,\Psi}$ are both non negative,

(iii) $\rho_{A_n,\Psi} \le \lambda_{(A_1\circ A_2\circ\dots\circ A_{n-1}),\Psi}$ and $\rho_{A_n,\Psi} \le \lambda_{f,\Psi}$ i.e. $\rho_{A_n,\Psi} \le \min\{\lambda_{(A_1\circ A_2\circ\dots\circ A_{n-1}),\Psi}, \lambda_{f,\Psi}\}$ and

(iv) A_n be of regular growth, then

$$\lim_{r\to\infty} \frac{\{\log T(r,(A_1\circ A_2\circ\dots\circ A_{n-1})\circ A_n)\}^2}{T(r,f)T(r,(A_1\circ A_2\circ\dots\circ A_{n-1}))} = 0.$$

Proof It is well known that for an entire function A_n, $T(r,A_n) \le \log^{+} M(r,A_n)$ .So in view of Lemma 1 ,we get for all sufficiently large values of r,

$$T(r,(A_1\circ A_2\circ\dots\circ A_{n-1})\circ A_n) \le (1+o(1))T(M(r,A_n),(A_1\circ A_2\circ\dots\circ A_{n-1}))$$

$$i.e., \log T(r,(A_1\circ A_2\circ\dots\circ A_{n-1})\circ A_n) \le \log(1+o(1)) + \log T(M(r,A_n),(A_1\circ A_2\circ\dots\circ A_{n-1}))$$

$$i.e., \log T(r,(A_1\circ A_2\circ\dots\circ A_{n-1})\circ A_n) \le o(1) + (\rho_{(A_1\circ A_2\circ\dots\circ A_{n-1}),\Psi} + \varepsilon)\log M(\Psi(r), A_n)$$

$$i.e., \log T(r,(A_1\circ A_2\circ\dots\circ A_{n-1})\circ A_n) \le o(1) + (\rho_{(A_1\circ A_2\circ\dots\circ A_{n-1}),\Psi} + \varepsilon)\Psi(r)^{\rho_{A_n,\Psi}+\varepsilon} \quad \dots\dots\dots\dots\dots\dots(1)$$

Also, we obtain for all sufficiently large values of r,

$$T(r,(A_1\circ A_2\circ\dots\circ A_{n-1})) \ge \Psi(r)^{\lambda_{(A_1\circ A_2\circ\dots\circ A_{n-1}),\Psi}-\varepsilon} \quad \dots\dots\dots\dots\dots(2)$$

Now combining (1) & (2) it follows for all sufficiently large values of r,

$$\frac{\log T(r,(A_1\circ A_2\circ.....A_{n-1})\circ A_n)}{T(r,(A_1\circ A_2\circ.....A_{n-1}))} \le \frac{o(1)+(\rho_{(A_1\circ A_2\circ.....A_{n-1}),\Psi}+\varepsilon)\Psi(r)^{\rho_{A_n,\Psi}+\varepsilon}}{\Psi(r)^{\lambda_{(A_1\circ A_2\circ.....A_{n-1}),\Psi}-\varepsilon}}$$

$$\Rightarrow \limsup_{r\to\infty}\frac{\log T(r,(A_1\circ A_2\circ.....A_{n-1})\circ A_n)}{T(r,(A_1\circ A_2\circ.....A_{n-1}))} \le \limsup_{r\to\infty}\frac{o(1)+(\rho_{(A_1\circ A_2\circ.....A_{n-1}),\Psi}+\varepsilon)\Psi(r)^{\rho_{A_n,\Psi}+\varepsilon}}{\Psi(r)^{\lambda_{(A_1\circ A_2\circ.....A_{n-1}),\Psi}-\varepsilon}}.$$

Since, $\rho_{A_n,\Psi} < \lambda_{(A_1\circ A_2\circ.....A_{n-1}),\Psi}$, we can choose $\varepsilon > 0$ in such a way that $\rho_{A_n,\Psi} + \varepsilon < \lambda_{(A_1\circ A_2\circ.....A_{n-1}),\Psi} - \varepsilon$ and $\Psi(r)$ is a non-decreasing function ,so it follows from above that

$$\limsup_{r\to\infty}\frac{\log T(r,(A_1\circ A_2\circ.....A_{n-1})\circ A_n)}{T(r,(A_1\circ A_2\circ.....A_{n-1}))} = 0$$

i.e. $$\lim_{r\to\infty}\frac{\log T(r,(A_1\circ A_2\circ.....A_{n-1})\circ A_n)}{T(r,(A_1\circ A_2\circ.....A_{n-1}))} = 0 \quad ....(3)$$

Again, for all sufficiently large values of r,

$$\log T(r,f) \ge (\lambda_{f,\Psi}-\varepsilon)\log\Psi(r) \quad ....(4)$$

$$\Rightarrow T(r,f) \ge \Psi(r)^{\lambda_{f,\Psi}-\varepsilon}$$

Since, $\rho_{A_n,\Psi} < \lambda_{f,\Psi}$,we can choose $\varepsilon > 0$ in such a way that

$\rho_{A_n,\Psi} + \varepsilon < \lambda_{f,\Psi} - \varepsilon$. ....(5)

Now combining (1),(4)& (5) it follows for all sufficiently large values of r,

$$\frac{\log T(r,(A_1\circ A_2\circ.....A_{n-1})\circ A_n)}{T(r,f)} \le \frac{o(1)+(\rho_{(A_1\circ A_2\circ.....A_{n-1}),\Psi}+\varepsilon)\Psi(r)^{\rho_{A_n,\Psi}+\varepsilon}}{\Psi(r)^{\lambda_{f,\Psi}-\varepsilon}}$$

i.e. $$\limsup_{r\to\infty}\frac{\log T(r,(A_1\circ A_2\circ.....A_{n-1})\circ A_n)}{T(r,f)} = 0$$

i.e. $$\lim_{r\to\infty}\frac{\log T(r,(A_1\circ A_2\circ.....A_{n-1})\circ A_n)}{T(r,f)} = 0 \quad ....(6)$$

Therefore in view of (3) and (6), we obtain that

$$\lim_{r\to\infty}\frac{\{\log T(r,(A_1\circ A_2\circ.....A_{n-1})\circ A_n)\}^2}{T(r,f)T(r,(A_1\circ A_2\circ.....A_{n-1}))}$$

i.e. $$\lim_{r\to\infty}\frac{\log T(r,(A_1\circ A_2\circ.....A_{n-1})\circ A_n)}{T(r,f)}\bullet\frac{\log T(r,(A_1\circ A_2\circ.....A_{n-1})\circ A_n)}{T(r,(A_1\circ A_2\circ.....A_{n-1}))} = 0$$

i.e. $$\lim_{r\to\infty}\frac{\{\log T(r,(A_1\circ A_2\circ.....A_{n-1})\circ A_n)\}^2}{T(r,f)T(r,(A_1\circ A_2\circ.....A_{n-1}))} = 0.$$

This proves the theorem.

Remark 1 The following example ensures the validity of the conclusion as drawn in Theorem 1.

Example 1 Let n=2 , $f = \Psi = z^2, A_1 = z, A_2 = z^2$.

Then

$$A_1 \circ A_2 = z^2,$$

$$\log T(r, A_1 \circ A_2) = \log\{2\log r + O(1)\},$$
$$\rho_{A_1,\Psi} = \limsup_{r\to\infty} \frac{\log\log T(r, A_1)}{\log \Psi(z)} = 0, \textit{ finite}.$$
$$\textit{Similarly}, \lambda_{z^2,\Psi} = 0,$$
$$\lambda_{f,\Psi} = 0.$$

Then
$$\lim_{r\to\infty} \frac{\{\log T(r, A_1 \circ A_2)\}^2}{T(r,f)T(r,A_1)} = 0.$$

Remark 2 We can choose A_i's as meromorphic function for i=1,2,…n-1, but A_n must be an entire function.

Thus in the next example we take A_1 as meromorphic function.

Example 2 Let n=2 , $f = \Psi = z^2, A_1 = \frac{1}{z-2}$ *and* $A_2 = z^2$.

$$A_1 \circ A_2 = \frac{1}{z^2 - 2},$$
$$\log T(r, A_1 \circ A_2) = \log\{2\log r + O(1)\},$$
$$\rho_{A_1,\Psi} = \limsup_{r\to\infty} \frac{\log\log T(r, A_1)}{\log \Psi(z)} = 0, \textit{ finite}.$$
$$\textit{Similarly}, \lambda_{A_1 A_2,\Psi} = 0,$$
$$\lambda_{f,\Psi} = 0.$$

Thus the conditions are satisfied

and $\lim_{r\to\infty} \frac{\{\log T(r, A_1 \circ A_2)\}^2}{T(r,f)T(r,A_1)} = 0.$

Theorem 2 Let f be an entire function satisfying the n[th] order linear differential equation

$$f^{(n)} + A_1(z) f^{(n-1)} + A_2(z) f^{(n-2)} + \dots\dots + A_n(z) f = 0,$$

where $A_i(z)$'s are non zero entire functions. If $\lambda_{(A_1 \circ A_2 \circ \dots \circ A_{n-1}),\Psi}$ =0 then

$$\rho_{(A_1 \circ A_2 \circ \dots \circ A_{n-1}) \circ A_n,\Psi} \geq \lambda^*_{(A_1 \circ A_2 \circ \dots \circ A_{n-1}),\Psi} \bullet \mu$$

where $0 < \mu < \rho_{(A_1 \circ A_2 \circ \dots \circ A_{n-1}),\Psi}$.

Proof : In view of Lemma 2 and for $0 < \mu < \rho_{(A_1 \circ A_2 \circ \dots \circ A_{n-1}),\Psi}$.,we get that

$$\rho_{(A_1 \circ A_2 \circ \dots \circ A_{n-1}) \circ A_n,\Psi} = \limsup_{r\to\infty} \frac{\log T(r, (A_1 \circ A_2 \circ \dots \circ A_{n-1}) \circ A_n)}{\log \Psi(r)}$$

$$\geq \liminf_{r\to\infty} \frac{\log T(\exp r^{\mu}, (A_1 \circ A_2 \circ \dots \circ A_{n-1}))}{\log \Psi(r)}$$

$$= \liminf_{r\to\infty} \frac{\log T(\exp r^{\mu}, (A_1 \circ A_2 \circ \dots \circ A_{n-1}))}{\log^{[2]}(\exp \Psi(r)^{\mu})} \bullet \liminf_{r\to\infty} \frac{\log^{[2]}(\exp \Psi(r)^{\mu})}{\log \Psi(r)}$$

$$= \lambda^*_{(A_1 \circ A_2 \circ \dots \circ A_{n-1}),\Psi} \bullet \liminf_{r\to\infty} \frac{\log (\Psi(r)^{\mu})}{\log \Psi(r)}$$

$$= \lambda^*_{(A_1 \circ A_2 \circ \cdots\cdots A_{n-1}),\Psi} \bullet \mu \liminf_{r\to\infty} \frac{\log(\Psi(r))}{\log \Psi(r)}$$

$$= \lambda^*_{(A_1 \circ A_2 \circ \cdots\cdots A_{n-1}),\Psi} \bullet \mu .$$

Thus the theorem is proved.

Theorem 3 Let f be an entire function satisfying the n[th] order linear differential equation

$$f^{(n)} + A_1(z) f^{(n-1)} + A_2(z) f^{(n-2)} + \cdots\cdots + A_n(z) f = F(z),$$

where $A_i(z)'s$ are non zero entire functions. If

$(i)\ \rho_{(A_1 \circ A_2 \circ \cdots\cdots A_{n-1}),\Psi}, \rho_{A_n,\Psi}$ are both finite,

$(ii)\ \rho_{A_n,\Psi} < \lambda_{F,\Psi}$ and $\lambda_{F,\Psi}$ is positive , then for any $\alpha \in (-\infty, \infty)$,

$$\lim_{r\to\infty} \frac{[\log\{T(r,((A_1 \circ A_2 \circ \cdots\cdots A_{n-1}) \circ A_n)) \log M(r, A_n)\}]^{1+\alpha}}{T(\exp r, F)} = 0 .$$

Proof : If $1+\alpha \le 0$, the theorem is obhious. So we suppose that $1+\alpha > 0$.In view of Lemma 1 , we have for all sufficiently large values of r,

$$\log\{T(r,(A_1 \circ A_2 \circ \cdots\cdots A_{n-1}) \circ A_n) \log M(r, A_n)\}$$

$$\le \log T(r, A_n) + \log T(M(r, A_n), (A_1 \circ A_2 \circ \cdots\cdots A_{n-1})) + \log\{1 + o(1)\}$$

$$\le (\rho_{A_n,\Psi} + \varepsilon) \log \Psi(r) + (\rho_{(A_1 \circ A_2 \circ \cdots\cdots A_{n-1}),\Psi} + \varepsilon) \Psi(r)^{\rho_{A_n,\Psi}+\varepsilon} + o(1)$$

$$\le \Psi(r)^{\rho_{A_n,\Psi}+\varepsilon} . \{ \rho_{(A_1 \circ A_2 \circ \cdots\cdots A_{n-1}),\Psi} + \varepsilon + \frac{(\rho_{A_n,\Psi} + \varepsilon) \log \Psi(r) + o(1)}{\Psi(r)^{\rho_{A_n,\Psi}+\varepsilon}} \} \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Again we have for all sufficiently large values of r,

$$\log T(r,F) \ge (\lambda_{F,\Psi} - \varepsilon) \log \Psi(r)$$

$$\Rightarrow T(r,F) \ge \Psi(r)^{(\lambda_{F,\Psi} - \varepsilon)}$$

$$\Rightarrow T(\exp r, F) \ge \Psi(r)^{(\lambda_{F,\Psi} - \varepsilon)} . \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Now combining (7) and (8) ,we have for all sufficiently large values of r

$$\frac{[\log\{T(r,(A_1 \circ A_2 \circ \cdots\cdots A_{n-1}) \circ A_n) \log M(r, A_n)\}]^{1+\alpha}}{T(\exp r, F)}$$

$$\le \frac{\Psi(r)^{(\rho_{A_n,\Psi}+\varepsilon)(1+\alpha)} \left\{ \rho_{A_1 A_2 \cdots A_{n-1},\Psi} + \varepsilon + \frac{(\rho_{A_n,\Psi} + \varepsilon) \log \Psi(r) + o(1)}{\Psi(r)^{(\rho_{A_n,\Psi}+\varepsilon)}} \right\}^{1+\alpha}}{\Psi(r)^{(\lambda_{F,\Psi} - \varepsilon)}}$$

Since $\rho_{A_n,\Psi} < \lambda_{F,\Psi}$,we can choose $\varepsilon > 0$ in such a way that

$$\rho_{A_n,\Psi} + \varepsilon < \lambda_{F,\Psi} - \varepsilon$$

i.e. $\limsup_{r\to\infty} \frac{[\log\{T(r,(A_1\circ A_2\circ .....A_{n-1})\circ A_n)\log M(r,A_n)\}]^{1+\alpha}}{T(\exp r,F)} = 0,$

from which the theorem follows.

Remark 3 :We choose f instead of F in the denominator of the statement then the analogous theorem also holds. The following example reveals the fact.

Example 3 : Let n=2 , $f = \Psi = z^2, A_1 = z, A_2 = z^2$ *and* $\alpha = 0.$

$$A_1\circ A_2 = z^2,$$
$$\log T(r, A_1\circ A_2) = \log(2\log r + O(1)),$$
$$\log M(r, z^2) = 2\log r,$$
$$T(\exp r, z^2) = 2r + O(1).$$

Then

$$\limsup_{r\to\infty} \frac{\{\log\{T(r,A_1\circ A_2)\log M(r,A_2)\}\}}{T(\exp r, f)} = 0.$$

Theorem 4 : Let f be an entire function satisfying the n[th] order linear differential equation

$$f^{(n)} + A_1(z)f^{(n-1)} + A_2(z)f^{(n-2)} + ... + A_n(z)f = 0,$$

where $A_i(z)'s$ are non zero entire functions. If

$$(i)\ 0<\overline{\lambda}_{(A_1\circ A_2\circ.....A_{n-1})\circ A_n,\Psi} \le \overline{\rho}_{(A_1\circ A_2\circ.....A_{n-1})\circ A_n,\Psi} < \infty$$
$$(ii)\ 0 < \overline{\rho}_{f,\Psi} < \infty$$

Then for any positive number α ,

$$\liminf_{r\to\infty} \frac{\log^{[2]}T(r,(A_1\circ A_2\circ .....A_{n-1})\circ A_n)}{\log^{[2]}T(r^\alpha, f)} \le \frac{\overline{\rho}_{(A_1\circ A_2\circ.....A_{n-1})\circ A_n,\Psi}}{\overline{\rho}_{f,\Psi}} \le \limsup_{r\to\infty} \frac{\log^{[2]}T(r,(A_1\circ A_2\circ .....A_{n-1})\circ A_n)}{\log^{[2]}T(r^\alpha, f)}.$$

Proof : From the definition of hyper Ψ -order we get for all sufficiently large values of r,

$$\log^{[2]}T(r,(A_1\circ A_2\circ .....A_{n-1})\circ A_n) \le (\overline{\rho}_{(A_1\circ A_2\circ.....A_{n-1})\circ A_n,\Psi} + \varepsilon)\log\Psi(r).$$

Again we have for a sequence of values of r tending to infinity,

$\log^{[2]}T(r^\alpha, f) \ge \left(\overline{\rho}_{f,\Psi} - \varepsilon\right)\log\Psi(r)$, as $\Psi(r)$ is equivalent to $\Psi(r^\alpha)$.

Now combining above two equations, it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[2]}T(r,(A_1\circ A_2\circ .....A_{n-1})\circ A_n)}{\log^{[2]}T(r^\alpha, f)} \le \frac{\{\overline{\rho}_{(A_1\circ A_2\circ.....A_{n-1})\circ A_n,\Psi} + \varepsilon\}\log\Psi(r)}{\{\overline{\rho}_{f,\Psi} - \varepsilon\}\log\Psi(r)}$$

Since $\varepsilon > 0$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r,(A_1 \circ A_2 \circ .....A_{n-1}) \circ A_n)}{\log^{[2]} T(r^\alpha, f)} \leq \frac{\overline{\rho}_{(A_1 \circ A_2 \circ .....A_{n-1}) \circ A_n), \Psi}}{\overline{\rho}_{f,\Psi}} ....(9)$$

Also for arbitrary positive ε and for all sufficiently large values of r,

$$\log^{[2]} T(r^\alpha, f) \leq \left(\overline{\rho}_{f,\Psi} + \varepsilon\right) \log \Psi(r) \qquad ....(.10)$$

Also for a sequence of values of r tending to infinity,

$$\log^{[2]} T(r,(A_1 \circ A_2 \circ .....A_{n-1}) \circ A_n) \geq \left(\overline{\rho}_{(A_1 \circ A_2 \circ .....A_{n-1}) \circ A_n), \Psi} - \varepsilon\right) \log \Psi(r) ....(11)$$

Now from (10) & (11) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(r,(A_1 \circ A_2 \circ .....A_{n-1}) \circ A_n)}{\log^{[2]} T(r^\alpha, f)} \geq \frac{\left(\overline{\rho}_{(A_1 \circ A_2 \circ .....A_{n-1}) \circ A_n), \Psi} - \varepsilon\right) \log \Psi(r)}{\left(\overline{\rho}_{f,\Psi} + \varepsilon\right) \log \Psi(r)}$$

Since $\varepsilon > 0$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(r,(A_1 \circ A_2 \circ .....A_{n-1}) \circ A_n)}{\log^{[2]} T(r^\alpha, f)} \geq \frac{\left(\overline{\rho}_{(A_1 \circ A_2 \circ .....A_{n-1}) \circ A_n), \Psi}\right)}{\left(\overline{\rho}_{f,\Psi}\right)} \qquad ....(12)$$

Then the theorem follows from (9) and (12).

Remark 4 If $\Psi(z) = z$,we may obtain the corollary.

Corollary 1 For $0 < \overline{\lambda}_{(A_1 \circ A_2 \circ .....A_{n-1}) \circ A_n), \Psi} \leq \overline{\rho}_{(A_1 \circ A_2 \circ .....A_{n-1}) \circ A_n), \Psi} < \infty$ and $0 < \overline{\rho}_{f,\Psi} < \infty$,then for any positive number α,

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r,(A_1 \circ A_2 \circ .....A_{n-1}) \circ A_n)}{\log^{[2]} T(r^\alpha, f)} \leq \frac{\overline{\rho}_{(A_1 \circ A_2 \circ .....A_{n-1}) \circ A_n), \Psi}}{\alpha.\overline{\rho}_{f,\Psi}} \leq \limsup_{r\to\infty} \frac{\log^{[2]} T(r,(A_1 \circ A_2 \circ .....A_{n-1}) \circ A_n)}{\log^{[2]} T(r^\alpha, f)}.$$

Remark 5 Theorem 4 and Corollary 1 shows that the middle part of the first one is independent of α, where as the second one is dependent on the same.

Theorem 5 Let f be an entire function satisfying the n[th] order linear differential equation

$$f^{(n)} + A_1(z) f^{(n-1)} + A_2(z) f^{(n-2)} + ... + A_n(z) f = 0,$$

where $A_i(z)'s$ are non zero entire functions. If

$(i)\ 0 < \sigma_{(A_1 \circ A_2 \circ .....A_{n-1}) \circ A_n, \Psi} < \infty,$

$(ii)\ 0 < \rho_{f,\Psi} < \infty,$

$(iii)\ \sigma_{f,\Psi} < \infty,$

$(iv)\ \rho_{(A_1 \circ A_2 \circ .....A_{n-1}) \circ A_n, \Psi} = \rho_{f,\Psi}.$

Then

$$\liminf_{r\to\infty}\frac{T(r,(A_1\circ A_2\circ.....A_{n-1})\circ A_n)}{T(r,f)}\le\frac{\sigma_{(A_1\circ A_2\circ.....A_{n-1})\circ A_n),\Psi}}{\sigma_{f,\Psi}}\le\limsup_{r\to\infty}\frac{T(r,(A_1\circ A_2\circ.....A_{n-1})\circ A_n)}{T(r,f)}.$$

Proof From the definition of $\Psi-$ type , we get for arbitrary positive ε and for all sufficiently large values of r,

$$T(r,(A_1\circ A_2\circ.....A_{n-1})\circ A_n)\le(\sigma_{(A_1\circ A_2\circ.....A_{n-1})\circ A_n,\Psi}+\varepsilon)\Psi(r)^{\rho_{(A_1\circ A_2\circ...A_{n-1})\circ A_n,\Psi}}. \quad .....(13)$$

Again we have for a sequence of values of r tending to infinity,

$$T(r,f)\ge(\sigma_{f,\Psi}-\varepsilon)\Psi(r)^{\rho_{f,\Psi}} \quad ....(14)$$

Since $\rho_{(A_1\circ A_2\circ.....A_{n-1})\circ A_n,\Psi}=\rho_{f,\Psi}$,so from (13) and (14) it follows for a sequence of values of r tending to infinity,

$$\frac{T(r,(A_1\circ A_2\circ.....A_{n-1})\circ A_n)}{T(r,f)}\le\frac{\{\sigma_{(A_1\circ A_2\circ.....A_{n-1})\circ A_n),\Psi}+\varepsilon\}}{\{\sigma_{f,\Psi}-\varepsilon\}}$$

Since $\varepsilon>0$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty}\frac{T(r,(A_1\circ A_2\circ.....A_{n-1})\circ A_n)}{T(r,f)}\le\frac{\sigma_{(A_1\circ A_2\circ.....A_{n-1})\circ A_n,\Psi}}{\sigma_{f,\Psi}} \quad ....(15)$$

Also for a sequence of values of r tending to infinity,

$$T(r,(A_1\circ A_2\circ.....A_{n-1})\circ A_n)\ge\left(\sigma_{(A_1\circ A_2\circ.....A_{n-1})\circ A_n),\Psi}-\varepsilon\right)\Psi(r)^{\rho_{(A_1\circ A_2\circ...A_{n-1})\circ A_n,\Psi}} \quad ....(16)$$

Now for all sufficiently large values of r,

$$T(r,f)\le(\sigma_{f,\Psi}+\varepsilon)\Psi(r)^{\rho_{f,\Psi}}. \quad ....(17)$$

Now from (16) & (17) we obtain for a sequence of values of r tending to infinity,

$$\frac{T(r,(A_1\circ A_2\circ.....A_{n-1})\circ A_n)}{T(r,f)}\ge\frac{\{\sigma_{(A_1\circ A_2\circ.....A_{n-1})\circ A_n,\Psi}-\varepsilon\}}{\{\sigma_{f,\Psi}+\varepsilon\}}$$

Since $\varepsilon>0$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty}\frac{T(r,(A_1\circ A_2\circ.....A_{n-1})\circ A_n)}{T(r,f)}\ge\frac{\sigma_{(A_1\circ A_2\circ.....A_{n-1})\circ A_n),\Psi}}{\sigma_{f,\Psi}} \quad ....(18)$$

Then the theorem follows from (15) and (18).

Theorem 6 Let f be a transcendental entire function satisfying the nth order linear differential equation

$$f^{(n)}+A_1(z)f^{(n-1)}+A_2(z)f^{(n-2)}+........+A_n(z)f=0,$$

where $A_i(z)'s$ are non zero entire functions. If

$(i)\, 0 < \lambda_{A_n,\Psi} \le \rho_{A_n,\Psi} < \infty,$

$(ii)\, \lambda_{A_1 \circ A_2 \circ \dots \circ A_{n-1},\Psi} > 0,$

$(iii)\, \rho_{f,\Psi} < \infty,$

$(iv)\, \delta(0; A_1 \circ A_2 \circ \dots \circ A_{n-1}) < 1,$

Then

$$\limsup_{r\to\infty} \frac{\log T(r,(A_1 \circ A_2 \circ \dots \circ A_{n-1}) \circ A_n)}{\log T(r^{\beta}, f)} = \infty,$$ where β is a real constant.

Proof We suppose that $\beta > 0$, otherwise the theorem is obvious.

For given $\varepsilon(0 < \varepsilon < 1 - \delta(0; A_1 \circ A_2 \circ \dots \circ A_{n-1})),$

$N(r,0;(A_1 \circ A_2 \circ \dots \circ A_{n-1})) > (1 - \delta(0; A_1 \circ A_2 \circ \dots \circ A_{n-1}) - \varepsilon) T(r, A_1 \circ A_2 \circ \dots \circ A_{n-1})$, for a sequence of values or r tending to infinity.

So from Lemma 3 , we get for a sequence of values or r tending to infinity,

$$T(r,(A_1 \circ A_2 \circ \dots \circ A_{n-1}) \circ A_n) + O(1)$$
$$\ge \left(\log \frac{1}{\eta}\right)\left[\frac{(1 - \delta(0; A_1 \circ A_2 \circ \dots \circ A_{n-1}) - \varepsilon) T\left(\left(M(\eta r)^{\frac{1}{1+\alpha}}, A_n\right), (A_1 \circ A_2 \circ \dots \circ A_{n-1})\right)}{\log\left(M(\eta r)^{\frac{1}{1+\alpha}}, A_n\right) - O(1)} - O(1)\right]$$

$$\Rightarrow \log T(r,(A_1 \circ A_2 \circ \dots \circ A_{n-1}) \circ A_n) + O(1) \ge O(\log r) + \log T\left(\left(M(\eta r)^{\frac{1}{1+\alpha}}, A_n\right), (A_1 \circ A_2 \circ \dots \circ A_{n-1})\right)$$
$$+ \log\left[1 - \frac{\log M((\eta r)^{\frac{1}{1+\alpha}}, A_n) + O(1)}{(1 - \delta(0;(A_1 \circ A_2 \circ \dots \circ A_{n-1})) - \varepsilon) T\left(\left(M(\eta r)^{\frac{1}{1+\alpha}}, A_n\right), (A_1 \circ A_2 \circ \dots \circ A_{n-1})\right)}\right]$$

Since f is transcendental ,it follows that

$$\lim_{r\to\infty} \frac{\log M((\eta r)^{\frac{1}{1+\alpha}}, A_n)}{T\left(\left(M(\eta r)^{\frac{1}{1+\alpha}}, A_n\right), (A_1 \circ A_2 \circ \dots \circ A_{n-1})\right)} = 0.$$

So from above we get for a sequence of values of r tending to infinity ,

$$\Rightarrow \log T(r,(A_1 \circ A_2 \circ \dots \circ A_{n-1}) \circ A_n) \ge O(\log r) + \log T\left(\left(M(\eta r)^{\frac{1}{1+\alpha}}, A_n\right), (A_1 \circ A_2 \circ \dots \circ A_{n-1})\right) + O(1) \quad \dots\dots\dots\dots\dots\dots(19)$$

Also we see that for all large values of r,

$$M(r, A_n) > \exp\left\{\Psi(r)^{\frac{1}{2}\lambda_{A_n,\Psi}}\right\} \quad \dots\dots\dots\dots\dots\dots(20)$$

$$\log T(r,(A_1 \circ A_2 \circ \dots \circ A_{n-1})) > \frac{1}{2}\lambda_{(A_1 \circ A_2 \circ \dots \circ A_{n-1}),\psi} \log \Psi(r) \dots\dots\dots\dots(21)$$

So from (19),using (20) & (21) we get for a sequence of values of r tending to infinity ,

$$\frac{\log T\left(r,(A_1\circ A2\circ.....\circ A_{n-1})\circ A_n\right)}{\log T\left(r^{\beta},f\right)} > \frac{O(\log r)}{(1+\rho_{f,\Psi})\log\Psi(r)} + \frac{\lambda_{A1\circ A2\circ.....\circ A_{n-1},\Psi}}{2} \bullet \frac{(\eta r)^{\frac{\lambda_{A_n,\Psi}}{2(1+\alpha)}}}{(1+\rho_{f,\Psi})\log\Psi(r)} + O(1)$$

$$\Rightarrow \limsup_{r\to\infty} \frac{\log T\left(r,(A_1\circ A2\circ.....\circ A_{n-1})\circ A_n\right)}{\log T\left(r^{\beta},f\right)} = \infty.$$

This proves the theorem.

Remark 6 If we consider Δ in the place of δ, then the analogous theorem is also true with 'limit superior' replaced by 'limit'.

Remark 7 In the theorem using Δ ,if we consider $\rho_{A_1,A_2,\ldots A_{n-1},\Psi} > 0$ instead of $\lambda_{(A_1\circ A2\circ.....\circ A_{n-1}),\Psi} > 0$ the theorem remains true with 'limit' replaced by 'limit superior'.

Conclusion

The results as deduced in this paper may be thought of from another angle of view and those can be carried out in case of difference polynomials of higher degree. Therefore several modified techniques may be adopted in order to solve the problems arisen and those can be regarded as a virgin area to the researchers in this field.

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